Chapter 29

Currents:

29-1 Pumping charges:

To make charge carriers flow through a resistor, one must establish a pot. difference between its ends.

i) one way

charged conducting sphere

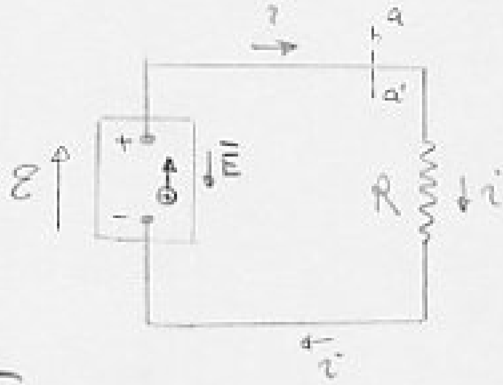
But this doesn't produce a steady current.

ii) We need a charge pump, a device that - by doing work on the charge carriers maintains a potential difference between the terminals of the pump.

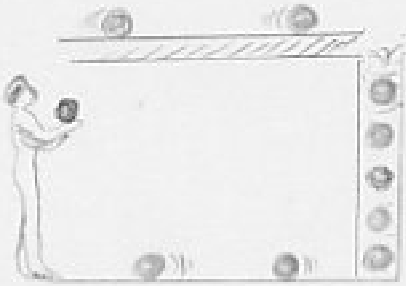
We call such a device an emf device, and the device is said to provide an emf $\mathcal{E}$, which means that, it does work on charge carriers.

Examples of emf device: Battery, Electric generator, solar cells

29-2 Work, Energy, And EMF



There must be some source of energy within the device, enabling it to do work on the charges and thus forcing them to move against $\bar{E}$.



This source of energy may be chemical (as in battery).

At any interval dt, dq passes aa'

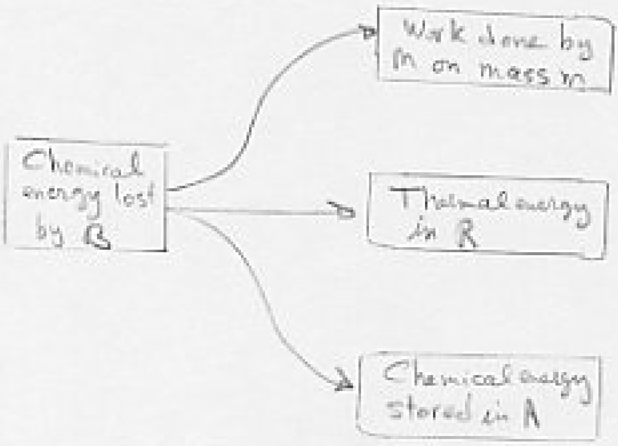
We define: $\mathcal{E} = \frac{dW}{dq}$ (def. of $\mathcal{E}$)

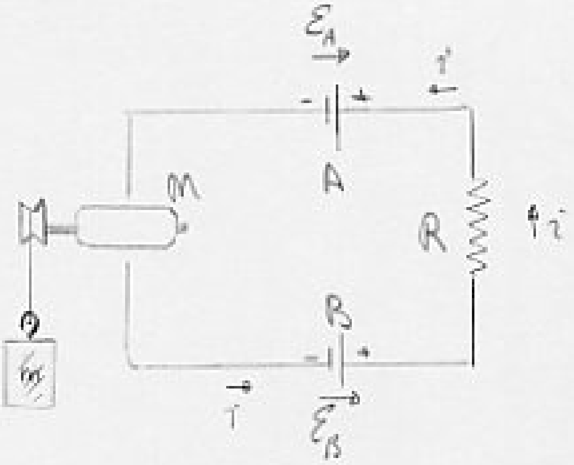
Def.: The emf of an emf device is the work per unit charge that the device does in moving charge from its low pot. terminal to its high pot. terminal.

Ideal emf device $r = 0$ $V_{+,-} = \mathcal{E}$

Real " " $r \neq 0$ $V_{+,-} \neq \mathcal{E}$

$\mathcal{E}_B > \mathcal{E}_A$





29-3 Calculating the current:

Energy method:

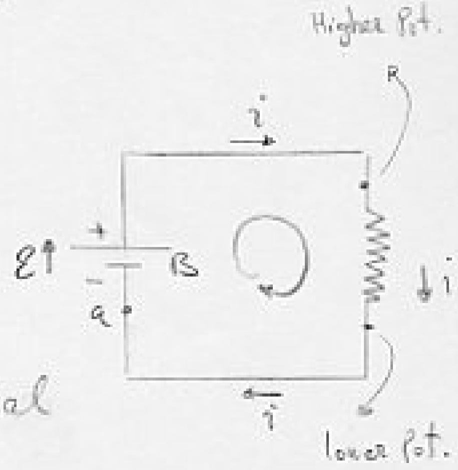
$P = i^2R \longrightarrow dW = i^2R\,dt$ Work done on R (within dt)

During the same time interval $dq = i\,dt$ have moved through the battery.

$dW = \mathcal{E}dq = \mathcal{E}i\,dt$ work done on dq by battery

Conservation of energy $\longrightarrow \mathcal{E}i\,dt = i^2R\,dt$

$\longrightarrow \mathcal{E} = iR \longrightarrow i = \dfrac{\mathcal{E}}{R}$



Potential method:

loop Rule: The algebraic sum of the changes in pot. encountered in a complete traversal of any circuit must be Zero.

This is often referred to as Kirchhoff's loop rule.

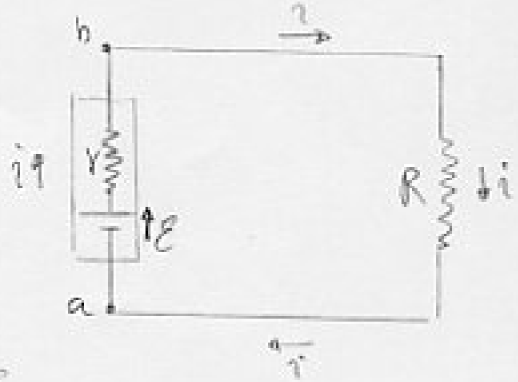
Resistance Rule: If you mentally pass through a resistance in the direction of the current, the change in potential is $-iR$.

EMF Rule: If you mentally pass through an ideal emf device in the direction of emf arrow, the change in potential is $+\mathcal{E}$.

$$V_a + \mathcal{E} - iR = V_a \longrightarrow \mathcal{E} - iR = 0$$

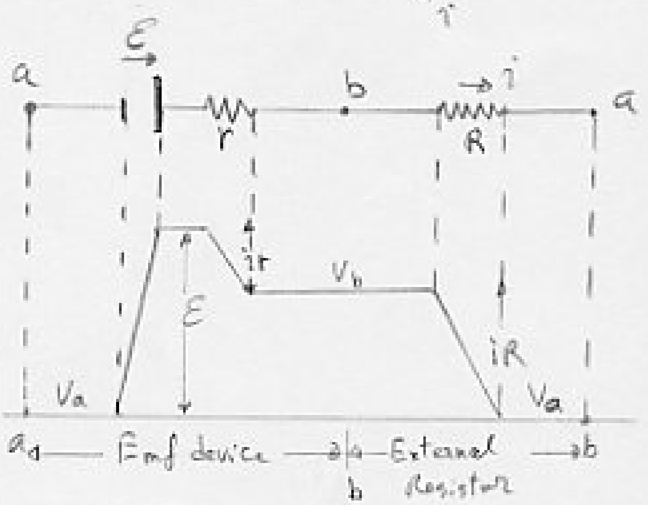
29-4 Other single-loop Circuits:

Internal Resistance:

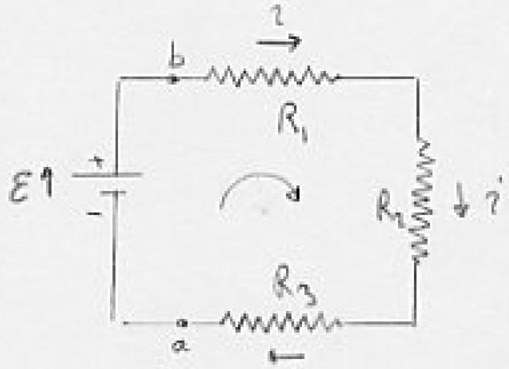


$$\mathcal{E} - ir - iR = 0$$

$$i = \frac{\mathcal{E}}{R+r}$$



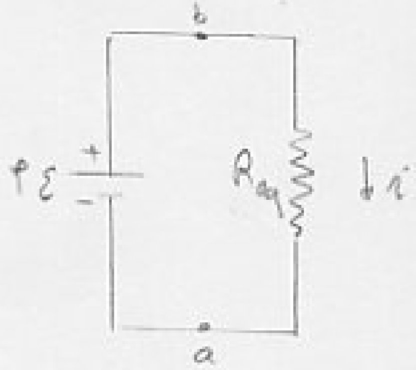
Resistors in Series:



$$\mathcal{E} - iR_1 - iR_2 - iR_3 = 0$$

$$\begin{cases} i = \dfrac{\mathcal{E}}{R_1 + R_2 + R_3} \\ i = \dfrac{\mathcal{E}}{R_{eq}} \end{cases} \longrightarrow R_{eq} = R_1 + R_2 + R_3$$

$$R_{eq} = \sum_{i=1}^{n} R_i$$



29-5 Potential Differences:

Ex. For Fig (P-139):

$$V_b - iR = V_a \quad \rightarrow \quad V_b - V_a = iR$$

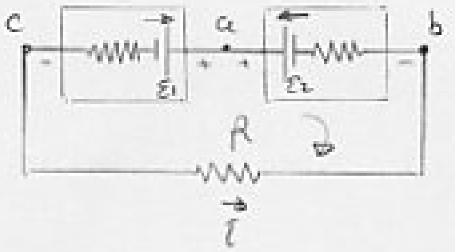
Combining with $i = \dfrac{\mathcal{E}}{R+r}$ $\longrightarrow V_b - V_a = \mathcal{E}\dfrac{R}{R+r}$

Alternatively:

$$V_b + ir - \mathcal{E} = V_a \quad \longrightarrow \quad V_b - V_a = \mathcal{E} - ir$$

Combining with $i = \dfrac{\mathcal{E}}{R+r}$ $\longrightarrow V_b - V_a = \mathcal{E}\dfrac{R}{R+r}$

Ex.



$\mathcal{E}_1 = 2.1\ V$, $\mathcal{E}_2 = 4.4\ V$

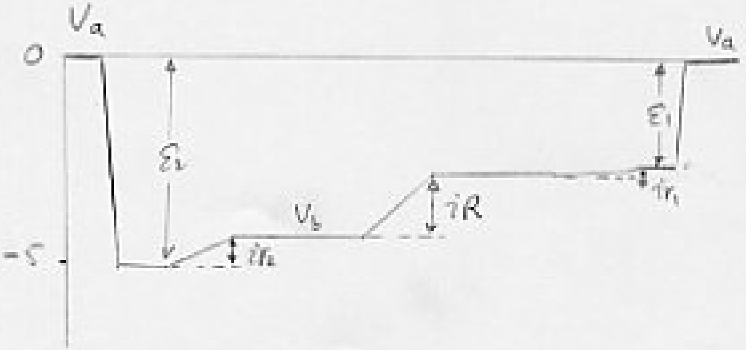
$r_1 = 1.8\ \Omega$ $r_2 = 2.3\ \Omega$ $R = 5.5\ \Omega$

$i = ?$

Sol.

$$-\mathcal{E}_2 + ir_2 + iR + ir_1 + \mathcal{E}_1 = 0$$

$$i = \frac{\mathcal{E}_2 - \mathcal{E}_1}{R + r_1 + r_2} = 0.2396\ A$$



$V_a - V_b = ?$

$V_b - ir_2 + \mathcal{E}_2 = V_a \longrightarrow V_a - V_b = -ir_2 + \mathcal{E}_2 = 3.84\ V$

Alternatively:

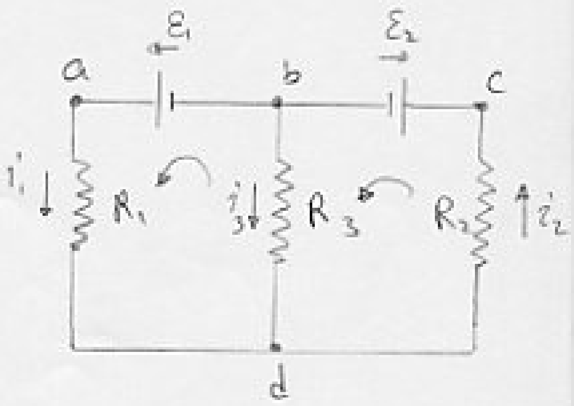
$V_b + iR + ir_1 + \mathcal{E}_1 = V_a \longrightarrow V_a - V_b = i(R + r_1) + \mathcal{E}_1 = 3.84\ V$

$V_a - V_c = ?$

$V_c + ir_1 + \mathcal{E}_1 = V_a \longrightarrow V_a - V_c = ir_1 + \mathcal{E}_1 = 2.5\ V$

29-6 Multiloop Circuits:

Junction Rule: The sum of the currents approaching any junction must be equal to the sum of the currents leaving that junction (Kirchhoff's junction rule).
This is a statement of conservation of charge.



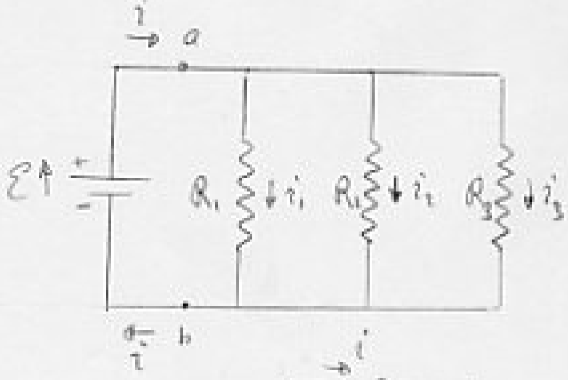
$$\begin{cases} \mathcal{E}_1 - i_1R_1 + i_3R_3 = 0 \\ -i_3R_3 - i_2R_2 - \mathcal{E}_2 = 0 \\ i_1 + i_3 = i_2 \end{cases}$$

$$i_1 = \frac{\mathcal{E}_1(R_2+R_3) - \mathcal{E}_2R_3}{R_1R_2 + R_2R_3 + R_1R_3}$$

$$i_2 = \frac{\mathcal{E}_1R_3 - \mathcal{E}_2(R_1+R_3)}{R_1R_2 + R_2R_3 + R_1R_3}$$

$$i_3 = \frac{\mathcal{E}_1R_2 + \mathcal{E}_2R_1}{R_1R_2 + R_2R_3 + R_1R_3}$$

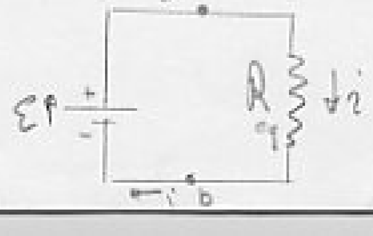
Resistance in Parallel:



$$i_1 = \frac{V}{R_1}, \quad i_2 = \frac{V}{R_2}, \quad i_3 = \frac{V}{R_3}$$

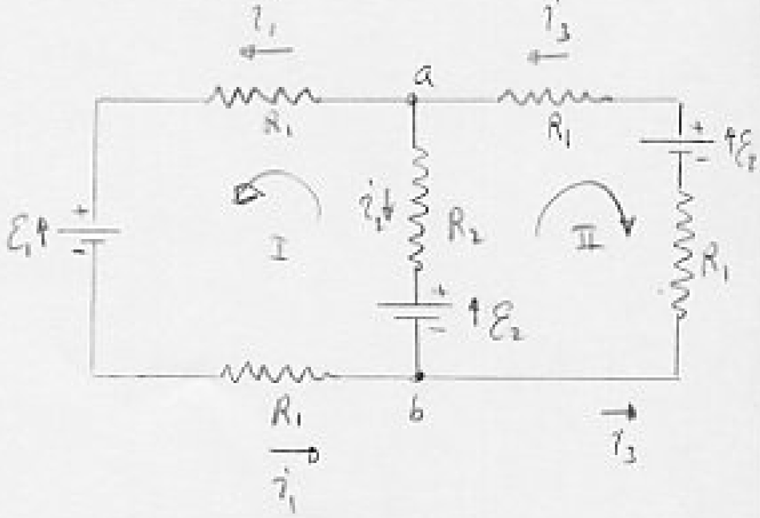
$$i = i_1 + i_2 + i_3 = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$i = \frac{V}{R_{eq}}$$



$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$\frac{1}{R_{eq}} = \sum_{i=1}^{n} \frac{1}{R_i}$$



<u>Ex.</u>

$\mathcal{E}_1 = 2.1\,V \quad \mathcal{E}_2 = 6.3\,V$

$R_1 = 1.7\,\Omega \quad R_2 = 3.5\,\Omega$

a) $i_1 = ?\quad i_2 = ?\quad i_3 = ?$

b) $V_a - V_b$

<u>Sol.</u>

a) $i_3 = i_1 + i_2$

I) $-i_1R_1 - \mathcal{E}_1 - i_1R_1 + \mathcal{E}_2 + i_2R_2 = 0$

II) $+i_3R_1 - \mathcal{E}_2 + i_3R_1 + \mathcal{E}_2 + i_2R_2 = 0$

$$\rightarrow \begin{cases} 2i_1R_1 - i_2R_2 = \mathcal{E}_2 - \mathcal{E}_1 \\ i_2R_2 + 2i_3R_1 = 0 \end{cases}$$

$$i_1 = \frac{(\mathcal{E}_2 - \mathcal{E}_1)(2R_1 + R_2)}{4R_1(R_1 + R_2)} = 0.82\,A$$

$$i_2 = -\frac{\mathcal{E}_2 - \mathcal{E}_1}{2(R_1 + R_2)} = -0.40\,A$$

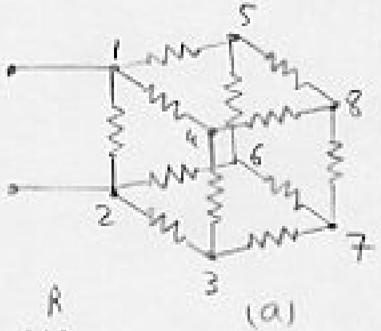
$$i_3 = \frac{(\mathcal{E}_2 - \mathcal{E}_1)R_2}{4R_1(R_1 + R_2)} = 0.42\,A$$

b) $V_a - i_2R_2 - \mathcal{E}_2 = V_b \rightarrow V_a - V_b = \mathcal{E}_2 + i_2R_2 = +4.9\,V$

Ex.

$V_4 = V_5$ → So if we connect the point 4 to 5, and 3 + 6, the configuration will not change.

$V_3 = V_6$



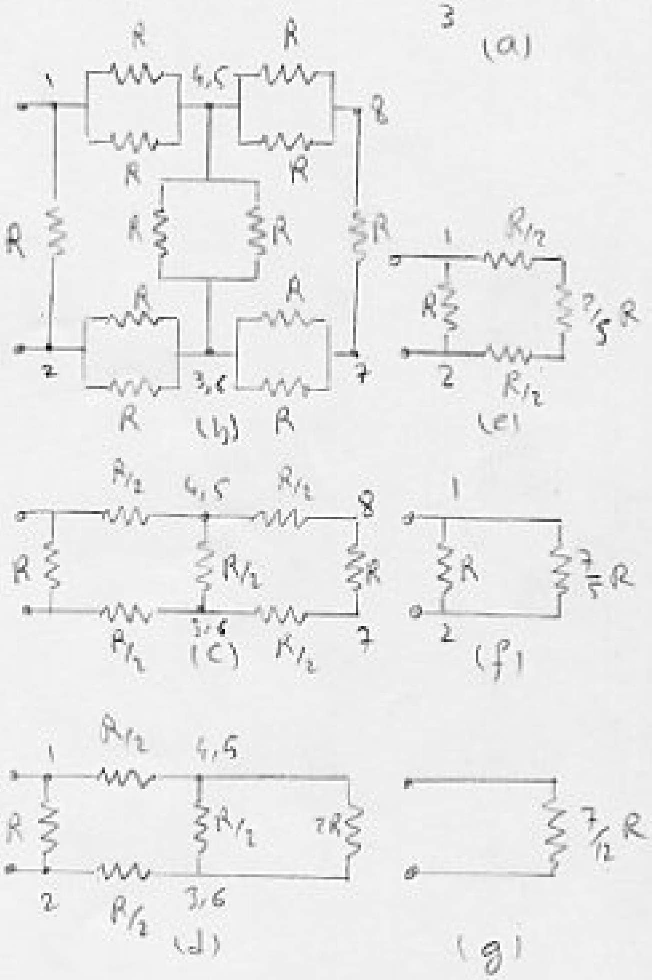
i.e. No current between 4, and 5

" " " 3 " 6

(a) → (b) → (c) → (d)

→ (e) → (f) → (g)

$$R_{eq} = \frac{7}{12} R$$



Ex.

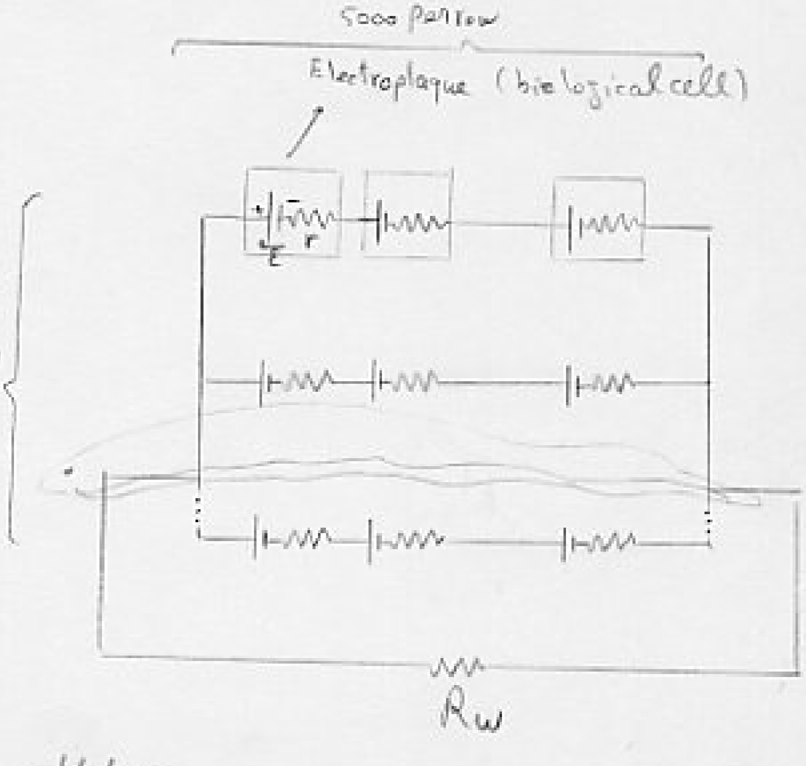
Electric fish:

$\mathcal{E} = 0.15\,V$ $r = 0.25\,\Omega$

$R_W = 800\,\Omega$ water resistance

a) $i_w = ?$ current through water

b) $i_{row} = ?$



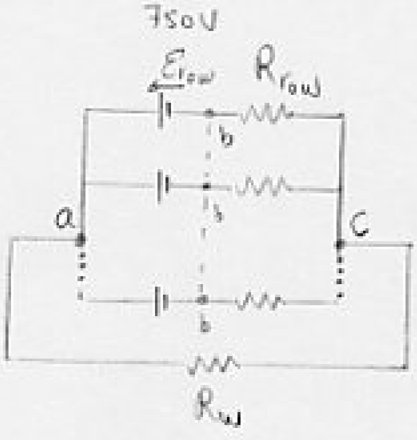
a)

$$\mathcal{E}_{row} = 5000\,(0.15) = 750\,V$$

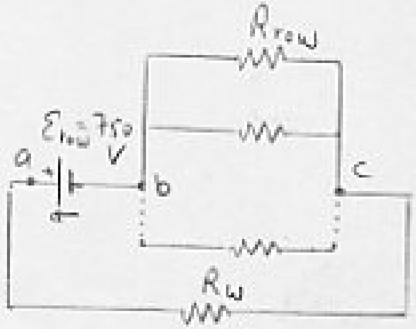
$$R_{row} = 5000\,(0.25) = 1250\,\Omega$$

All b points have the same pot.

So (b) ⟶ (c)



(b)

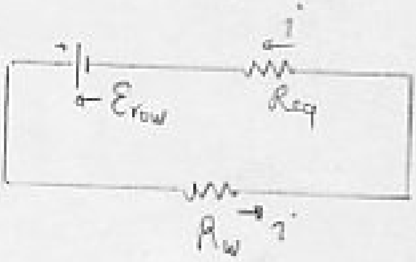


(c)

$$\frac{1}{R_{eq}} = \sum_{j=1}^{140} \frac{1}{R_j} = 140\,\frac{1}{R_{row}}$$

$$R_{eq} = \frac{R_{row}}{140} = \frac{1250}{140} = 8.93\,\Omega$$

$$\mathcal{E}_{row} - iR_w - iR_{eq} = 0 \qquad i = \frac{\mathcal{E}_{row}}{R_w + R_{eq}}$$



(d)

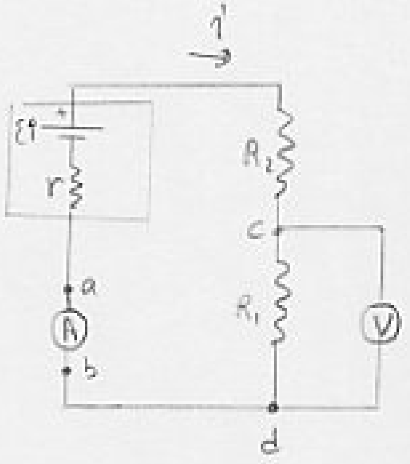
$$i = \frac{750}{800 + 8.93} = 0.93\,A$$

b) $$i_{row} = \frac{i}{140} = 6.6 \times 10^{-3}\,A \quad \text{(small)}$$

⟶ This means that eel need not stun or kill itself when it stuns or kills a fish.

29-7 Measuring Instruments:

1- The ammeter



$R_A << R$ (in the circuit)

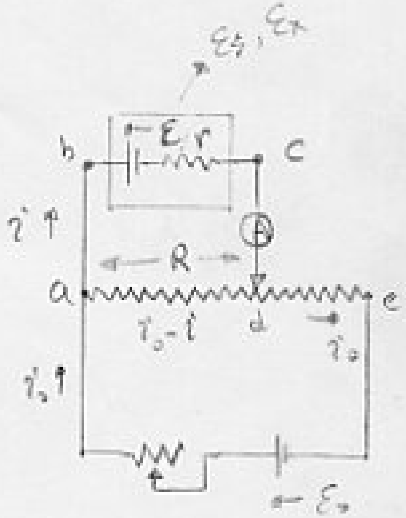
Example: $R_A << (r + R_1 + R_2)$

2- The Voltmeter

$R_V >> R$ (in the circuit)

Example: $R_V >> R_1$

3- Potentiometer



Aim: To find unknown $\mathcal{E}_x$.

1) Put $\mathcal{E}_s$ (standard emf) between b and c

2) Adjust potentiometer for zero current

At this moment $R \equiv R_s$, $\mathcal{E}_s = i_0 R_s$ (Since $i = 0$, r does not enter)

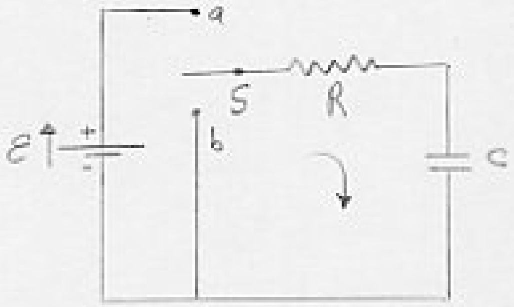
3) Replace $\mathcal{E}_s$ with $\mathcal{E}_x$ and repeat the same steps.

At this moment $R \equiv R_x$, $\mathcal{E}_x = i_0 R_x$

$$\longrightarrow \mathcal{E}_x = \mathcal{E}_s \frac{R_x}{R_s}$$

29-8 RC Circuits

Charging a Capacitor:



i) S on a position;

loop rule (clockwise)

$$\mathcal{E} - iR - \frac{q}{c} = 0$$ where $V_c = -\frac{q}{c}$ (the top plate being at higher pot.)

$$\rightarrow \begin{cases} iR + \frac{q}{c} = \mathcal{E} \\ i = \frac{dq}{dt} \end{cases} \rightarrow R\frac{dq}{dt} + \frac{q}{c} = \mathcal{E}$$ (charging equ.)

This equ can be solved with the inital condition ($q=0$ at $t=0$)

Now: $\frac{dq}{dt} + \frac{1}{RC}q = \frac{\mathcal{E}}{R}$ is especial form of

$$y'(x) + f(x)\,y = r(x)$$

$$\rightarrow (fy - r)\,dx + dy = 0$$

We have to find integrating factor $F(x)$, such that:

$$F(x)(fy - r)\,dx + F(x)\,dy = 0$$

to be exact.

i.e. $\frac{\partial}{\partial y}\left(F(fy - r)\right) = \frac{\partial F}{\partial x} \rightarrow Ff = \frac{\partial F}{\partial x}$

By separating variables: $\quad f\,dx = \frac{dF}{F}$

$$\rightarrow \ln|F| = \int f(x)\,dx \quad \rightarrow F(x) = e^{h(x)}$$

where $h(x) = \int f(x)\,dx$

Thus;

$$e^{h(x)}(y' + fy) = e^{h} r$$

$$\rightarrow \frac{d}{dx}(y e^{h}) = e^{h} r \qquad (\text{since } h' = f)$$

$$y e^{h} = \int e^{h} r\,dx + c \quad \rightarrow y(x) = e^{-h}\left[\int e^{h} r\,dx + c\right]$$

Now;

$$\left\{\begin{array}{l} q(t) \longrightarrow y(x) \\ \frac{1}{RC} \longrightarrow f(x) \\ \frac{\mathcal{E}}{R} \longrightarrow r(x) \\ t \longrightarrow x \\ h = \int \frac{1}{RC}\,dt \longrightarrow h = \int f(x)\,dx \end{array}\right.$$

$$h = \int \frac{1}{RC}\,dt = \frac{t}{RC}$$

$$q(t) = e^{-\frac{t}{RC}}\left[\int e^{\frac{t}{RC}} \frac{\mathcal{E}}{R}\,dt + b\right]$$

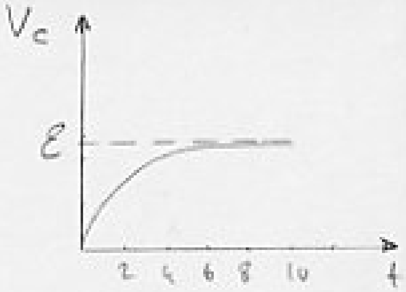
$$q(t) = e^{-\frac{t}{RC}}\left[\frac{\mathcal{E}}{R} RC\, e^{t/RC} + b\right]$$

$$q(t) = \mathcal{E}C + b e^{-t/RC}$$

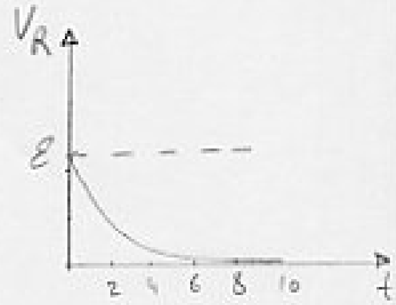
$$q(0) = 0 \quad \rightarrow 0 = \mathcal{E}C + b \quad \rightarrow b = -\mathcal{E}C$$

$$q(t) = \mathcal{E}C(1 - e^{-t/RC}) \qquad \text{charging capacitor}$$

$$i = \frac{dq}{dt} = \frac{\mathcal{E}}{R} e^{-t/RC}$$ Charging Capacitor



$$\begin{cases} V_c = \frac{q}{c} = \mathcal{E}(1 - e^{-t/RC}) \\ V_R = iR = \mathcal{E} e^{-t/RC} \end{cases}$$



The Time Const.:

$\tau \equiv RC$: capactive time const. (dimension of t)

At $t = \tau$; $q(\tau) = \mathcal{E}C(1 - e^{-1}) = 0.63\, \mathcal{E}C$

At $t \to \infty$; $q(\infty) = \mathcal{E}C$

$\to$ τ is the time at which the charge on the capacitor has increased about 63% of its fully charged value.

Discharging a Capacitor;

Assume now the capacitor is fully charged to the potential difference of the battery.

At a new $t = 0$, the switch S is thrown from a, to b,

$\to R\frac{dq}{dt} + \frac{q}{C} = 0 \quad \to q(t) = q_0\, e^{-t/RC} \quad q_0 = \mathcal{E}C$

charging capacitor in our case

At $t = \tau \equiv RC$. The capacitor charge is reduced about 37% of its initial charge.

$$i = \frac{dq}{dt} = -\frac{q_0}{RC} e^{-t/RC} = -i_0 e^{-t/RC}$$ discharging capacitor.

$$\frac{q_0}{RC} = \frac{\mathcal{E}C}{RC} = \frac{\mathcal{E}}{R} = i_0$$ (current at $t=0$)

Ex.

A capacitor C is discharging through R.

a) $t = ?$ for $q = \frac{1}{2} q_0$

$$q = q_0 e^{-t/RC} \qquad q = \frac{1}{2} q_0 \rightarrow \frac{1}{2} q_0 = q_0 e^{-t/RC}$$

$$\rightarrow \ln \frac{1}{2} = -\frac{t}{RC} \qquad t = (-\ln \tfrac{1}{2}) RC = 0.69\,\tau$$

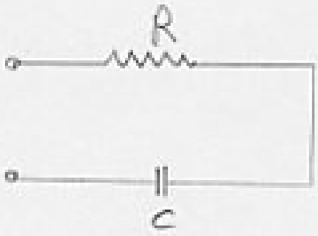
b) $t = ?$ $U = \frac{1}{2} U_0$

$$U = \frac{q^2}{2C} = \frac{q_0^2}{2C} e^{-2t/RC} = U_0 e^{-2t/RC}$$

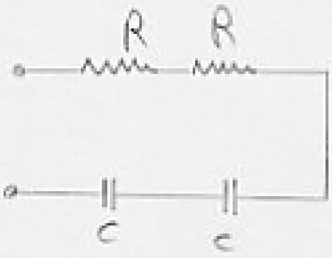
$$U = \frac{1}{2} U_0 \rightarrow \frac{1}{2} U_0 = U_0 e^{-2t/RC} \rightarrow \ln \frac{1}{2} = -\frac{2t}{RC}$$

$$t = -RC \frac{\ln \frac{1}{2}}{2} = 0.35\,\tau$$

Ex.



(a)



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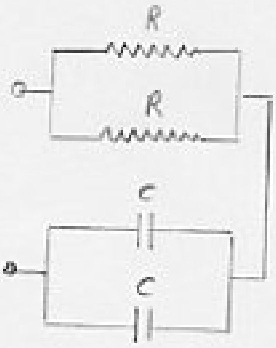


(c)

$\tau_a = RC$ for (a)

a) $\tau_b = ?$ " (b)

$\tau = (2R)(\frac{1}{2}C) = RC$

b) $\tau_c = ?$

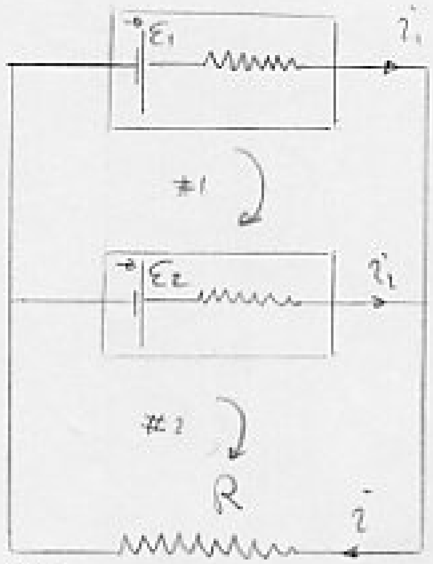
$\tau_c = (\frac{R}{2})(2C) = RC$

These results are evidence of the reciprocal nature of the formulas for combining resistors and capacitors in series and in parallel.

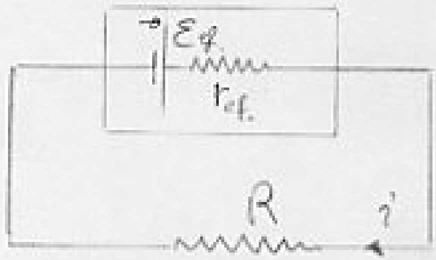
Ex.

Show that $\mathcal{E}_{ef} = r_{ef}\left(\frac{\mathcal{E}_1}{r_1} + \frac{\mathcal{E}_2}{r_2}\right)$

where $\frac{1}{r_{ef}} = \frac{1}{r_1} + \frac{1}{r_2}$



(a)



(b)

Sol.

(1) $\mathcal{E}_1 - i_1 r_1 + i_2 r_2 - \mathcal{E}_2 = 0$ #(1)

(2) $\mathcal{E}_2 - r_2 i_2 - iR = 0$ #(2)

(3) $i = i_1 + i_2$

$$i_1 = \frac{\mathcal{E}_1 r_2 - \mathcal{E}_2 R + \mathcal{E}_1 R}{r_1 r_2 + r_2 R + r_1 R}, \quad i_2 = i - i_1$$

$$\left\{\begin{array}{l} i = \dfrac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{r_1 r_2 + R(r_1 + r_2)} \\ r_{ef} = \dfrac{r_1 r_2}{r_1 + r_2} \longrightarrow i = \dfrac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{(r_1 + r_2)(r_{ef} + R)} \end{array}\right.$$

$(a) \underset{\text{equ.}}{\sim} (b)$

on the otherhand (b) $\Rightarrow i = \frac{\mathcal{E}_{ef}}{r_{ef} + R}$

$$\longrightarrow \frac{\mathcal{E}_{ef}}{r_{ef} + R} = \frac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{(r_1 + r_2)(r_{ef} + R)} \Rightarrow \mathcal{E}_{ef} = \frac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{r_1 + r_2}$$

$$\mathcal{E}_{ef} = \frac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{r_1 r_2} r_{ef} \rightarrow \mathcal{E}_{ef} = r_{ef}\left(\frac{\mathcal{E}_1}{r_1} + \frac{\mathcal{E}_2}{r_2}\right)$$

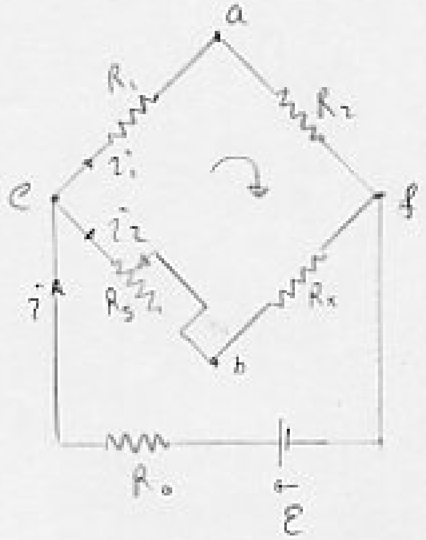
63P

Wheatston Bridge

R_s is adjusted in a value until $V_a = V_b$.

Show that at this moment

$$R_x = \frac{R_2}{R_1} R_x$$



$$\begin{cases} V_e - i_1 R_1 - i_1 R_2 = V_f \\ V_e - i_2 R_s - i_2 R_x = V_f \end{cases} \rightarrow \begin{cases} V_e - V_f = i_1 (R_1 + R_2) \\ V_e - V_f = i_2 (R_s + R_x) \end{cases}$$

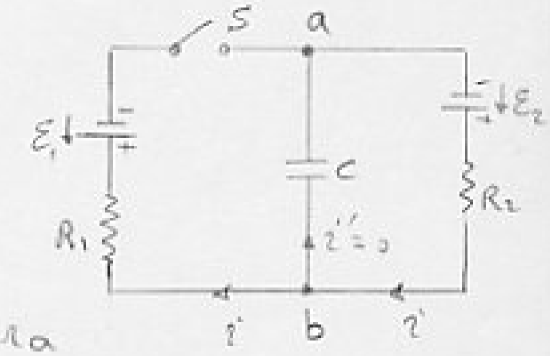
$$\rightarrow \quad i_1 (R_1 + R_2) = i_2 (R_s + R_x)$$

$$V_a = V_b \quad \rightarrow \quad i_1 R_1 = i_2 R_s \quad \rightarrow \quad i_1 = \frac{R_s}{R_1} i_2$$

$$\rightarrow \quad \frac{R_s}{R_1} (R_1 + R_2)\, i_2 = (R_s + R_x)\, i_2$$

$$\rightarrow R_x = \frac{R_2}{R_1} R_s$$

77P



$C = 10\,\mu F$, $\mathcal{E}_1 = 1.0\,V$, $\mathcal{E}_2 = 3.0\,V$ $R_1 = 0.2\,\Omega$
$R_2 = 0.4\,\Omega$

S; Open for a long time, then is closed for a long time $q_1 - q_2 = ?$

Sol.

i) S open ; $q_1 = C\mathcal{E}_2$ $q_1 = 10 \times 10^{-6} \times 3 = 3 \times 10^{-5}\,C$

ii) S closed ; $\mathcal{E}_2 - iR_2 - iR_1 - \mathcal{E}_1 = 0$ $3 - (0.4 + 0.2)i - 1 = 0$

$i = 3.3\,A$ ($i' = 0$ after a long time) $V_a + \mathcal{E}_2 - iR_2 = V_b$ $V_b - V_a = 1.7\,V$

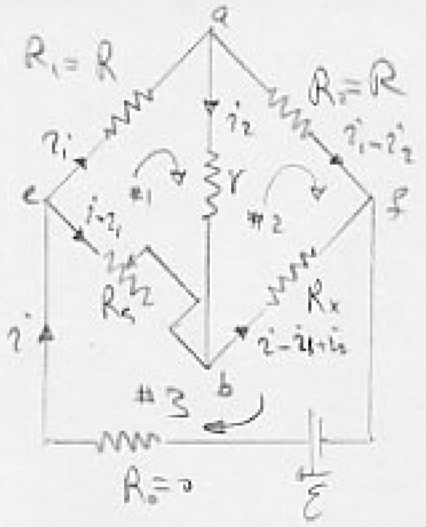
$q_2 = C(V_b - V_a) = 1.7 \times 10^{-5}$

$q_1 - q_2 = 1.3 \times 10^{-3}\,C$

64P

If $R_1 = R_2 = R$ $R_0 = 0$

Show that $i = \dfrac{\mathcal{E}(R_s - R_x)}{(R+2r)(R_s+R_x)+2R_sR_x}$



Sol.

(1) $i_1R + i_2r - (i - i_1)R_s = 0$

(2) $(i_1 - i_2)R - (i - i_1 + i_2)R_x - i_2r = 0$

(3) $\mathcal{E} - (i - i_1)R_s - (i - i_1 + i_2)R_x = 0$

(3)' $\mathcal{E} - i_1R - (i_1 - i_2)R = 0$

Solving $i_2 = \dfrac{\mathcal{E}(R_s - R_x)}{(R+2r)(R_s+R_x)+2R_sR_x}$

If $V_a = V_b \rightarrow V_a - V_b = i_2r = 0 \rightarrow i_2 = 0$

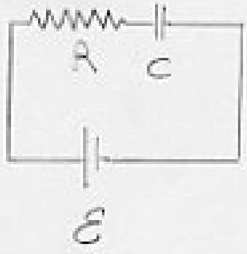
$\rightarrow R_s = R_x$

In the previous prob $R_x = \dfrac{R_2}{R_1} \cdot R_s$

Since $R_1 = R_2 \rightarrow R_x = R_s$

The results are consistent.

72P $R = 3\,M\Omega$ $C = 1\,\mu F$ $\mathcal{E} = 4\,V$



At $t = 1\,s$ after the connection is made, what are the rates at which:

a) the charge of the capacitor is increasing

b) energy is being stored in C ; c) thermal energy in R

d) energy is being delivered by the battery.

Sol.

$$q = C\mathcal{E}(1 - e^{-t/RC})$$

a) $\dot{q} = \frac{dq}{dt} = \frac{\mathcal{E}}{R} e^{-t/RC} = \frac{4}{3\times10^6} e^{-1/3\times10^6\times10^{-6}} = 9.6\times10^{-7}\ C/s$

b) $U = \frac{1}{2} q^2 C$ $\frac{dU}{dt} = \frac{q}{C}\dot{q} = \mathcal{E}(1 - e^{-t/RC})\dot{q}$

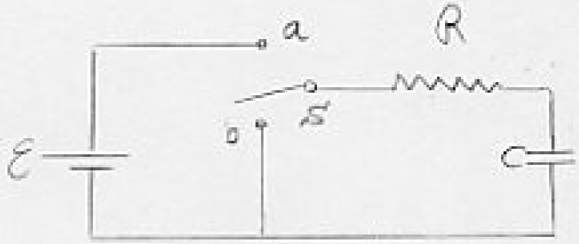
$$\frac{dU}{dt} = 4(1 - e^{-1/3})(0.96\times10^{-6}) = 1.1\times10^{-6}\ W$$

c) $P_R = \frac{dW}{dt} = i^2 R = (\dot{q})^2 R = (0.96\times10^{-6})^2\, 3\times10^{6} = 2.7\times10^{-6}\ W$

d) $P_{\mathcal{E}} = P_R + P_C = 2.7\times10^{-6} + 1.1\times10^{-6} = 3.8\times10^{-6}\ W$

Ex.

S: For a long time on a-position, Now it is thrown to b-position.



$W_R = ?$

Sol. $dW = i^2 R\, dt$

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(C\mathcal{E}\, e^{-t/RC}\right) = \frac{\mathcal{E}}{R}\, e^{-t/RC}$$

$$W = \int_0^\infty \frac{\mathcal{E}^2}{R^2}\, e^{-2t/RC}\, R\, dt = \frac{\mathcal{E}^2}{R}\left[-\frac{RC}{2}\, e^{-2t/RC}\right]_0^\infty = \frac{\mathcal{E}^2}{R}\,\frac{RC}{2}$$

$$W = \frac{1}{2}\mathcal{E}^2 C$$

which is exactly the energy stored in C,

i.e. $\quad U_C = \frac{1}{2}\mathcal{E}^2 C = \frac{1}{2}\frac{q^2}{C}$

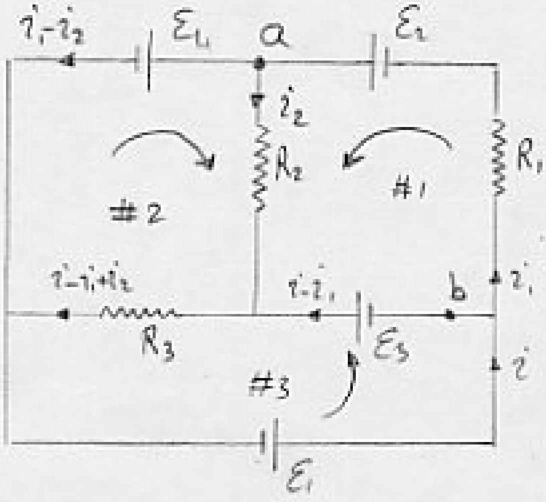
Exam Prob.

$\mathcal{E}_1 = 18V$, $\mathcal{E}_2 = 15V$, $\mathcal{E}_3 = 9V$, $\mathcal{E}_4 = 6V$

$R_1 = 2\Omega$, $R_2 = 4\Omega$, $R_3 = 3\Omega$

Current through $\mathcal{E}_1, R_1, R_2$?

$V_b - V_a = ?$



sol.

#1) $\mathcal{E}_2 - i_2 R_2 - \mathcal{E}_3 - i_1 R_1 = 0$

#2) $\mathcal{E}_4 - i_2 R_2 - (i - i_1 + i_2) R_3 = 0$

#3) $\mathcal{E}_1 + \mathcal{E}_3 - (i - i_1 + i_2) R_3 = 0$

$i_1 = 13.5A$ $\quad i_2 = -5.25A$, $\quad i = 27.75A$

$$\begin{cases} 15 - 4i_2 - 9 - 2i_1 = 0 \\ 6 - 4i_2 - 3(i - i_1 + i_2) = 0 \\ 18 + 9 - 3(i - i_1 + i_2) = 0 \end{cases}$$

$V_b - \mathcal{E}_1 + \mathcal{E}_4 = V_a$ $\quad \rightarrow \quad V_b - V_a = \mathcal{E}_1 - \mathcal{E}_4 = 18 - 6 = 12V$