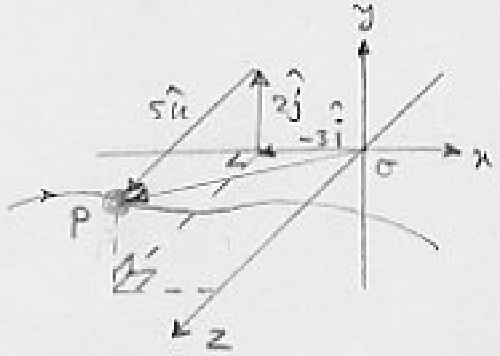
Chapter 4

Motion in 2- and 3-dim.

4-2 Position and Displacement:

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ position vector (say of a particle)

<u>Ex.</u> $\bar{r} = -3\hat{i} + 2\hat{j} + 5\hat{k}$

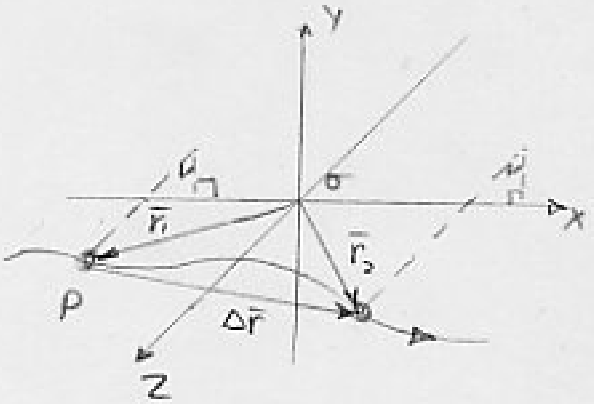


Displacement:

Position of the object at t_1 : r_1

" " " " " $t_1 + \Delta t$: r_2

$\Delta\bar{r} = \bar{r}_2 - \bar{r}_1$ displacement



Sample Prob 4-1

$\bar{r}_1 = -3\hat{i} + 2\hat{j} + 5\hat{k}$

$\bar{r}_2 = 9\hat{i} + 2\hat{j} + 8\hat{k}$

$\Delta\bar{r} = ?$

<u>Sol.</u> $\Delta\bar{r} = (9\hat{i} + 2\hat{j} + 8\hat{k}) - (-3\hat{i} + 2\hat{j} + 5\hat{k})$

$= 12\hat{i} + 3\hat{k}$

4-3 Velocity and Average Velocity:

$$\bar{\vec{V}} = \frac{\Delta \vec{r}}{\Delta t} \qquad \bar{\vec{V}} = \frac{\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k}$$

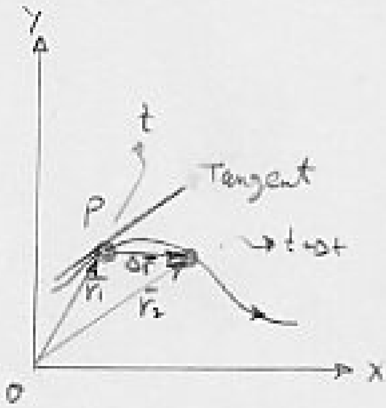
average velocity

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

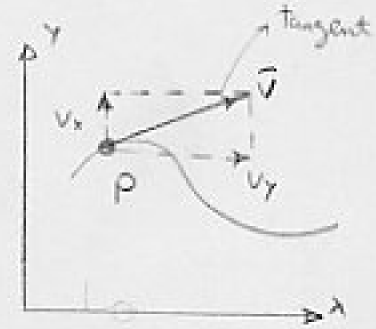
instantaneous velocity

$$\rightarrow \vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

where $v_x = \frac{dx}{dt}$, $V_y = \frac{dy}{dt}$, $V_z = \frac{dz}{dt}$



$$\text{As } \Delta t \to 0 \begin{cases} \bar{r}_2 \xrightarrow{\text{moves toward}} \bar{r}_1 \Rightarrow \Delta\bar{r} \to 0 \\ \text{dir. of } \Delta\bar{r}\ (\text{dir. of } \bar{\vec{V}}) \to \text{dir. of tangent line} \\ \bar{\vec{V}} \longrightarrow \vec{V} \end{cases}$$



Remark: Dir. of $\bar{\vec{V}}$ = Dir. of $\Delta\vec{r}$

Dir of $\vec{V}$ = Dir. of tangent line

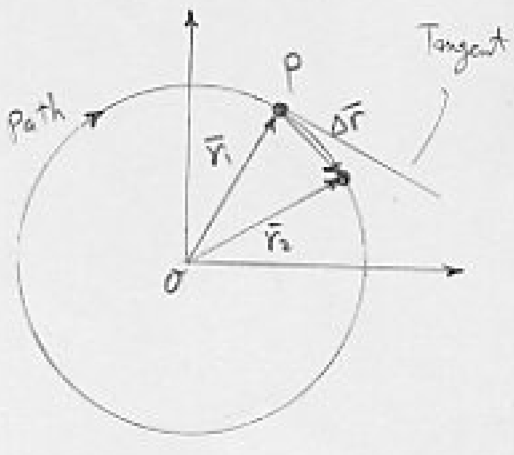
(i.e. $\vec{V}$ is always tangent to the particle's path)

Ex.

Particle moving in a circular path,

As $\Delta t \to 0$ $\Delta\bar{r} \to$ tangent line

If the string is cut, the particle would move in a straight line along $\vec{V}$ at that moment.



4-4 Acceleration and Average Acceleration;

$$\vec{a} = \frac{V_2 - V_1}{\Delta t} = \frac{\Delta \vec{V}}{\Delta t} \qquad \text{average acc.}$$

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{V}}{\Delta t} = \frac{d\vec{V}}{dt} \qquad \text{instantaneous acc.}$$

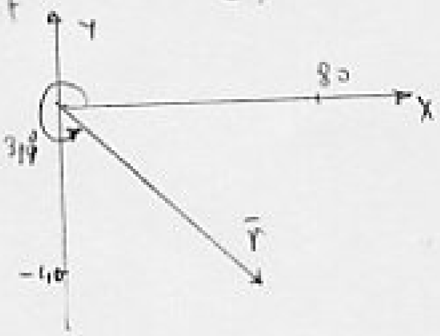
If the velocity changes in $\begin{cases} \text{either mag.} \\ \text{or dir.} \end{cases}$ (or both)

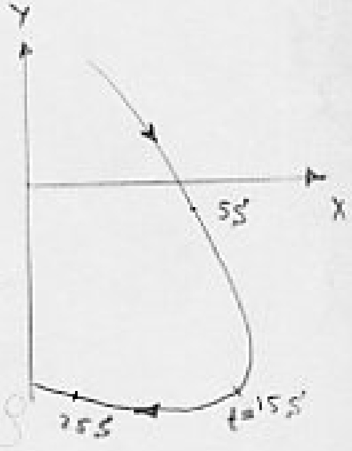
$\longrightarrow$ there is an acc.

$$\vec{a} = \frac{d}{dt}\left(V_x \hat{i} + V_y \hat{j} + V_z \hat{k}\right) = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

$$\rightarrow \vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

where $a_x = \frac{dv_x}{dt}$, $a_y = \frac{dv_y}{dt}$, $a_z = \frac{dv_z}{dt}$





Sample prob. 4.2

Motion of a particle;

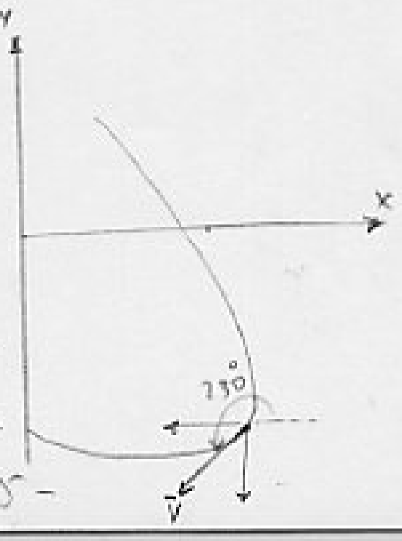
$x = -0.31t^2 + 7.2t + 28$

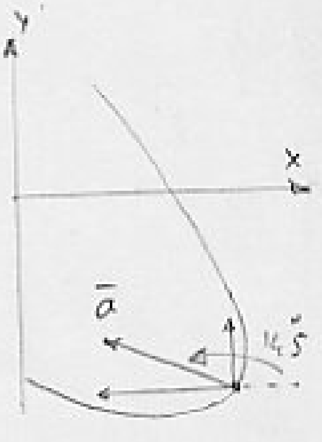
$y = 0.22t^2 - 9.1t + 30$

a) $x, y = ?$ at $t = 15$

$x = -0.31(15)^2 + 7.2(15) + 22 = 66\text{ m}$

$y = 0.22(15) - 9.1(15) + 30 = -57\text{ m}$





$|\bar{r}| = \sqrt{x^2+y^2} = 87\,m \quad \theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-57}{66}\right) = -41^\circ \sim 319^\circ$

$\cancel{+139^\circ}$

Sample prob. 4.3 -

In Prob. 4.2, $\bar{V} = ?$ at $t = 15\,s$

Sol.

$V_x = \frac{dx}{dt} = -0.62t + 7.2 \qquad V_x(15) = -2.1\ m/s$

$V_y = \frac{dy}{dt} = 0.44t - 9.1 \qquad V_y(15) = -2.5\ m/s$

$|\bar{V}| = \sqrt{V_x^2+V_y^2} = 3.3\ m/s \qquad \theta = \tan^{-1}\frac{V_y}{V_x} = \tan^{-1}\left(\frac{-2.5}{-2.1}\right) = -130^\circ$

$\cancel{50^\circ}$

$\sim 230^\circ$

Sample prob. 4.4 -

In Prob. 4.2, $\bar{a} = ?$ at $t = 15$

Sol.

$a_x = \frac{dv_x}{dt} = -0.62\ m/s^2 \qquad a_y = \frac{dv_y}{dt} = 0.44\ m/s^2$

$|\bar{a}| = \sqrt{a_x^2+a_y^2} = 0.76\ m/s^2 \qquad \theta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\left(\frac{0.44}{-0.62}\right) = 145^\circ$

Sample Prob 4.5

A particle is motion;

$\bar{V}_0 = -2.0\hat{i} + 4.0\hat{j}$

At $t=0$ the particle undergoes a const. acc $|\bar{a}| = 3.0\ m/s^2$ at an angle $\theta = 130^\circ$ from the positive x-dir.

$\bar{V} = ?$ at $t = 2.0\,s$

Sol.

$V_x = V_{0x} + a_x t$

$V_y = V_{0y} + a_y t$

$a_x = a\cos\theta = (3\,m/s^2)\cos(130^\circ) = -1.93\,m/s^2$

$a_y = a\sin\theta = (3\,m/s^2)\sin(130^\circ) = +2.3\,m/s^2$

$V_x = -2.0 - 1.93t$ $\qquad V_x(2.0) = -5.9\,m/s$

$V_y = 4.0 + 2.3t$ $\qquad V_y(2.0) = 8.6\,m/s$

$\bar{V} = -5.9\hat{i} + 8.6\hat{j}$ $\qquad V = \sqrt{V_x^2 + V_y^2} = 10\,m/s$

$\theta = \tan^{-1}\frac{8.6}{-5.9} = 124^\circ$

4-5 Projectile Motion

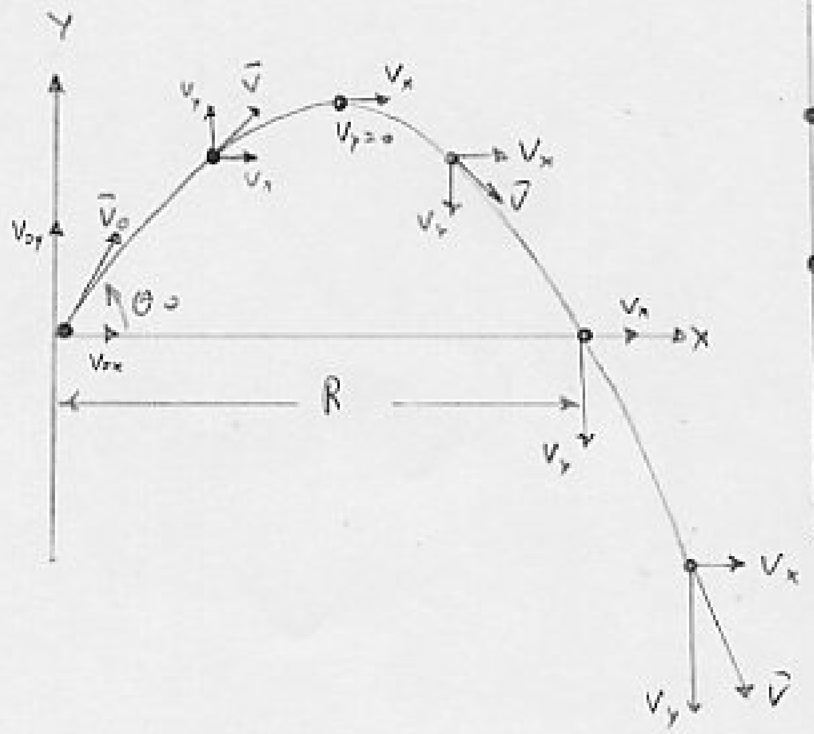
Motion in two-dim., with free-fall acc. $\bar{g}$.

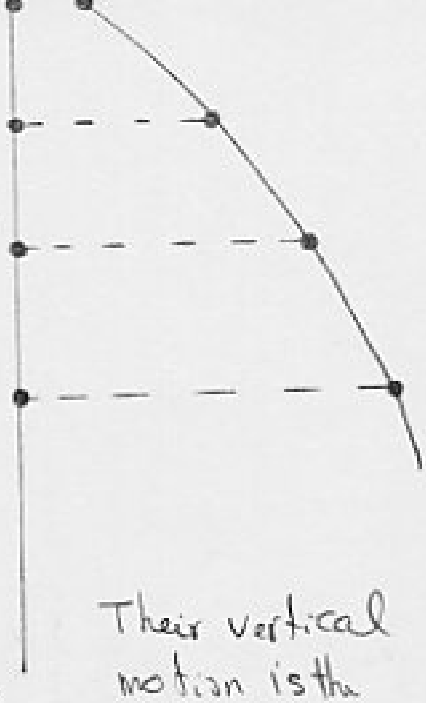
We assume the air has no effect.

$\bar{V}_0 = V_{0x}\hat{i} + V_{0y}\hat{j}$ $\qquad V_{0x} = V_0\cos\theta_0$ $\qquad V_{0y} = V_0\sin\theta_0$

V_x = Const. in mag. and dir.

V_y = changes " "





Their vertical motion is the same.

4.6 Projectile Motion Analyzed

The Horizontal Motion

$$a_x = 0 \Rightarrow V_x = V_{0x}$$

$$x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2 \rightarrow x - x_0 = (v_0 \cos\theta_0)t \quad (1)$$

The Vertical Motion,

$$y - y_0 = v_{0y}t + \frac{1}{2}a_y t^2 \rightarrow y - y_0 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 \quad (2)$$

$$v = v_0 + at \rightarrow v_y = v_0 \sin\theta_0 - gt$$

$$v^2 = v_0^2 - 2g(y - y_0) \rightarrow v^2 = (v_0 \sin\theta_0)^2 - 2g(y - y_0)$$

The Equ. of the Path;

Eliminating t between (1) and (2);

$$y = (\tan\theta_0)x - \left(\frac{g}{2(v_0\cos\theta_0)^2}\right)x^2 \quad \text{trajectory}$$

$$\begin{cases} x_0 = 0 \\ y_0 = 0 \end{cases}$$

This equ. is of the form;

$$y = ax + bx^2 \quad \text{Parabola}$$

The Horizontal Range;

$$\begin{cases} x - x_0 = R \\ y - y_0 = 0 \end{cases} \qquad \begin{cases} x - x_0 = (v_0\cos\theta_0)\,t = R \\ y - y_0 = (v_0\sin\theta_0)t - \frac{1}{2}gt^2 = 0 \end{cases}$$

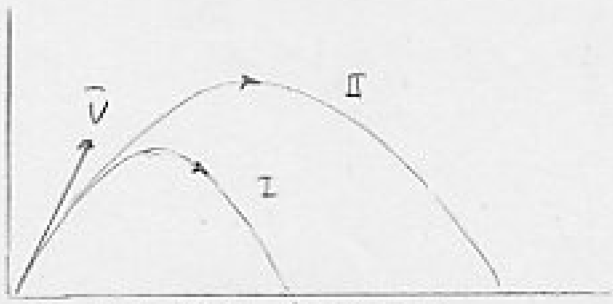
$$\xrightarrow{\text{Eliminating } t} R = \frac{2v_0^2}{g}\sin\theta_0\cos\theta_0 = \frac{v_0^2}{g}\sin 2\theta_0$$

$$2\theta_0 = 90^\circ \longrightarrow \theta_0 = 45 \longrightarrow R_{max}$$

The Effect of the Air;

The effect of air increases;

as v increases



I - The effect of air $\neq 0$

II " " " $= 0$

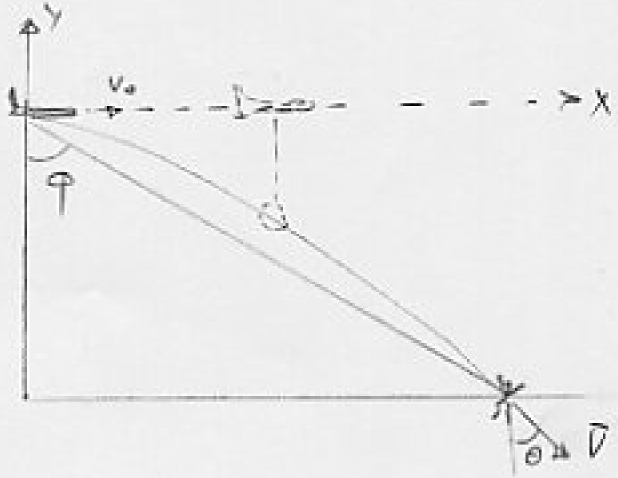
Sample Prob 4-6

Rescue Plane;

$h = 1200\,m \quad V_x = 430\ km/h$

At what angle φ should the pilot release a rescue capsule if it is to strike (very close to) the person in the water?

Sol.



$y - y_0 = (v_0 \sin\theta_0)t - \frac{1}{2} g t^2$

$y - y_0 = -1200 \quad \theta_0 = 0$

$-1200 = 0 - \frac{1}{2}(9.8)t^2$

$t = 15.65\,s$

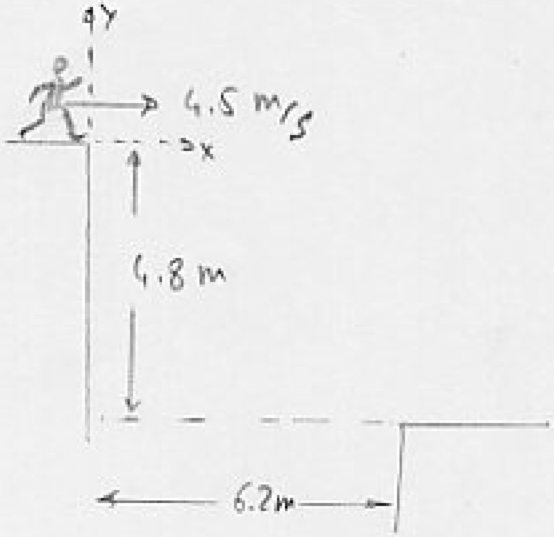
$x - x_0 = (v_0 \cos\theta_0)t = 1869\,m$

$\varphi = \tan^{-1}\frac{x}{h} = 57°$

Sample prob. 4-7

Jump or not to jump?

Sol.



$y - y_0 = -4.8\,m \quad \theta_0 = 0$

$y - y_0 = (v_0 \sin\theta_0)t - \frac{1}{2} g t^2$

$t = 0.990\,s$

$x - x_0 = (v_0 \cos\theta_0)t = 4.5\,m \rightarrow$ Don't jump!

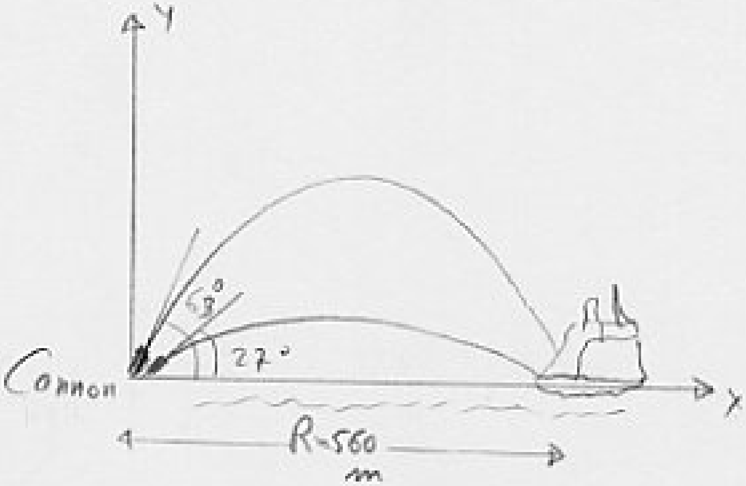
Sample prob. 4.8

muzzle velocity = 82 m/s

a) To hit the ship $\theta = ?$

b) $t = ?$ flight time

c) How far should the ship be from the fort if it is to be beyond range of the cannon?



Sol.

a) $R = \frac{v_0^2}{g} \sin 2\theta_0 \rightarrow 2\theta_0 = \sin^{-1} \frac{gR}{v_0^2} = \sin^{-1}(0.816)$

$\theta_0 = 27°, \theta_0 = 63°$

b) $x - x_0 = (v_0 \cos\theta_0) t \qquad t = \frac{x - x_0}{v_0 \cos\theta_0} = \frac{560}{82 \cos 27} = 7.7\,s$

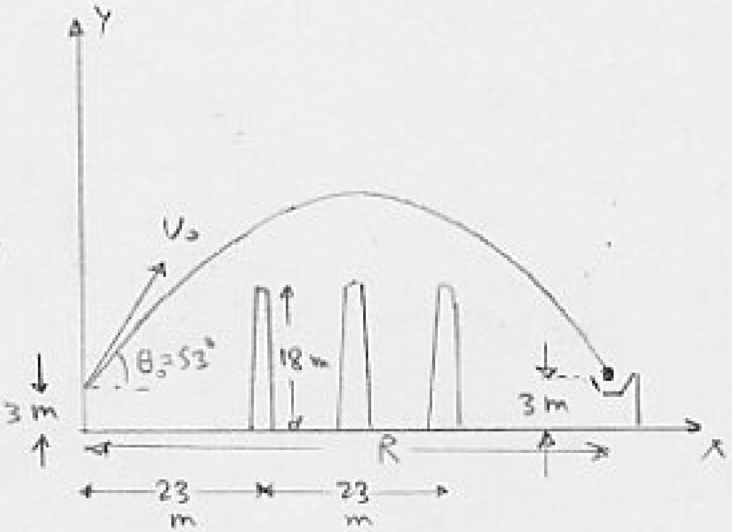
$t = \frac{560}{82 \cos 63} = 15\,s$

c) $R_{max} = \frac{v_0^2}{g} (\sin 2\theta_0)_{max} = \frac{v_0^2}{g} \sin(2 \times 45) = 690\,m$

The ship distance > 690 m

Sample prob. 4.9

$V_0 = 26.5\ m/s$



a) Does the ball clear the first barrier?

$$\begin{cases} y_0 = 3 \\ x_0 = 0 \\ x = 23 \end{cases}$$

$$y - y_0 = (\tan\theta_0)x - \frac{g x^2}{2(v_0\cos\theta_0)^2} = (\tan 53°)(23) - \frac{(9.8)(23)^2}{2(26.5)^2(\cos 53)^2} = 20.3\ m$$

$y = 3 + 20.3 = 23.3\ m \qquad 23.3 - 18 = 5.3\ m$

The ball clears the barrier about 5.3 m.

b) If the ball reaches its max. height when it is over the middle barrier, what is it clearance above it?

$V_y^2 = (V_0 \sin\theta_0)^2 - 2g(y - y_0) = 0$ at max. height

$y = 25.9\ m \qquad 25.9 - 18 = 7.9\ m$ clearance

c) $t = ?$ the time of flight

$y - y_0 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 = 0 \qquad t = 4.3\ s$

d) The position of basket (to score)?

$x - x_0 = (v_0 \cos\theta_0)t$

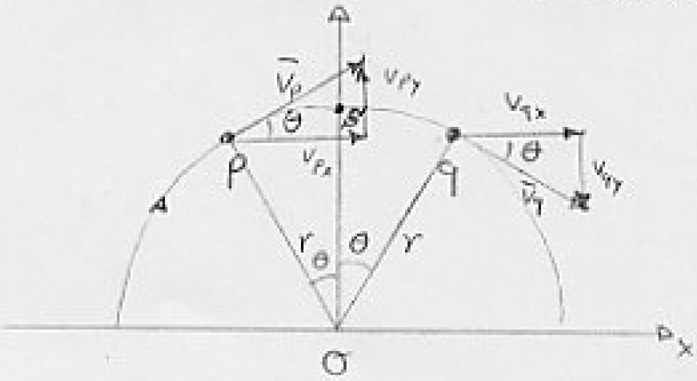
$x - 0 = (26.5)(\cos 53)(4.3) \qquad x = 69\ m = R$

4-7 Uniform Circular Motion

- A particle is in uniform circular motion if it travels in a circle or circular arc at const. speed.

Although the speed does not vary, the particle is accelerating

$|\bar{V}| = \text{const.}$ but the dir. of $\bar{V}$ changes $\longrightarrow$ causes also acceleration



q, p : symmetric

$$\begin{cases} V_{px} = +V\cos\theta \\ V_{py} = +V\sin\theta \end{cases}$$

$$\begin{cases} V_{qx} = +V\cos\theta \\ V_{qy} = -V\sin\theta \end{cases}$$

$$\Delta t = \frac{arc(pq)}{V} = \frac{r(2\theta)}{V}$$

$$\bar{a}_x = \frac{V_{qx}-V_{px}}{\Delta t} = \frac{V\cos\theta - V\cos\theta}{\Delta t} = 0$$

This result is not surprizing (from the symmetry $V_{qx} = V_{px}$)

$$\bar{a}_y = \frac{V_{qy}-V_{py}}{\Delta t} = \frac{-V\sin\theta - V\sin\theta}{\Delta t} = -\frac{2V\sin\theta}{2r\theta/V} = -\left(\frac{V^2}{r}\right)\left(\frac{\sin\theta}{\theta}\right)$$

(−) sign $\longrightarrow$ $\bar{a}_y$: downward

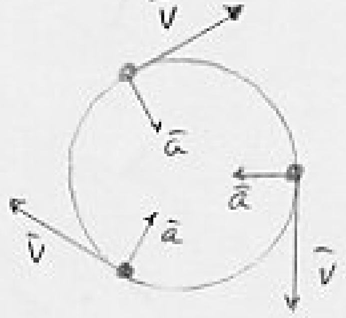
Now let $\theta \to 0$

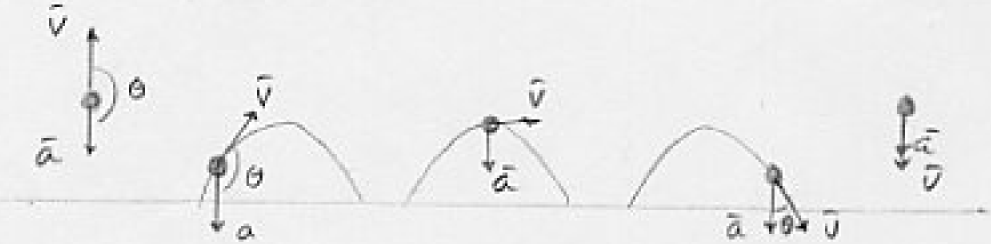
$\longrightarrow$ both $\begin{cases} q \\ p \end{cases} \longrightarrow S'$

$\lim_{\theta \to 0} \frac{S\theta}{\theta} = 1$

$\begin{cases} a_x = 0 & \longrightarrow \text{acc. tangent to the path} \\ |a_y| = \frac{v^2}{r} & \longrightarrow \text{" perpendicular " "} \end{cases}$

$\longrightarrow a = \frac{v^2}{r}$ centripetal acc.





Note that the acc. and velocity don't have any fixed directional relation to each other.

Sample prob. 4-10

A satellite motion

Altitude ; h = 200 km $\qquad g = 9.20\ m/s^2$

$V = ?$

Soln

$r = R_E + h \qquad a = g$

$a = \frac{v^2}{r} \rightarrow g = \frac{v^2}{R_E + h} \qquad v = \sqrt{g(R_E+h)}$

$v = \sqrt{(9.20\,m/s^2)(6.37\times10^6\,m + 200\times10^3\,m)} = 7.77\ km/s$

$\rightarrow \quad T = 1.47\,h$ for one orbital revolution

Although $g = 9.20$, the astronaut would experience weightlessness.

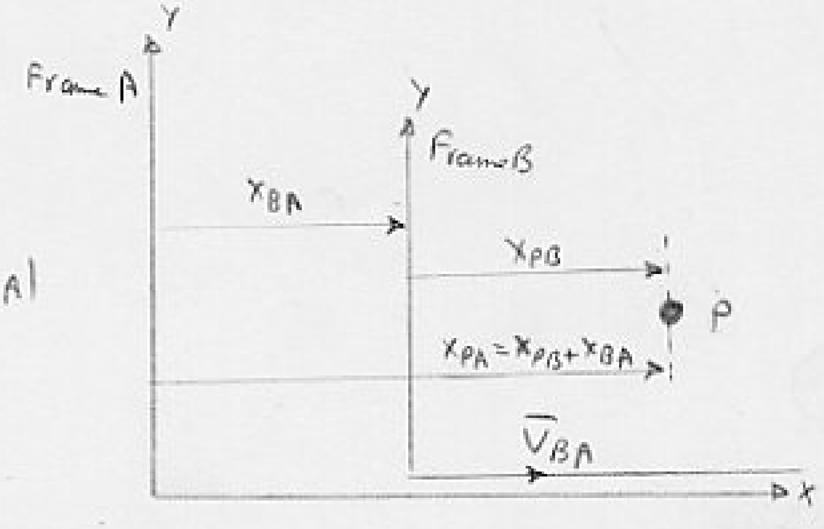
Explanation: Both the astronaut and the spacecraft are accelerating toward the center of Earth at $9.20\ m/s^2$. They are both in free fall, just like a passenger in a freely falling elevator cab.

4-8 Relative Motion in One-dim.

$X_{PA} = X_{PB} + X_{BA}$

$\frac{d}{dt}(X_{PA}) = \frac{d}{dt}(X_{PB}) + \frac{d}{dt}(X_{BA})$

$V_{PA} = V_{PB} + V_{BA}$



We consider only frames that move at a const. velocity w.r.t. each other.

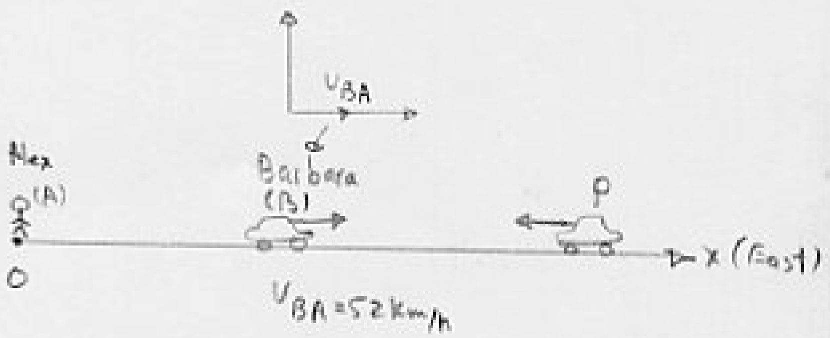
Such frames are called <u>inertial</u> <u>reference</u> <u>frames</u>.

$$\frac{d}{dt}(V_{PA}) = \frac{d}{dt}(V_{PB}) + \frac{d}{dt}(V_{BA})$$

$$\rightarrow a_{PA} = a_{PB}$$

Sample prob. 4-11

$V_{PA} = 78\ km/h$ (Alex measures)



a) What velocity will Barbara measure ?

$$V_{PB} = V_{PA} - V_{BA} = -78-(52) = -130\ km/h$$

b) If Alex sees car P brake to a halt in 10 sec, What acc. (const.) will he measure for it ?

$$V = V_0 + at \qquad a = \frac{V-V_0}{t} = \frac{0-(-78)}{10} = \left(\frac{78\ km/h}{10s}\right)\left(\frac{1m/s}{3.6\,km/h}\right)$$

$$= 2.2\ m/s^2$$

c) What acc. would Barbara measure for the braking car?

$$a = \frac{V-V_0}{t} = \frac{-52-(-130)}{10} = 2.2\ m/s^2$$

4-9 Relative Motion in Two Dimensions:

A moving particle P is observed by two observers from A- and B- reference frames.

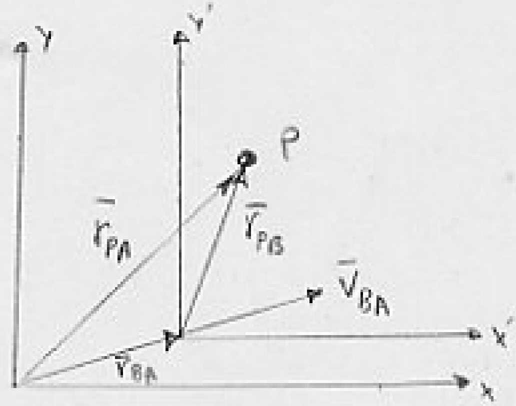
$\bar{V}_{AB}$ = const. relative vlocity of the frames

↓

(the frames are inertial)

For simplicity, we assume $x \parallel x'$, $y \parallel y'$

$\bar{r}_{PA} = \bar{r}_{PB} + \bar{r}_{BA}$

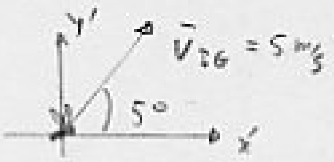


$\rightarrow \bar{V}_{PA} = \bar{V}_{PB} + \bar{V}_{BA}$

$\rightarrow \bar{a}_{PA} = \bar{a}_{PB}$

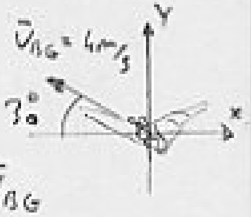
→ All observers on <u>inertial</u> reference frames will measure the <u>same</u> acc.

These equs. can be generalized in 3-dim.

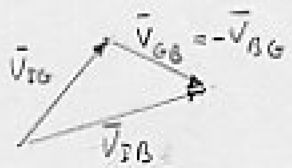


Sample prob. 4-12

A bat detects an insect.



V_{BG} : the velocity of bat w.r.t the ground



V_{IG} : " " " insect " " "

$\bar{V}_{IG}$ = ? (relative)

Sol.

$$\bar{V}_{IG} = 5\cos 50^\circ\,\hat{i} + 5\sin 50^\circ\,\hat{j}$$

$$V_{BG} = 4\cos 150^\circ\,\hat{i} + 4\sin 150^\circ\,\hat{j}$$

$$\bar{V}_{IB} = \bar{V}_{IG} + \bar{V}_{GB} = \bar{V}_{IG} - \bar{V}_{BG}$$

$$\bar{V}_{IB} = 6.7\,\hat{i} + 1.8\,\hat{j}$$

Sample prob. 4-13

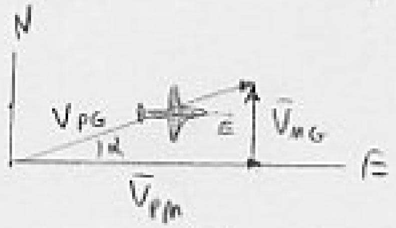
The compass in the plane indicates that it is headed due <u>east</u>; its airspeed indicator reads 215 Km/h. (Air speed is speed relative to the air.) A steady wind of 65 Km/h is blowing due north.

a) $V_{PG} = ?$ the velocity of the plane w.r.t. the ground

Reference A: ground (G)

" B: air mass (M)

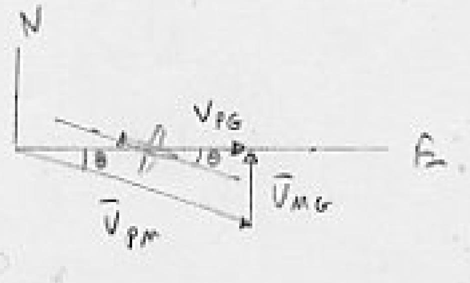
$$\rightarrow \bar{V}_{PG} = \bar{V}_{PM} + \bar{V}_{MG}$$



(a)

$$|\bar{V}_{PG}| = \sqrt{(\bar{V}_{PM})^2 + (\bar{V}_{MG})^2}$$

$$= \sqrt{(215)^2 + (65)^2} = 225 \text{ km/h}$$



(b)

$$\alpha = \tan^{-1} \frac{V_{MG}}{V_{PM}} = \tan^{-1} \frac{65}{215} = 16.8^\circ$$

b) If the pilot wishes to fly due east, what must be the heading? That is what must the compass read?

Fig. b. $\longrightarrow$ $\bar{V}_{PG} = \bar{V}_{PM} + \bar{V}_{MG}$

$$\longrightarrow V_{PG} = \sqrt{V_{PM}^2 - V_{MG}^2} = \sqrt{(215)^2 - (65)^2} = 205 \text{ km/h}$$

$$\theta = \sin^{-1} \frac{V_{MG}}{V_{PM}} = \sin^{-1} \frac{65}{215} = 17.6^\circ$$

4-10 Relative Motion and High Speed:

$C = 299,792,458$ m/s

$$v_{PA} = \frac{V_{PB} + V_{BA}}{1 + V_{PB} V_{BA}/c^2}$$ valid in all speeds

Sample prob. 4-14

(Slow speeds) $v_{PB} = v_{BA} = 0.0001c$ ($= 67{,}000$ mi/h)

$v_{PA} = v_{PB} + v_{BA} = 0.0002c$, and $v_{PA} = \frac{v_{PB} + v_{BA}}{1 + v_{PB} v_{BA}/c^2} \simeq 0.0002c$

Sample prob 4-15

(High speeds) $v_{PB} = v_{BA} = 0.65c$

$v_{PA} = v_{PB} + v_{BA} = 1.3c$ but $v_{PA} = \frac{v_{PB} + v_{BA}}{1 + v_{PB} v_{BA}/c^2} \simeq 0.91c$ Correct