Chapter 7

Nuclear Reactions

7-1 Coulomb Excitation:

$$\eta = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{zZe^2}{\hbar v} = \alpha z Z \frac{c}{v}$$

strength of the Coulomb
field between these two particles

(Sommerfield number)

classically; $\left[\frac{1}{4\pi\epsilon_0}\right]\frac{zZe^2}{R_s} = \frac{1}{2}\mu v^2$

R_s : closest distance between two particles.

$$\frac{1}{2}R_s = \frac{\alpha\hbar c z Z}{\mu v^2} = \eta\,\frac{\hbar}{\mu v} \quad \text{(in terms of } \eta\text{)} \qquad \alpha = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{e^2}{\hbar c}$$

$$\bar{\lambda} = \frac{\hbar}{p} = \frac{\hbar}{\mu v}$$ reduced de Broglie wave length

$$\rightarrow \eta = \frac{\frac{1}{2}R_s}{\bar{\lambda}}$$

η : small $\longrightarrow$ Coulomb field weak compared with the available kinetic energy in the scattering

η : small $\longrightarrow$ the wave func. of the incident particle is not greatly modified by the Coulomb field

$\longrightarrow$ Born approx. applies.

In the Coulomb excitation we are interested in the limit with $\eta \gg 1$

$\longrightarrow$ in this limit the two particles are never close enough to each other (no effect of nucl. forces).

$\longrightarrow$ The excitation of the target or projectile is accomplished.

It may be regarded as the inverse of the electromagnetic decay.

Very strong fields can be created by bombarding nuclei with a beam of heavy ions.

Multipole expansion:

$$V(t) = \pm\left[\frac{1}{4\pi\varepsilon_0}\right]\frac{3Ze^2}{|r_p(t) - r|}$$ time-dep. int.

$r_p(t)$: Projectile

r : target

More precise;

$$V(t) = \pm Z e^2 \sum_{i=1}^{Z} \frac{1}{|r_p(t) - r_i|}$$

$$\frac{1}{|r_p(t) - r|} = \sum_{\lambda=0}^{\infty} \frac{r^\lambda}{r_p^{\lambda+1}} P_\lambda(\hat{r}\cdot\hat{r}_p)$$

$$= \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \frac{4\pi}{2\lambda+1} \frac{r^\lambda}{r_p^{\lambda+1}(t)} Y_{\lambda\mu}(\theta,\varphi) Y^*_{\lambda\mu}(\theta_p,\varphi_p)$$

Scattering cross-section is now affected by the <u>various</u> states of the <u>target</u>

$$\left(\frac{d\sigma}{d\Omega}\right)_{fi} = \frac{1}{2J_i+1} \sum_{M_f M_i} |P_{M_f M_i}|^2 \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}}$$

This is similar to;

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} |F(q^2)|^2 \leftarrow \text{form factors}$$

J_i: initial spin of target

$|P_{M_f M_i}|^2$: probabilities ,

Acc. to the first order perturbation theory ;

$$C_n^1(t) = -\frac{i}{\hbar} \int_{t_0}^{t} e^{i\omega_{ni}t'} V_{ni}(t')\,dt'$$

$P_{i\to n} \simeq |C_n'|^2$

$$P_{M_f M_i} = -\frac{i}{\hbar}\int_{-\infty}^{\infty} \langle J_f M_f \xi | V(t') | J_i M_i \xi \rangle \, e^{i(E_f - E_i)t'/\hbar} \, dt'$$

substituting the expansion of $\frac{1}{|r_p(t)-r|}$ into $V(t)$;

$$\rightarrow P_{M_f M_i} = \frac{4\pi Z e}{i\hbar} \sum_{\lambda\mu} \frac{1}{2\lambda+1} \underbrace{\langle J_f M_f \xi | Q_{\lambda\mu}(E\lambda) | J_i M_i \xi \rangle}_{\text{(1) the dependence of } \frac{d\sigma}{d\Omega} \text{ on nuclear states}} S_{\lambda\mu}(E\lambda)$$

$$Q_{\lambda\mu}(E\lambda) = Z e r^{\lambda} Y_{\lambda\mu}(\theta, \varphi)$$

$$\left(\text{in general } Q_{\lambda\mu}(E\lambda) = \sum_{i=1}^{Z} e(i)\, r_i^{\lambda}\, Y_{\lambda\mu}(\theta_i, \varphi_i)\right)$$

$$B(\lambda : J_i \xi \rightarrow J_f \xi) = \sum_{\mu M_f} |\langle J_f M_f \xi | Q_{\lambda\mu} | J_i M_i \xi \rangle|^2$$

$$= \frac{1}{2J_i+1} |\langle J_f \xi \| Q_\lambda \| J_i \xi \rangle|^2$$

$$S_{\lambda\mu}(E\lambda) = \int_{-\infty}^{\infty} e^{i(E_f - E_i)t'/\hbar} \frac{1}{r_p^{\lambda+1}(t')} Y_{\lambda\mu}(\theta_p, \varphi_p) \, dt'$$

$S_{\lambda\mu}(E\lambda)$: indep. of nucl. states,
depended to θ_p, φ_p (scattering angles)

$\theta_p = \theta_p(t)$, $\varphi_p = \varphi_p(t)$

For $\lambda=0$ (monopole) $P_0 = 1$, $\overset{\circ}{r} = 1$

$\rightarrow \langle J_f \cdots | O_{00}(E\lambda) | J_i \cdots \rangle = 0$ (because $i \neq f$, orthogonality)

i.e. if $i = f \longrightarrow$ no excitation (only scattering)

For $\lambda = 1$: ($2^\lambda = 2^1$-pole, dipole tr.)

If the plane of the orbit is taken to be xz-plane $\rightarrow \varphi = 0$

(Equ1, P190) $\rightarrow$

$$(P_{M_f M_i})_{\lambda=1} = \frac{3Ze^2}{i\hbar} \int_{-\infty}^{\infty} e^{i\omega_{fi} t'} \frac{\langle J_f M_f \xi | r' P_1(\hat{r} \cdot \hat{r}_p(t')) | J_i M_i \xi \rangle}{r_p^2(t')} dt'$$

$$P_1(x) = x \quad \rightarrow r P_1(\hat{r} \cdot \hat{r}_p) = r(\hat{r} \cdot \hat{r}_p) = r \cdot \hat{r}_p(t)$$
$$= x \sin\theta(t) + z \cos\theta(t)$$

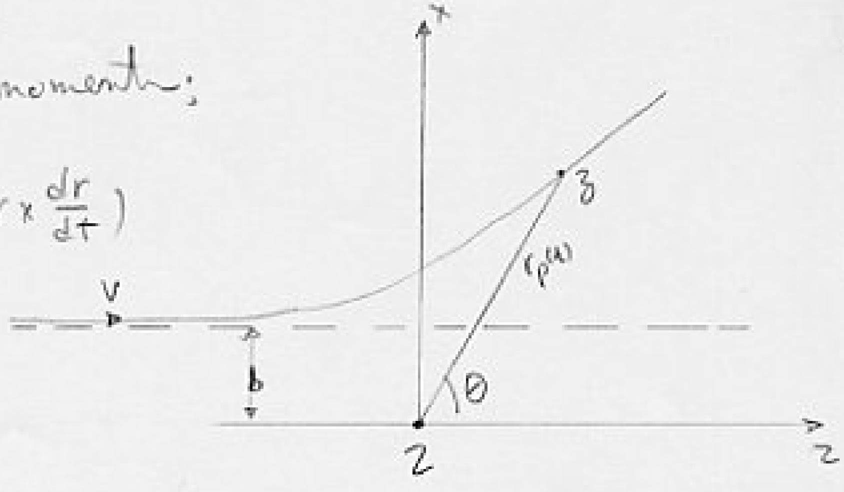
For central pot., we have conservation of the angular momentum;

$$L = r \times p = m(r \times v) = m(r \times \frac{dr}{dt})$$

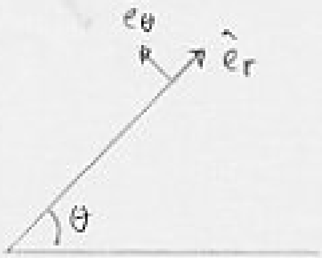
$\bar{r} = r e_r$

$\bar{v} = \frac{d\bar{r}}{dt} = \dot{r} \hat{e}_r + r\dot{\theta} \hat{e}_\theta$

$L = m r^2 \dot{\theta} \hat{y}$



On the other hand



$L = m v b \hat{y}$

$\rightarrow r^2 \dot{\theta} = bv \qquad r_p^2(t) \dfrac{d\theta}{dt} = bv$

$$\frac{dt}{r_p^2(t)} = \frac{d\theta}{bv}$$

$$\rightarrow (P_{M_f M_i})_{\lambda=1} = \frac{3Ze^2}{i\hbar bv} \int_{-\pi}^{\theta(b)} e^{\omega_{fi} t'} (X_{fi} S\theta + Z_{fi} C\theta)\, d\theta$$

where $X_{fi} = \langle J_f M_f \xi | X | J_i M_i \xi \rangle$ x-component of dipole matrix-element

Usually, $t' = f(\theta)$ is complicated.

$\rightarrow$ the integral can be evaluated if the classical motion of the projectile is known.

In the case of Coulomb excitation the projectile performs a Kepler motion under the influence of the Coulomb force.

$\int \longrightarrow$ in terms of Hankel funcs. of imaginary argument

If we assume $e^{i\omega_{fi} t'} \simeq 1$

$\rightarrow$ the integration will be simple.

This approx. is valid when,

$$\text{collision time} \ll \frac{1}{\omega_{fi}}$$

Collision time: The time during which the projectile is close enough to the target to be effective exciting the latter.

$$\left(P_{m_f m_i}\right)_{\lambda=1} \simeq \frac{3Ze^2}{i\hbar bv}\int_{-\pi}^{\theta(b)} \left(X_{fi}\sin\theta + Z_{fi}\cos\theta\right)d\theta$$

$$= -\frac{3Ze^2}{i\hbar bv}\left\{X_{fi}\left[\cos\theta(b)+1\right] - Z_{fi}\sin\theta(b)\right\}$$

Niels Bohr has shown under what conds. this is a good approx. in the treatment of collisions of heavy charged particles

Neglecting the rare sharply deflected collisions;

$$\theta(b) \simeq 0$$

$$\left(P_{m_f m_i}\right)_{\lambda=1} \simeq -\frac{2 3Ze^2}{i\hbar bv} X_{fi}$$

The energy loss:

$$T = \sum_f (E_f - E_i)\,|P_{fi}|^2 = \frac{4 3^2 Z^2 e^4}{b^2 v^2 \hbar^2}\sum_f |X_{fi}|^2 (E_f - E_i)$$

This can be simplified.

Consider the expression;

$$\frac{1}{2}\left[[x,H_0],x\right] = xH_0x - \frac{1}{2}H_0x^2 - \frac{1}{2}x^2H_0$$

$$\frac{1}{2}\langle i|\left[[x,H_0],x\right]|i\rangle = \langle i|xH_0x|i\rangle - E_i\langle i|x^2|i\rangle$$

$$= \sum_f\left(\langle i|x|f\rangle E_f\langle f|x|i\rangle - E_i\langle i|x|f\rangle\langle f|x|i\rangle\right)$$

$$= \sum_f (E_f - E_i)|X_{if}|^2$$

If $H_0 = \frac{p^2}{2\mu} \rightarrow [x,H_0] = \frac{i\hbar}{\mu}p_x \Rightarrow \frac{1}{2}\left[[x,H_0],x\right] = \frac{\hbar^2}{2\mu}I$

$$\rightarrow \frac{2\mu}{\hbar^2}\sum_f (E_f - E_i)|X_{if}|^2 = 1$$

by virtue of the normalization of the wave fncs.

$$\rightarrow T = \frac{2z^2Z^2e^4}{\mu b^2v^2}$$

(excitation of nucleus)

(for Z-charges of the target concentrated at one point.)

Instead if we consider $V(t) = \pm ze^2\sum_{i=1}^{Z}\frac{1}{|r_i - r_p(t)|}$,

this leads to $X = X_1 + X_2 + \cdots + X_Z$

$$\rightarrow \frac{2\mu}{\hbar^2}\sum_f (E_f - E_i)|X_{if}|^2 = Z$$

$$\Rightarrow T = \frac{2z^2Ze^4}{\mu b^2v^2}$$

(excitation of atom)

This is the <u>average</u> loss in the <u>inelastic</u> collision.

General case:

$$S_{\lambda\mu}(E\lambda) = \frac{1}{va^{\lambda}} Y_{\lambda\mu}\left(\frac{\pi}{2},0\right) f_{\lambda\mu}(\theta,\xi)$$

ξ: adiabaticity parameter $\quad \xi = \eta \dfrac{E_f - E_i}{2E_i}$

$$f_{\lambda\mu}(\theta,\xi) = \int_{-\infty}^{\infty} e^{i\xi(\epsilon\sinh x + x)} \frac{(\cosh x + \epsilon + i\sqrt{\epsilon^2-1}\,\sinh x)^{\mu}}{(\epsilon\cosh x + 1)^{\lambda+\mu}}\, dx$$

where $\epsilon = \left(\sin\frac{\theta}{2}\right)^{-1}$

$$Y_{\lambda\mu}\left(\frac{\pi}{2},0\right) = \begin{cases} (-)^{(\lambda+\mu)/2} \sqrt{\dfrac{2\lambda+1}{4\pi}} \dfrac{\sqrt{(\lambda-\mu)!\,(\lambda+\mu)!}}{(\lambda-\mu)!!\,(\lambda+\mu)!!} & \text{for } \lambda+\mu = \text{even} \\ 0 & \text{otherwise} \end{cases}$$

Ref.: Alder et al. (Rev. Mod. Phys. 28(1956)432; 30(1958)353)

$$\left(\frac{d\sigma}{d\Omega}\right)_{fi} = \sum_{\lambda=1}^{\infty} \left(\frac{Z_1 e}{a^{\lambda}\hbar v}\right)^2 B(E\lambda; J_i\xi \to J_f\xi)\, \frac{df(E\lambda,\xi)}{d\Omega}$$

$a = \dfrac{R_s}{2}$

differential scattering cross-section for Coulomb excitation.

$$\frac{df(E\lambda,\rho)}{d\Omega} = \frac{4\pi^2}{(2\lambda+1)^3} \sum_{\mu} \left| Y_{\lambda\mu}\left(\frac{\pi}{2},0\right) f_{\lambda\mu}(\theta,\rho) \right|^2 \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}}$$

$$f(E\lambda,\rho) = \int \frac{df(E\lambda,\rho)}{d\Omega}\, d\Omega$$ total excitation cross-section

Magnetic multipole excitations are also present in a Coulomb excitation.

Multiple Scattering:

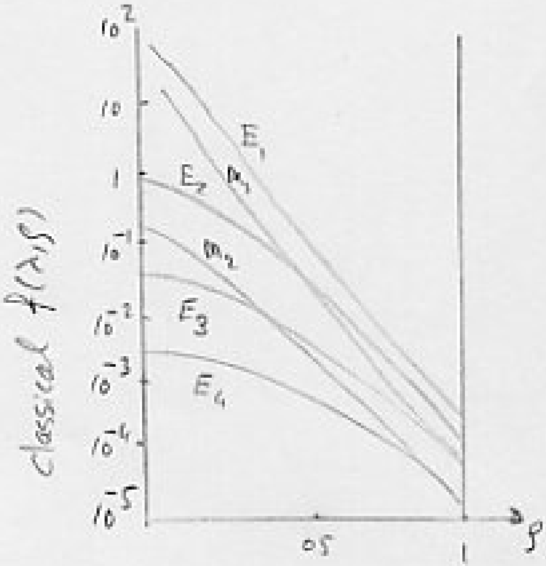
Because of the <u>intense</u> electromagnetic fields brought along by <u>heavy-ion</u> scattering $\longrightarrow$ Very high <u>multipolarity</u> excitations may be attained.

As seen from the Fig.;

$$P(E,\lambda) \gg P(E,\lambda+1)$$

$$P(M,\lambda) \gg P(M,\lambda+1)$$

P: Probability of excitation



Furthermore;

Since $< |Q_{\lambda\mu}| > \gg < |Q_{\lambda+1,\mu}| >$

$\longrightarrow$ the strengths for <u>high</u> multipol trs. are <u>weaker</u>.

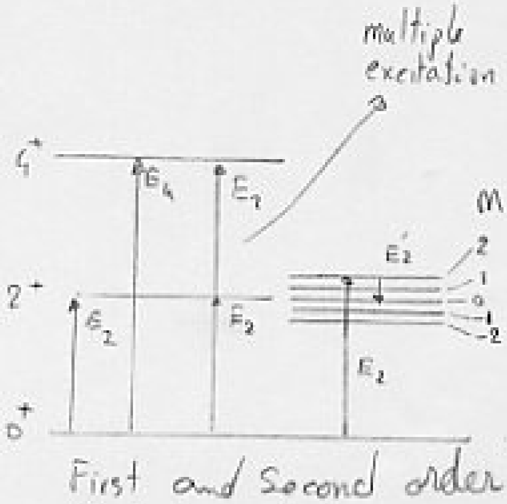
Ex. – Consider a hypothetical <u>even-even</u> nucleus

E2-excitations $\gg$ E4-excitation
stronger

$(E2-E'2)$- " $\sim$ or $>$ E4- "

$\downarrow$

$\underbrace{P(E2)\cdot P'(E2)}_{\text{probability}}$



First and Second order Coulomb excitation

→ Multiple processes through <u>successive</u> low multipolarity Coulomb excitations may become the <u>preferred path</u> for a nucleus to reach final states of relatively high angular momenta.

Reorientation effect:

Coulomb excitation from $(J^\pi, M) \longrightarrow (J^\pi, M')$

In our example (Fig.) $2^+, M \longrightarrow 2^+, M'$ (E2-excitation)

Such a process is sensitive to $\begin{cases} \text{i- } B(E2) \sim |\langle \| O_\lambda \| \rangle|^2 \\ \text{ii- } \langle M' | O_{\lambda\mu} | M \rangle \end{cases}$ reduced tr. prob.

Such a matrix-element $\xrightarrow{\text{is related to}}$ the quadrupole mom. of the excited state

⟶ quadrupole mom. of the excited state may be deduced from the $\frac{d\sigma}{d\Omega}$ of the Coulomb excitation process.

The value of the quadrupole mom. $\xrightarrow{\text{depends}}$ somewhat on the <u>nucl. model</u> used.

However, this <u>is not</u> a serious <u>prob.</u>, since the <u>reliability</u> of the model can be checked against several other properties of the nucleus.

7-2 Compound Nucleus Formation;

We have seen;

$\begin{cases} \text{i- Single-particle} \\ \text{ii- Collective} \end{cases}$ deg. of freedom (two-extreme points of view of the nucl. structure).

A parallel situation exists in the nucl. reaction studies (i.e. two limiting situations of direct reaction and compound nucleus formation).

i- Direct reation,

In this case one assumes the only one nucleon in the projectile interacts with one of the nucleons in the target.
(The rest nucleons in both projectile and target remain unaffected).
This is because $t \sim 10^{-22}$ sec available time for the projectile and target to interact

Since;

$t \underset{\text{comparable}}{\sim} t'$: transit time for an incident particle with Kinetic energy $\geq$ 1 MeV/nucleon to travel a distance of the order of nuclear diameter.

→ The probability for interacting more than once is small.

ii) Compound Nucleus;

On the other if the Kinetic energy of the projectile << 1 MeV / nucleon

→ the projectile and target may fuse together for a much longer time ($\sim 10^{-14}$ s)

→ a compound nucleus is formed (as the intermediate state).

Notation:

particular exit channel

$$A(a,b)B \quad \text{or} \quad a + A \longrightarrow b + B$$

projectile; target; particle emerging; nucleus remaining behind

Ex. $^{48}Ca(p,n)^{48}Sc$ or $p + {}^{48}Ca \longrightarrow n + {}^{48}Sc$

other exit channel: $^{48}Ca(p,p)^{48}Ca^*$

Possible or (open) exit channels are governed by

$\left\{ \begin{array}{l} \text{conservation laws} \\ \text{selection rules} \end{array} \right.$ operating in the scatt.

Ex. $\quad {}^{48}Ca(d,pn)^{48}Ca \quad$ or $\quad d + {}^{48}Ca \rightarrow \underbrace{p + n + {}^{48}Ca}_{\text{more than two-particles in the exit channel}}$

It is often possible to arrive at the same exit channel using different incident channels.

Ex:

$$p + {}^{48}Ca \xrightarrow{\text{scatt.}} n + {}^{48}Sc$$
$$n + {}^{48}Sc \xrightarrow{\text{scatt.}} n + {}^{48}Sc$$

Since we are dealing with microscopic objects $\longrightarrow$ Time-Reversal invariance holds.

Thus, $\quad p + {}^{48}Ca \longrightarrow n + {}^{48}Sc \quad$ may also take place with time-order going in the opposite dir.,

$$n + {}^{48}Sc \longrightarrow p + {}^{48}Ca$$

On occasion it may be more convenient to speak of a reaction channel without specifying whether it is an incident channel or an exit channel.

At low energies τ = long   life-time of the excited states

because: the number of <u>open exit</u> channels is <u>small</u>

$\rightarrow \Gamma_{\text{compound nucl. state}}$ == narrow

$$a + A \begin{cases} \rightarrow b + B \\ \rightarrow c + C \\ \rightarrow d + D \end{cases}$$

At the same time; the density of states at such low energies is small $\rightarrow$ D = large

With $D \gg \Gamma$ <u>isolated</u> resonances dominate the <u>reaction</u> cross-section.

high energy region $\rightarrow$ D{

low " " $\rightarrow$ D{

G

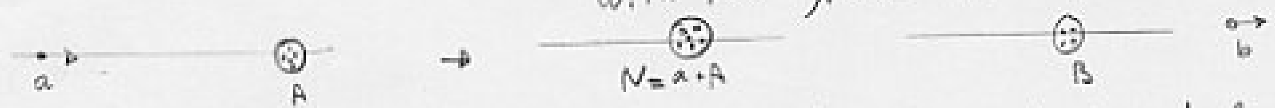
Scattering Cross-Section:

In the <u>compound</u> nucleus (low energy scatt.) there is <u>no relation</u>

between $\begin{cases} \text{formation} \\ \text{and} \\ \text{decay} \end{cases}$ of the compound nucleus.

This is expected;

Since the excitation energy (through incident) is <u>shared</u> among <u>many nucleons</u> $\longrightarrow$ <u>large</u> number of <u>collision</u> between nucleons in the compound nucleus occurs (in a relatively long <u>lifetime</u>, compared with the typical time for a nucl. int.)



At low energies when a and A fuse together, they loose their initial identification.

At low energies (< 1 MeV/nucleon) λ_d > nucleus dim. $\longrightarrow$ Scatt. is <u>not</u> sensitive to the <u>details</u> of the structure of the nuclei

de Broglie

Now, let;

σ_α : Cross-section for forming a compound nucleus N through incident channel α

and;

	decay product	exict channel	tr. probability
	$N \longrightarrow a + A$	α	W_α (it may be the same as enterance channel)
	$N \longrightarrow b + B$	β	W_β
	$N \longrightarrow c + C$	γ	W_γ
	......	...	..

$\tau = \frac{1}{W} \rightarrow \Gamma_\beta = \hbar W_\beta$ partical width through β-channel $(\Delta E \Delta t \sim \hbar)$

...

$\Gamma = \Gamma_\alpha + \Gamma_\beta + \Gamma_\gamma + \cdots$

Γ_β / Γ : the probability for decaying through β channel.

$$\sigma_{\beta\alpha} = \sigma_\alpha \frac{\Gamma_\beta}{\Gamma}$$

reaction cross-section (enterance channel α, exit channel β) for a compound nucleus N

Def: R_c: channel radius for each reaction channel

such that;

$r > R_c$: No int. between scattered particle and residual nucleus (ignoring Coulomb int.)

In this region;

wave funcs. = plane waves (or coulomb wave fucs.)

$r < R_c$ int. exists.

$r > R_c \longrightarrow \begin{cases} \psi_{out} \sim e^{ik\cdot x} & \text{(free particles)} \\ \text{or } \psi_{out} = \text{Coulomb wave fucs.} & \text{(in more general case)} \end{cases}$

$r < R_c \longrightarrow \psi_{in} = \text{complicated}$

$\xrightarrow{\text{Sol.}}$ at $r = R_c \begin{cases} r \frac{d}{dr} \log R_\ell = \frac{r}{R_\ell}\frac{dR_\ell}{dr} \\ \text{or } \frac{r}{u_\ell}\frac{du_\ell}{dr} \end{cases}$ i.e. $B_\ell = \left(\frac{r}{u_\ell}\frac{du_\ell}{dr}\right)_{r=R_c}$

the logarithmic derivative of the radial wave func. (of each channel)

must be continuous. $\left(R_\ell = \frac{u_\ell}{r}\right)$

$$B_{c(in)} = B_{c(out)} \quad (\text{at } r = R_c)$$

We assume (for simplicity) only s-wave scatt. $\neq 0$

and all the others $\simeq 0$

$$\psi^A(r,\theta) = \sum_{\ell=0}^{\infty} \frac{-(2\ell+1)}{2ikr} i^\ell \left[e^{-i(kr - \frac{\ell\pi}{2})} - S_\ell(k)\, e^{i(kr - \frac{\ell\pi}{2})} \right] \cdot P_\ell(\cos\theta)$$

$$u_0(r) \sim \left(e^{-ikr} - \eta_\ell(k)\, e^{ikr} \right) \qquad \eta_\ell = e^{2i\delta_\ell}$$

(δ_ℓ: Complex in inelastic scatt.)

$$\beta_0 = \left(\frac{r}{u_0}\frac{du_0}{dr}\right)_{r=R_c} \rightarrow \quad \beta_0 = R_c \frac{-ike^{-ikR_c} - ik\eta_0 e^{ikR_c}}{e^{-ikR_c} - \eta_0 e^{ikR_c}}$$

$$\eta_0 = \frac{\beta_0 + ikR_c}{\beta_0 - ikR_c} e^{-2ikR_c}$$

$$\sigma^{el} = \frac{\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)|1-\eta_0|^2 = \frac{\pi}{k^2}\sum_{\ell}(2\ell+1)\left|1-e^{2i\delta_\ell}\right|^2$$

$$\sigma^{el} = \frac{\pi}{k^2}|1-\eta_0|^2 = \frac{\pi}{k^2}\left|1-\frac{\beta_0+ikR_c}{\beta_0-ikR_c}e^{-2ikR_c}\right|^2$$

$$= \frac{\pi}{k^2}\left|e^{2ikR_c} - \frac{\beta_0+ikR_c}{\beta_0-ikR_c}\right|^2\left|e^{-2ikR_c}\right|^2 \quad = \frac{\pi}{k^2}\left|e^{2ikR_c} - \frac{\beta_0+ikR_c}{\beta_0-ikR_c}\right|^2$$

$$= \frac{\pi}{k^2}\left|e^{2ikR_c} - 1 - \frac{2ikR_c}{\beta_0-ikR_c}\right|^2 \qquad (1)$$

$$\sigma^{re} = \frac{\pi}{k^2}\sum_{\ell}(2\ell+1)(1-|\eta_\ell|^2) \qquad \text{reaction cross-section}$$

$$\sigma^{re} = \frac{\pi}{k^2}(1-|\eta_0|^2) = \frac{\pi}{k^2}\left(1-\left|\frac{(\mathrm{Re}\,\beta_0)+i(\mathrm{Im}\,\beta_0)+ikR_c}{(\mathrm{Re}\,\beta_0)+i(\mathrm{Im}\,\beta_0)-ikR_c}e^{-2ikR_c}\right|^2\right)$$

$$\sigma^{re} = \frac{\pi}{k^2}\frac{-4kR_c(\mathrm{Im}\,\beta_0)}{(\mathrm{Re}\,\beta_0)^2+(\mathrm{Im}\,\beta_0-kR_c)^2} \qquad (2)$$

For purely <u>elastic</u> scatt. (Real potential) $\longrightarrow \beta_0$ = real

$$\longrightarrow \sigma^{re} = 0$$

From (2) $\longrightarrow$ Since $\underset{\substack{\uparrow \\ \text{must be}}}{\alpha^{re} \geq 0}$ $\longrightarrow$ $\begin{cases} |\eta_0|^2 \leq 1 \\ \mathrm{Im}\,\beta_0 < 0 \end{cases}$

Breit - Wigner formula for isolated resonances:

From

$$\left[-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{\ell(\ell+1)}{r^2} + \frac{2\mu}{\hbar^2}V(r) - k^2\right] R_{\ell,k}(r) = 0$$

or $\qquad R = \dfrac{u}{r}$

$$\left[-\frac{d^2}{r^2} + \frac{2\mu}{\hbar^2}V(r) + \frac{\ell(\ell+1)}{r^2} - k^2\right] u_{\ell,k}(r) = 0$$

$$(E_2-E_1) R_{\ell,k_1}(r) R_{\ell,k_2}(r)\, r^2 + \frac{\hbar^2}{2\mu}\frac{d}{dr}\left\{r^2\left[R_{\ell,k_1}(r)\frac{dR_{\ell,k_2}(r)}{dr} - \frac{dR_{\ell,k_1}(r)}{dr}R_{\ell,k_2}(r)\right]\right\} = 0$$

Integrating this equ. from 0 to a and dividing by

$R_{\ell,k_1}(r) R_{\ell,k_2}(r)$;

$$\beta_\ell(E_2) - \beta_\ell(E_1) = \frac{-2\mu(E_2-E_1)\int_0^a R_{\ell,k_1}(r) R_{\ell,k_2}(r)\, r^2 dr}{\hbar^2 a\, R_{\ell,k_1}(a) R_{\ell,k_2}(a)}$$

or $$\beta_\ell(E_2) - \beta_\ell(E_1) = \frac{-2\mu(E_2-E_1)\int_0^a u_{\ell,k_1}(r) u_{\ell,k_2}(r)\, a\, dr}{\hbar^2 u_{\ell,k_1}(a) u_{\ell,k_2}(a)}$$

where $E = \dfrac{\hbar^2 k^2}{2\mu}$

On the limit; $E_1 \to E_2$

$$\frac{\partial}{\partial E} B_\ell(E) = -\frac{2\mu}{a\hbar^2 |R_{\ell k}(a)|^2} \int_0^a |R_{\ell,k}(r)|^2 r^2 dr$$

$\frac{\partial B_\ell}{\partial E} < 0 \to B_\ell$ is a monotonically decreasing func. of E,

except that it jumps discontinuously from $-\infty$ to ∞

whenever $R_{\ell,k}(a) \to 0$

qualitative behavior of $B_\ell \sim \cot(ka)$

$$\to B_\ell(E) \approx B_\ell(E_c^\ell) + (E - E_c^\ell)\left(\frac{\partial B_\ell}{\partial E}\right)_{E=E_c^\ell} + \cdots$$

E_c^ℓ : Resonance energy

$$\to B_\ell(E) = c - b(E - E_c^\ell) \qquad b = -\left(\frac{\partial B_\ell}{\partial E}\right)_{E_c^\ell} > 0$$

Now, the radial wave func. for $r > R_c$

$$\psi^+ = \frac{1}{(2\pi)^{3/2}} \sum i^\ell (2\ell+1) A_\ell(r) P_\ell(\cos\theta), \quad A_\ell(r) = C_\ell^{(1)} h_\ell^{(1)}(kr) + C_\ell^{(2)} h_\ell^{(2)}(kr)$$

where $\begin{cases} h_\ell^{(1)} = j_\ell + i n_\ell \\ h_\ell^{(2)} = j_\ell - i n_\ell \end{cases}$ $\qquad \begin{cases} C_\ell^{(1)} = \frac{1}{2} e^{2i\delta_\ell} \\ C_\ell^{(2)} = \frac{1}{2} \end{cases}$ (for both elastic an inelastic cases)

(Quant. Notes P 140)

$$A_\ell(kr) = e^{i\delta_\ell}\left[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, n_\ell(kr)\right] \qquad r > R$$

(for elastic case) δ_ℓ real

$$B_\ell \equiv \left(\frac{r}{A_\ell}\frac{dA_\ell}{dr}\right)_{r=R} = kR\left[\frac{j_\ell'(kR)\cos\delta_\ell - n_\ell'(kR)\sin\delta_\ell}{j_\ell(kR)\cos\delta_\ell - n_\ell(kR)\sin\delta_\ell}\right]$$

$$\rightarrow \tan\delta_\ell = \frac{kR\, j_\ell'(kR) - B_\ell\, j_\ell(kR)}{kR\, n_\ell'(kR) - B_\ell\, n_\ell(kR)}$$

$$\rightarrow C = B_\ell(E_c^\ell) = kR\,\frac{n_\ell'(kR)}{n_\ell(kR)} \qquad \left(\delta_\ell = \frac{\pi}{2}\right)$$

This approx. is valid if $n_\ell(kR) \neq 0$

$$\rightarrow \tan\delta_\ell = \frac{1}{n_\ell(kR)}\,\frac{kR\, j_1'(kR) - B_\ell\, j_\ell(kR)}{kR\,\frac{n_\ell'(kR)}{n_\ell(kR)} - B_\ell} = \frac{kR\, j_\ell' - B_\ell j_\ell}{n_\ell C - B_\ell n_\ell}$$

$$\simeq \frac{kR\, j_\ell' - B_\ell j_\ell}{n_\ell C - [C - b(E-E_c)]\, n_\ell} = \frac{kR\, j_\ell' - B_\ell j_\ell}{n_\ell\, b(E-E_c)}$$

$$= \frac{kR\, j_\ell' - [C - b(E-E_c)]\, j_\ell}{n_\ell\, b(E-E_c)} \simeq \frac{kR\, j_\ell' - C j_\ell}{n_\ell\, b(E-E_c)}$$

$$= \frac{kR\,(j_\ell' n_\ell - n_\ell' j_\ell)}{n_\ell^2\, b(E-E_c)}$$

By virtue of Wronskian relation

$$j_\ell(z)\, n_\ell'(z) - j_\ell'(z)\, n_\ell(z) = \frac{1}{z^2}$$

$$\rightarrow \tan\delta_\ell \simeq \frac{-\frac{1}{kR}}{\eta_\ell^2\, b(E-E_c)}$$

Defining: $\frac{1}{2}\Gamma = \frac{1}{kRb\,\eta_\ell^2(kR)}$, $\frac{\hbar^2k^2}{2\mu} = E_c$

$$\tan\delta_\ell \simeq \frac{\frac{1}{2}\Gamma}{E_c - E}$$

Using $\frac{a}{b} = \frac{c}{d} \rightarrow \frac{a}{b-ia} = \frac{c}{d-ic}$

$$\rightarrow \sin\delta_\ell\, e^{i\delta_\ell} = \frac{\Gamma}{2(E_r-E)-i\Gamma} \quad \rightarrow \sin^2\delta_\ell = \frac{\Gamma^2}{4(E_r-E)^2+\Gamma^2}$$

$$\rightarrow \sigma_\ell = \frac{4\pi(2\ell+1)}{k^2}\frac{\Gamma^2}{4(E_r-E)^2+\Gamma^2}$$ (in elastic scatt.)

Now come back in inelastic case;

$$Re(B_0) = c + a(E-E_c) + \cdots$$

The max. of σ^{ie} occurs at $Re(B_0)=0$

$$\rightarrow c \simeq 0$$

Similarly; $Im(B_0) = -b + \cdots$ leading order term in the imaginary part

$b>0$

(Eq. 1, P 205) $\rightarrow$

$$\sigma^{el} = \frac{\pi}{k^2}\left| e^{2ikR_c} - 1 - \frac{2ikR_c}{a(E-E_c) - i(b+kR_c)}\right|^2$$

$$\sigma^{re} = \frac{\pi}{k^2} \frac{4kR_c b}{[a(E-E_c)]^2 + (b + kR_c)^2}$$

Now define; $\Gamma = 2\,\frac{b+kR_c}{a}$ $\qquad \Gamma_\alpha = \frac{2kR_c}{a}$ $\qquad \Gamma^{re} = 2\frac{b}{a}$

Γ : total width $\qquad$ Γ_α : partial width for the enterance channel

$$\Gamma^{re} = \sum_{i \neq \alpha} \Gamma_i$$

Remark: In the limit of elastic scatt. $b \to 0 \to \begin{cases} \Gamma_\alpha \to \Gamma \\ \Gamma^{re} \to 0 \end{cases}$

$$\Gamma = \Gamma_\alpha + \Gamma^{re}$$

$$\sigma^{el} = \frac{\pi}{k^2} \Big| \underbrace{e^{2ikR_c} - 1}_{\text{nonresonating part}} - \underbrace{\frac{i\Gamma_\alpha}{(E-E_c) - i\frac{1}{2}\Gamma}}_{\text{resonating part}} \Big|^2$$

$$\sigma^{re} = \frac{\pi}{k^2} \frac{\Gamma^{re}\Gamma_\alpha}{(E-E_c)^2 + (\frac{1}{2}\Gamma)^2}$$

The contribution of the nonresonating part in σ^{el} corresponds to a smooth background in the cross-section (usually called shape-elastic or potential scattering)

At $E \simeq E_c$ $\quad \sigma^{el} \to \sigma^{cel} \simeq \frac{\pi}{k^2} \frac{\Gamma_\alpha^2}{(E-E_c)^2 + (\frac{1}{2}\Gamma)^2}$

Compound elastic scatt.

the elastic cross-section is dominated by the resonating part.

$$\sigma_\alpha = \sigma^{el} + \sigma^{re} = \frac{\pi}{k^2}\,\frac{\Gamma^{re}\Gamma^{\alpha} + \Gamma_\alpha^2}{(E-E_c)^2 + (\Gamma/2)^2} = \frac{\pi}{k^2}\,\frac{\Gamma\Gamma_\alpha}{(E-E_c)^2 + (\Gamma/2)^2}$$

which is the cross-section for forming the compound nucleus through entrance channel α.

We have ignored the shape-elastic scattering, because it does not involve the formation of a compound nucleus.

$$\sigma_{\beta\alpha} = \sigma_\alpha \frac{\Gamma_\beta}{\Gamma} = \frac{\pi}{k^2}\,\frac{\Gamma_\beta\Gamma_\alpha}{(E-E_c)^2 + (\Gamma/2)^2}$$

Breit-Wigner one-level formula

The reaction with entrance channel α and exit channel β

Overlapping Resonances:

So far the discussion has been confined to the low-energy region $\xrightarrow{\text{where}}$ the density of states is low

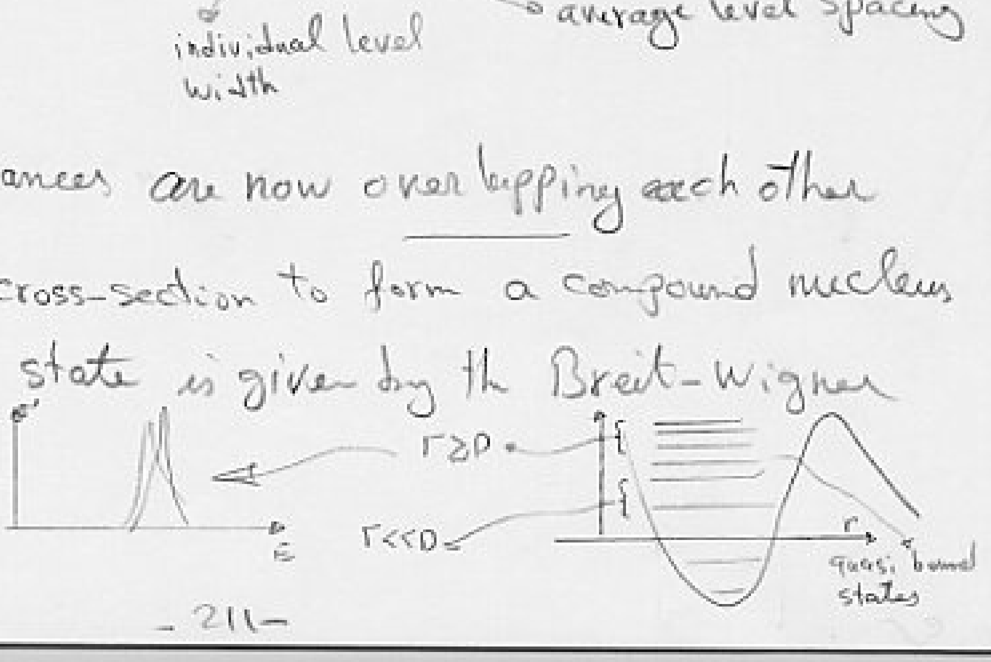
At higer energy region; $\Gamma \gtrsim D$

Γ: individual level width

D: average level spacing

$\longrightarrow$ The resonances are now overlapping each other

Assuming that the cross-section to form a compound nucleus for a particular state is given by the Breit-Wigner formula;



$$\bar{\sigma}_\alpha = \frac{1}{W}\int_{E-\frac{W}{2}}^{E+\frac{W}{2}} \sum_i \frac{\pi}{k^2}\frac{\Gamma^i\Gamma_\alpha^i}{(E'-E_i)^2+(\Gamma^i/2)^2}\,dE'$$

average cross-section in a small energy interval W

$$\bar{\sigma}_\alpha = \frac{\pi}{k^2}\frac{2\pi}{W}\sum_i \Gamma_\alpha^i$$

Γ^i : total width of the i-th resonance

Γ_α^i : partial width for decaying into channel α.

$\sum_i^n$: over all resonance in the energy region W centered at E
(n : number of virtual bound states in this region)

W : i) small enough that the underlying conds. for the resonances are not too different from each other

ii) yet large enough so that $W \gg \Gamma^i$ (in order for the average to have meaning).

In general case (states other than s-wave scatt.) angular momentum coupling factors enter the relations

Also $\quad \overline{\Gamma}_\alpha = \frac{D}{W} \sum_i \Gamma_\alpha^i \quad$ mean width

where

$\frac{W}{D}$: number of levels in the interval W

$\frac{\overline{\Gamma}_\alpha}{D}$: is known as the (s-wave) strength func.

$$\overline{\sigma_\alpha} = \frac{\pi}{k^2} 2\pi \frac{\overline{\Gamma}_\alpha}{D}$$

Since the density of the states $\left(\frac{W}{D}\right)$ is now high

$\longrightarrow$ the probability for the compound nucleus to decay through the entrance channel is small.

$\longrightarrow$ The nucleus appears to be black to the incident channel.

In the limit of completely absorptive nucleus, $\longrightarrow$ (low energy scatt.) (with respect to incident channel i.e. $a + A \longrightarrow b+B$, $c+C$, ... $\not\to a+A$ not)

$$\Psi_{inside} \approx \left(u_0(r) \sim e^{-iKr} \right)$$ incoming term

$$\rightarrow \beta_0 = \left(\frac{r}{u_0} \frac{du_0}{dr} \right)_{r=R_c} = -iKR_c$$

K : wave number for $r < R_c$

$$\eta_0 = \frac{B_0 + ikR_c}{B_0 - ikR_c} e^{-2ikR_c} = \frac{-iR_c(K-k)}{-iR_c(K+k)} e^{-2ikR_c}$$

$$\sigma^{re} = \frac{\pi}{k^2}\left(1 - \underbrace{|\eta_0|^2}_{\to\simeq 1}\right) = \frac{\pi}{k^2}\left(1 - \frac{(K-k)^2}{(K+k)^2}\right) = \frac{\pi}{k^2}\frac{4Kk}{(K+k)^2}$$

Since $k \ll K$ for low-energy scatt. from an attractive pot. well; $\longrightarrow$ Since $\sigma^\alpha = \sigma^{el} + \sigma^{re}$

$$\begin{cases} \sigma^{re} = \dfrac{4\pi}{Kk} \\ \text{Also (Equ.1 P205)} \to \sigma^{el} = \dfrac{4\pi}{K^2} \simeq 0 \end{cases} \rightarrow \sigma^\alpha = \frac{4\pi}{Kk} \quad (1)$$

average value of compound nucleus formation cross-section for channel α in the energy region

Now, $$\begin{cases} \bar{\sigma}^\alpha = \dfrac{4\pi}{Kk} \\ \bar{\sigma}^\alpha = \dfrac{\pi}{k^2} 2\pi \dfrac{\bar{\Gamma}_\alpha}{D} \end{cases} \rightarrow \frac{\bar{\Gamma}_\alpha}{D} = \frac{2k}{\pi K}$$

Furthermore, no resonance can be expected from (1).

In practice, resonances are observed at high energies. These are due primarily to the coupling of a large number of small resonances, for example, to a state in the vicinity that is strongly excited due to some special reasons related to the nucl. structure.

Such a strongly excited states is often called a doorway state.

7-3 Direct Reaction:

Stripping and pick up reactions:

A good example: (d, p) process

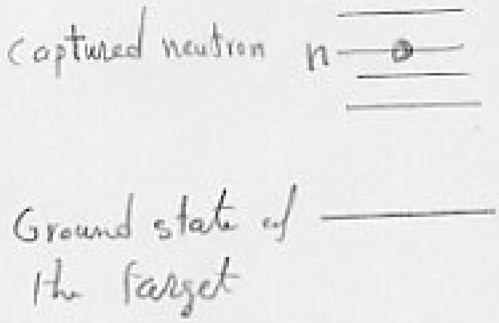
$$A + d \longrightarrow p + B$$

$T_d >$ a few Mev

Deutron is a loosely bound system of P and n

$\longrightarrow$ n is captured into a single-particle orbit of the target without disturbing the rest of the nucleons and p continuous on to become the scattered particle.

$\longrightarrow$ Known as one-neutron stripping reaction



Other stripping reactions:

(t, p)

$\longrightarrow$ two-neutrons are transfered

States in the final nucleus that are strongly excited by such a stripping reaction, are those formed predominantly by a nucleon coupled to the ground state of target nucleus.

Remark: triton $\begin{cases} \text{tritium}, {}^3_1t \\ {}^3_2He \end{cases}$

Complicated stripping reactions:

A cluster of nucleons is transfered.

In this cases it is called <u>direct</u> reaction if:

both { i- Target nucleus
ii- The internal structure of the cluster }

are <u>undisturbed</u>.

residual nucleus { Coupled cluster
Ground state of the target }

This cond. is generally <u>difficult</u> to meet when the <u>number</u> of nucleons in the cluster is <u>large</u>.

Pickup Reaction:

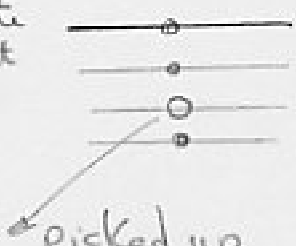
One or more nucleons are <u>taken away</u> from the target nucleus <u>without</u> changing the structure among the <u>rest</u> of the nucleons

Ex.

$^{40}Ca\,(p,d)\,^{39}Ca$ $\qquad$ $^{40}Ca + p \longrightarrow d + {}^{39}Ca$

The states in the residual nucleus, that are strongly excited by a pickup reaction are the one-hole states

39 nucleons are left unchanged in their relative motion.



The Schrödinger equ. for the scatt.:

$$(\nabla^2 + K^2)\,\psi(r) = \frac{2\mu}{\hbar^2} V(r)\,\psi(r) \qquad K^2 = \frac{2\mu E}{\hbar^2}$$

formal sol. $\longrightarrow$ $$\psi(\bar r) = e^{i\bar K_i \cdot \bar r} + \frac{2\mu}{\hbar^2}\int G(\bar r, \bar r')\,V(\bar r')\,\psi(\bar r')\,d^3 r'$$

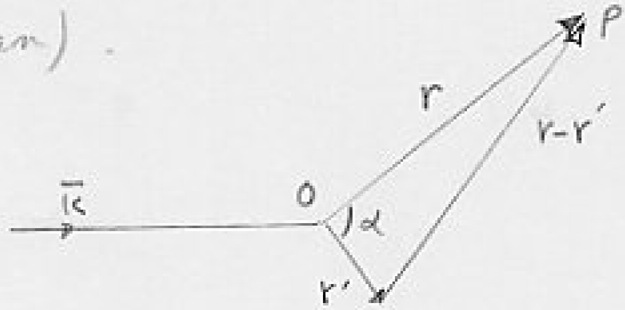
where $$G(\bar r, \bar r') = -\frac{1}{4\pi}\frac{e^{ik|\bar r - \bar r'|}}{|\bar r - \bar r'|}$$

The Green's func. is the sol. of:

$$(\nabla^2 + k^2)\,G(\bar r, \bar r') = \delta(\bar r - \bar r')$$

Remark: We can include within the defining equ. for $G(\bar r, \bar r')$ a part of $V(r)$, the part representing the average effect of the target nucleons on the incident particle.

This is similar in spirit to mean field approach used in nucl. structure investigations to obtain the single particle states in shell-model calculations (Hartree-Fock single particle Hamiltonian).



Now;

$$|\bar{r}-\bar{r}'| = \sqrt{r^2 - 2rr'\cos\alpha + r'^2} \simeq r - \hat{r}\cdot r'$$

$$k|r-r'| = kr - k_f \cdot r' \qquad (k_f = k\,\hat{r})$$

$$\psi(r) \simeq e^{ik_i \cdot r} - \frac{\mu}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-ik_f\cdot r'}\, V(r')\,\psi(r')\, d^3r'$$

where $|r-r'| \simeq r$ in the denominator

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2}\int e^{-ik_f\cdot r'}\, V(r')\,\psi(r')\,d^3r'$$

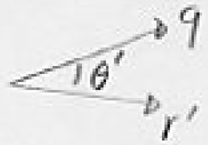
In Born approx.; $\psi(r') \simeq e^{ik_i\cdot r'}$ PWBA (Plane Wave Born Approx.)

$$f^{(1)}(\theta) = -\frac{\mu}{2\pi\hbar^2}\int e^{-ik_f\cdot r'}\, V(r')\, e^{+ik_i\cdot r'}\, d^3r'$$

$\bar{q} = \bar{k}_i - \bar{k}_f$ momentum transfer

spherical Bessel func.

$$e^{iq\cdot r'} = \sum_l i^l \sqrt{4\pi(2l+1)}\, j_l(qr')\, Y_{l0}(\theta')$$



Angular distribution:

In stripping and pick up reactions we are dealing with a <u>change</u> in the structure of nuclei involved

$\longrightarrow$ the wave funcs. of <u>both</u> <u>initial</u> nuclei and <u>final</u> nuclei must enter into the expressions for the scatt. amplitude.

$$\begin{cases} \Psi_i \to e^{ik_i \cdot r} \varphi_i \\ \Psi_f \to e^{ik_f \cdot r} \varphi_f \end{cases}$$ asymptotic forms of initial and final wave funcs. of the scatt. system

φ_i : the product of the <u>internal</u> wave funcs. of the <u>incident</u> and <u>target</u>

φ_f : " " " " " " " " scattered " residual

Ex.— For the case $^{40}Ca(d,p)^{41}Ca$

$$\varphi_i = [\varphi(d) \times \varphi(^{40}Ca)]_{J,T}$$

$$\varphi_f = [\varphi(p) \times \varphi(^{41}Ca)]_{J',T'}$$

$\varphi(d) = [\varphi(p) \times \varphi(n)]$ weakly coupled

$\varphi(^{41}Ca) = [\varphi(n) \times \varphi(^{40}Ca)]$

In order to simplify the argument and to avoid complications due to angular momentum recoupling, we shall treat the proton purely as a spectator in the entire scatt. process.

$$\varphi(^{41}Ca) \sim (\varphi(n)\,\varphi(^{40}Ca)\,Y_{l_t m_t}(\theta',\varphi'))$$

Note: Neutron is captured into a single-particle state of the target (^{40}Ca) with orbital angular momentum l_t.

$$f(\theta) \simeq -\frac{\mu}{2\pi\hbar^2}\int e^{-iq\cdot r'} \left\langle [\varphi(p) \times (\varphi(n)\,\varphi(^{40}Ca)\,Y_{l_t m_t}(\theta',\varphi'))] \right|$$

$$V(r') \left| [\varphi(^{40}Ca) \times [\varphi(p) \times \varphi(n)]] \right\rangle d^3r'$$

$V(r')$: strips a neutron from deutron and put it into the residual nucleus.

Assume; $V(r') = V_0\,\delta(r'-R)$ (stripping on contact)

R: radius of residual nucleus

V_0 : the probability of the process.

When we integrate over the coords., the wave-funcs. drop out.

$$f(\theta) \simeq -\frac{2\mu}{\hbar^2} V_0 \sum_{\ell} i^{\ell} \sqrt{\frac{2\ell+1}{4\pi}}\, j_{\ell}(qR) \int Y_{\ell 0}(\theta') Y^{*}_{\ell_t m_t}(\theta',\varphi') \sin\theta'\, d\theta'\, d\varphi'$$

$$= -\frac{2\mu}{\hbar^2} V_0 \sum_{\ell} i^{\ell} \sqrt{\frac{2\ell+1}{4\pi}}\, j_{\ell}(qR)\, \delta_{\ell\ell'}\, \delta_{m_t,0}$$

$$= -\frac{2\mu}{\hbar^2} V_0\, i^{\ell_t} \sqrt{\frac{2\ell_t+1}{4\pi}}\, j_{\ell_t}(qR) \qquad (1)$$

A more rigorous derivation may be found in the following refrences:

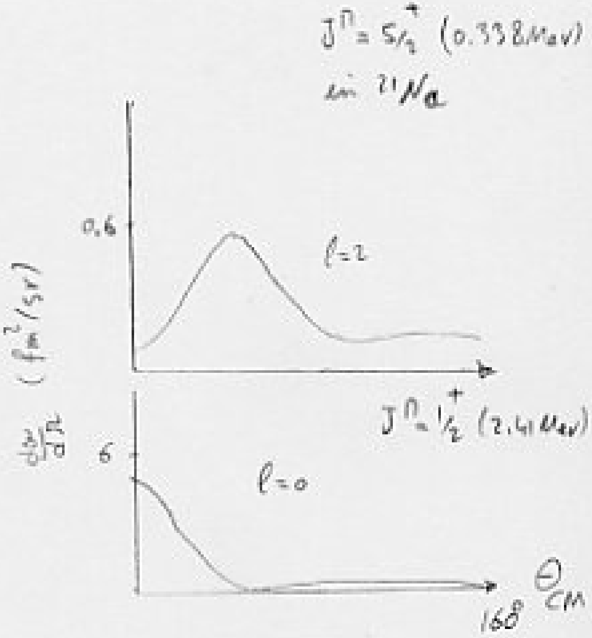
Theory of Direct Nucl. Int., Oxford Univ. Press, London 1961 (Toboeman)
Direct " " " " ", Oxford 1983 (Satchler)

$$q = \sqrt{k_i^2 + k_f^2 - 2k_i k_f \cos\theta} \simeq 2k \sin\left(\frac{\theta}{2}\right)$$

where $k \simeq k_i \simeq k_f$ In high energy scatt.

$$(1) \rightarrow \frac{d\sigma}{d\Omega} \sim |j_{\ell_t}(qR)|^2$$

Ex. — $^{20}Ne(d,n)^{21}Na$



For $l_t = 0$, $j_0(\rho) = \frac{\sin\rho}{\rho}$

(Max. at $\rho = 0$)

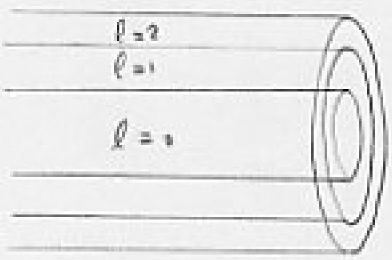
For $l_t = 1$, $j_1(\rho) \sim \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}$

(Max. not at $\rho = 0$)

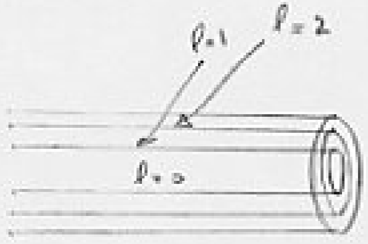
Using DWBA

(Distored Wave Born Approx.)

For l_t = large → (Max. at larger θ)



E(scatt.) ~ 0



E(scatt.) = larger

→ $\delta_l = 0$ for $l \neq 0$ when $E \sim 0$

$L = r \times p$

→ low E → low p → larger r (for a certain l)

Kinematics of the Stripping and Pick up Reactions

The stripping and pick up reactions can in general be written in the following form;

$$(1+2)+3 \longrightarrow 1+(2+3)$$

Ex.- Stripping (d,p)-reaction

$$d + {}^{A}X \longrightarrow p + {}^{A+1}X$$

$$\longrightarrow 1 \equiv p \;,\; 2 \equiv n \;,\; 3 \equiv {}^{A}X$$

Ex. - Pick up reaction (p,d);

$$p + {}^{A}X \longrightarrow d + {}^{A-1}X$$

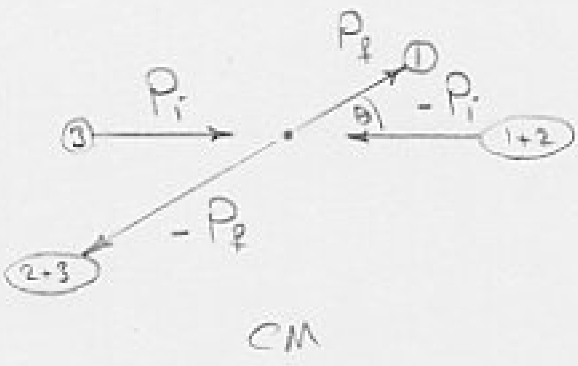
$$1 \equiv {}^{A-1}X \;,\; 2 \equiv n \;,\; 3 \equiv p$$

Ex. - Pick up reaction (p,α)

$$p + {}^{A}X \longrightarrow \alpha + {}^{A-3}X$$

$$1 \equiv {}^{A-3}X \;,\; 2 \equiv {}^{3}H \;,\; 3 \equiv p$$

$-P_i \in (1+2)$ in CM



$$\rightarrow \begin{cases} -\dfrac{m_1}{m_1+m_2} P_i & \in (1) \text{ in CM} \\ -\dfrac{m_2}{m_1+m_2} P_i & \in (2) \;\; " \;\; " \end{cases} \qquad (1)$$

Similarly $-P_f \in (2+3)$ in CM

$$\rightarrow \begin{cases} -\dfrac{m_2}{m_2+m_3} P_f & \in (2) \text{ in CM} \\ -\dfrac{m_3}{m_2+m_3} P_f & \in (3) \;\; " \;\; " \end{cases} \qquad (2)$$

Thus the momentum transfer q_k for the kth particle $(k=1,2,3)$ are;

$$\begin{aligned} q_1 &= P_f - \left(-\frac{m_1}{m_1+m_2}\right) P_i \\ q_2 &= -\frac{m_2}{m_2+m_3} P_f - \left(-\frac{m_2}{m_1+m_2} P_i\right) \qquad (3) \\ q_3 &= -\frac{m_3}{m_2+m_3} P_f - P_i \end{aligned}$$

Note that reduced masses are different in the initial and final states;

$$\begin{aligned} \frac{1}{\mu_i} &= \frac{1}{m_1+m_2} + \frac{1}{m_3} \qquad (4) \\ \frac{1}{\mu_f} &= \frac{1}{m_1} + \frac{1}{m_2+m_3} \end{aligned}$$

The total energy;

$$(4) \rightarrow \quad E = \frac{P_i^2}{2\mu_i} - \epsilon_{12} = \frac{P_f^2}{2\mu_f} - \epsilon_{23} \qquad (5) \quad \text{in CM.}$$

ϵ_{12}: binding energy of (1+2) system

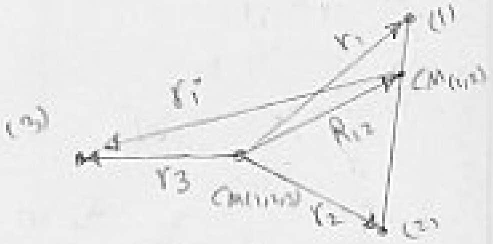
ϵ_{23}: " " (2+3) "

$$Q = \frac{P_f^2}{2\mu_f} - \frac{P_i^2}{2\mu_i} = \epsilon_{23} - \epsilon_{12} \qquad (6)$$

The CM. coords of particles (1+2) and (2+3) are;

$$\bar{R}_{12} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1+m_2}, \quad \bar{R}_{23} = \frac{m_2\bar{r}_2 + m_3\bar{r}_3}{m_2+m_3} \qquad (6')$$

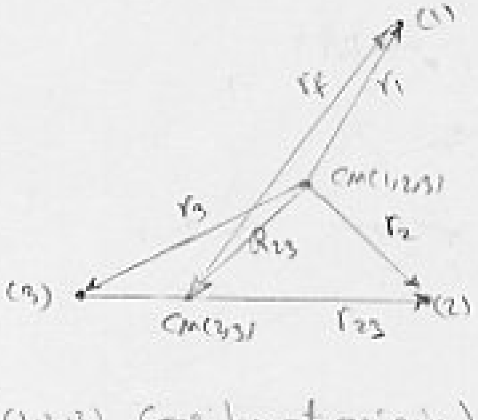
Since the center-of-mass remains fixed, the independent sets of coords. are;



$$\begin{cases} \bar{r}_{12} = \bar{r}_1 - \bar{r}_2 \\ \bar{r}_i = \bar{r}_3 - \bar{R}_{12} \end{cases} \quad \text{initially}$$

$$(7)$$

$$\begin{cases} \bar{r}_{23} = \bar{r}_2 - \bar{r}_3 \\ \bar{r}_f = \bar{r}_1 - \bar{R}_{23} \end{cases} \quad \text{finally}$$



with the cond.

$$m_1\bar{r}_1 + m_2\bar{r}_2 + m_3\bar{r}_3 = 0$$

(CM of the whole sys. is at rest)

(CM(1,2,3) considers at origin)

Momenta conjugates:

$$\begin{cases} P_{12} = \frac{\hbar}{i}\frac{\partial}{\partial r_{12}} = \sum_{k=1,2} P_k \frac{\partial \bar{r}_k}{\partial \bar{r}_{12}} \\ P_i = \frac{\hbar}{i}\frac{\partial}{\partial r_i} = \sum_{k=1,2,3} P_k \frac{\partial \bar{r}_k}{\partial \bar{r}_i} \end{cases}$$

$$\begin{cases} P_{23} = \frac{\hbar}{i}\frac{\partial}{\partial r_{23}} = \sum_{k=2,3} P_k \frac{\partial \bar{r}_k}{\partial \bar{r}_{23}} \\ P_f = \frac{\hbar}{i}\frac{\partial}{\partial r_f} = \sum_{k=1,2,3} P_k \frac{\partial \bar{r}_k}{\partial \bar{r}_f} \end{cases} \tag{8}$$

$$\begin{cases} \bar{r}_{12} = \bar{r}_1 - \bar{r}_2 \\ \bar{r}_1 = \frac{m_2}{m_1+m_2}\bar{r}_{12} = \frac{\mu_{12}}{m_1}\bar{r}_{12} \\ \bar{r}_2 = -\frac{m_1}{m_1+m_2}\bar{r}_{12} = -\frac{\mu_{12}}{m_2}\bar{r}_{12} \end{cases} \longrightarrow \begin{cases} \frac{\partial \bar{r}_1}{\partial \bar{r}_{12}} = \frac{\mu_{12}}{m_1} \\ \frac{\partial \bar{r}_2}{\partial \bar{r}_{12}} = -\frac{\mu_{12}}{m_2} \end{cases} \tag{9}$$

$$\begin{cases} \bar{r}_i = \bar{r}_3 - \bar{R}_{12} \\ \bar{r}_3 = \frac{\mu_{12}}{\mu_{12}+m_3}\bar{r}_i \\ \bar{R}_{12} = -\frac{m_3}{\mu_{12}+m_3}\bar{r}_i \\ m_1\bar{r}_1 + m_2\bar{r}_2 + m_3\bar{r}_3 = 0 \end{cases} \quad \text{where } \bar{R}_{12} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1+m_2} \tag{10}$$

We have similar equs. for $\bar{r}_{23}$ and $\bar{r}_f$ .. etc.

$$\begin{cases} \bar{P}_{12} = \mu_{12}\left(\dfrac{\bar{P}_1}{m_1} - \dfrac{\bar{P}_2}{m_2}\right) \\ \bar{P}_i = \mu_i\left(-\dfrac{\bar{P}_1+\bar{P}_2}{m_1+m_2} + \dfrac{\bar{P}_3}{m_3}\right) \end{cases}$$

$$\begin{cases} \bar{P}_{23} = \mu_{23}\left(\dfrac{\bar{P}_2}{m_2} - \dfrac{\bar{P}_3}{m_3}\right) \\ \bar{P}_f = \mu_f\left(\dfrac{\bar{P}_1}{m_1} - \dfrac{\bar{P}_2+\bar{P}_3}{m_2+m_3}\right) \end{cases} \tag{11}$$

where $\dfrac{1}{\mu_{12}} = \dfrac{1}{m_1} + \dfrac{1}{m_2}$, $\dfrac{1}{\mu_{23}} = \dfrac{1}{m_2} + \dfrac{1}{m_3}$

The Kinetic energy op. T_{cm} in the center-of-mass system is therefore:

$$T_{cm} = \frac{P_{12}^2}{2\mu_{12}} + \frac{P_i^2}{2\mu_i} = \frac{P_{23}^2}{2\mu_{23}} + \frac{P_f^2}{2\mu_f} \tag{12}$$

Note: $\dfrac{P_{12}^2}{2\mu_{12}}$ and $\dfrac{P_{23}^2}{2\mu_{23}}$ contributions are in ϵ_{12} and ϵ_{23} in equ. 5.

Theory of the Stripping and Pick-up Reactions;

$$H = T_{CM} + V_{12} + V_{13} + V_{23}$$

$H\Psi = E\Psi$ $\qquad$ E: total energy in CM. (13)

If,

φ_{12}: wave func. of initial bound state (1+2)

φ_{23}: " " " final " " (2+3)

$$\left(\frac{P_{12}^2}{2\mu_{12}} + V_{12}\right)\varphi_{12}(\bar{r}_{12}) = -\epsilon_{12}\,\varphi_{12}(\bar{r}_{12})$$

$$\left(\frac{P_{23}^2}{2\mu_{23}} + V_{23}\right)\varphi_{23}(\bar{r}_{23}) = -\epsilon_{23}\,\varphi_{23}(\bar{r}_{23}) \qquad (14)$$

and if;

$\chi_i(\bar{r}_i)$: wave func of free particle 3 w.r.t. the CM of (1+2)

$\chi_f(\bar{r}_f)$: " " " " " 1 " " " (2+3)

$$\rightarrow \begin{cases} \Psi(\bar{r}_{12}, \bar{r}_i) = \sum_\alpha \varphi_\alpha(\bar{r}_{12})\,\chi_\alpha(\bar{r}_i) \\ \Psi(\bar{r}_{23}, \bar{r}_f) = \sum_\alpha \varphi_\alpha(\bar{r}_{23})\,\chi_\alpha(\bar{r}_f) \end{cases} \qquad (15)$$

where α signifies a complete set of states of the bound system and the free particle.

$$\chi_i(\bar{r}_i) = \int \varphi_{12}^*(\bar{r}_{12})\, \psi(\bar{r}_{12}, \bar{r}_i)\, d^3 r_{12} \qquad (16)$$

$$\chi_f(\bar{r}_f) = \int \varphi_{23}^*(\bar{r}_{23})\, \psi(\bar{r}_{23}, \bar{r}_f)\, d^3 r_{23}$$

(12)(13) $\rightarrow$ $H\, \psi(\bar{r}_{23}, \bar{r}_f) = \left(\frac{P_{23}^2}{2\mu_{23}} + \frac{P_f^2}{2\mu_f} + V_{12} + V_{13} + V_{23} \right) \psi(\bar{r}_{23}, \bar{r}_f)$

(5) $\rightarrow$ $H\, \psi(\bar{r}_{23}, \bar{r}_f) = \underbrace{\left(\frac{P_f^2}{2\mu_f} - \epsilon_{23} \right)}_{\text{eigenvalue}} \psi(\bar{r}_{23}, \bar{r}_f)$

Since $P_f^2 \equiv -\nabla_f^2$

$$\rightarrow \frac{1}{2\mu_f}\left(\nabla_f^2 + P_f^2 \right) \psi(\bar{r}_{23}, \bar{r}_f) = \left(\frac{P_{23}^2}{2\mu_{23}} + V_{12} + V_{13} + V_{23} + \epsilon_{23} \right) \psi(\bar{r}_{23}, \bar{r}_f) \qquad (17)$$

We apply $(\nabla_f^2 + P_f^2)$ to $\chi_f(\bar{r}_f)$ in (16) and use (4) and (17)

$$(\nabla_f^2 + P_f^2)\, \chi_f(\bar{r}_f) = 2\mu_f \int \varphi_{23}^*(r_{23}) \left[V_{12} + V_{13} \right] \psi(\bar{r}_{23}, \bar{r}_f)\, d^3 r_{23}$$

The sol. can be obtained by using the outgoing Green func.;

$$G(\bar{r}_f,\bar{r}) = -\frac{1}{4\pi}\frac{e^{+ip_f|r_f-r|}}{|r_f-r|}$$

$$\chi_f(r_f) = -\frac{\mu_f}{2\pi}\int \frac{e^{+ip_f|r_f-r|}}{|r_f-r|}\,\varphi^*_{23}(r_{23})\,[V_{12}+V_{13}]\,\psi\, d^3r_{23}\, d^3r \qquad (18)$$

Comparing with the boundary cond.;

$$\chi_f(r_f) \xrightarrow[r_f\to\infty]{} f(\theta,\varphi)\,\frac{e^{ip_f r_f}}{r_f}$$

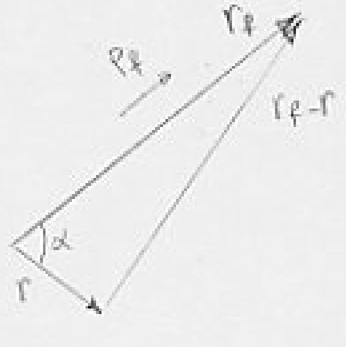
$$\rightarrow \quad f(\theta,\varphi) = -\frac{\mu_f}{2\pi}\int \varphi^*_{23}(r_{23})\, e^{-i\bar{p}_f\cdot\bar{r}_f}\,[V_{12}+V_{13}]\,\psi\, d^3r_{23}\, d^3r_f \qquad (19)$$

where we have have approximated;

$$|r_f-r| = (r_f^2+r^2-2r_f r\cos\alpha)^{1/2} \simeq r_f\left(1-\frac{r}{r_f}\cos\alpha\right) = r_f - \hat{r}_f\cdot\bar{r}$$

$$e^{+ip_f|r_f-r|} \simeq e^{ip_f r_f}\, e^{-ip_f\cdot r}$$

and $\dfrac{1}{|r_f-r|} \simeq \dfrac{1}{r_f}$



Born Approx. (Butler Theory)

In this approx. Ψ is replaced by ;

$$\Psi(r_{23}, r_f) \to \varphi_{12}(r_{12})\, e^{-iP_i\cdot R_{12}}\, e^{iP_i\cdot r_3}$$

$$= \varphi_{12}(r_{12})\, e^{iP_i\cdot r_i}$$

$$\to f(\theta,\varphi) \approx -\frac{\mu_f}{2\pi}\int \varphi_{23}^*(r_{23})\, e^{-iP_f\cdot r_f}\, [V_{12}+V_{13}]\, e^{iP_i\cdot r_i}\, \varphi_{12}(r_{12})\, d^3r_{23}\, d^3r_f$$

$$= -\frac{\mu_f}{2\pi}\int \varphi_{23}^*(r_{23})\, e^{i(q_3\cdot r_{23} - q_1\cdot r_{12})}\, [V_{12}+V_{13}]\, \varphi_{12}(r_{12})\, d^3r_{12}\, d^3r_{23} \qquad (20)$$

where we have used the equalities,

$$-P_f\cdot r_f + P_i\cdot r_i = q_3\cdot r_{23} - q_1\cdot r_{12} \qquad (\text{see } 4, 6', 7)$$

and $d^3r_{23}\, d^3r_f = d^3r_{23}\, d^3r_{12}$

Also we have assumed;

$$V_{12} \neq V_{12}(P)_{avg}$$

$$V_{13} \neq V_{13}(P_{avg})$$

The Born approx. ignores the (incident $\xleftrightarrow{int.}$ target) in expressing Ψ as the product of the incident plane wave and the wave func of Target nucleus.

This is reasonable if the reaction is direct int. (taking place near the surface of the target nucleus).

If the reaction occurs through the formation of compound nucleus, this will not be true.

In the later case;

The compound-nuclear eigenfuncs. describe Ψ.

In this case the int. between the incident and target is strong.

In equ (20), the V_{12} term contributes a nonvanishing result provided the final state contains components of the core, the target nucleus, left in its ground state.

On the other hand, the contribution from V_{13} will be nonvanishing if the final state corresponds to excitation of the core.

Generally, in stripping;

$$V_{13} \text{ contribution} \ll V_{12} \text{ contribution}$$

unless the final state involve almost purely the excitation of the core.

$$\longrightarrow \sigma(\theta,\varphi) \simeq \frac{\mu_i \mu_f}{(2\pi)^2} \frac{P_f}{P_i} \left| \int \varphi_{23}^{*}(\bar{r}_{23})\, e^{i q_3 \cdot r_{23}} d^3 r_{23} \int e^{-i q_1 \cdot r_{12}}\, V_{12}(\bar{r}_{12})\, \varphi_{12}(\bar{r}_{12})\, d^3 r_{12} \right|^2 \qquad (21')$$

Equ (14) for the bound state wave func $\varphi_{12}(r_{12})$;

$$\longrightarrow \left[-\frac{\nabla_{12}^2}{2\mu_{12}} + V_{12}(\bar{r}_{12}) + \epsilon_{12} \right] \varphi_{12}(\bar{r}_{12}) = 0 \qquad (22)$$

$(\hbar = 1)$

Also $\left(-\frac{\nabla_{12}^2}{2\mu_{12}} - \frac{q_1^2}{2\mu_{12}}\right) e^{-iq_1\cdot r_{12}} = 0$

(23)

Multiply (22) on the left by $e^{-iq_1\cdot r_{12}}$ and (23) on the left by $\varphi_{12}(\bar{r}_{12})$ and <u>subtracting</u> and <u>integrating</u> over $\bar{r}_{12}$;

$$\int e^{-iq_1\cdot r_{12}}\left[V_{12}(\bar{r}_{12}) + \epsilon_{12} + \frac{q_1^2}{2\mu_{12}}\right]\varphi_{12}(\bar{r}_{12})\,d^3r_{12}$$

$$= \frac{1}{2\mu_{12}}\int \left[e^{-iq_1\cdot r_{12}}\nabla_{12}^2\varphi_{12}(\bar{r}_{12}) - \varphi_{12}(\bar{r}_{12})\nabla_{12}^2 e^{-iq_1\cdot r_{12}}\right]d^3r_{12} = 0$$

where we have used the <u>property</u> that surface contribution vanish.

$$\int e^{-iq_1\cdot r_{12}} V_{12}(\bar{r}_{12})\,\varphi_{12}(\bar{r}_{12})\,d^3r_{12} = -\left(\epsilon_{12} + \frac{q_1^2}{2\mu_{12}}\right)\int e^{-iq_1\cdot r_{12}}\varphi_{12}(r_{12})\,d^3r_{12}$$

(24)

(20) (21') (24) $\longrightarrow$

$$f(\theta,\varphi) \simeq \frac{\mu_f}{2\pi}\left(\epsilon_{12} + \frac{q_1^2}{2\mu_{12}}\right) I_1(q_1)\, I_3(q_3)$$

$$\sigma(\theta,\varphi) \simeq \frac{\mu_i\mu_f}{(2\pi)^2}\,\frac{P_f}{P_i}\left(\epsilon_{12} + \frac{q_1^2}{2\mu_{12}}\right)^2 \left|I_1(q_1)\, I_3(q_3)\right|^2$$

(25)

where $$I_1(q_1) = \int e^{-iq_1\cdot r_{12}}\, \varphi_{12}(\bar{r}_{12})\, d^3 r_{12}$$

$$I_3(q_3) = \int \varphi^*_{23}(r_{23})\, e^{iq_3\cdot r_{23}}\, d^3 r_{23} \qquad (26)$$

If the bound nuclei (1+2) and (2+3) are in definite orbital angular states (l_{12}, m_{12}) and (l_{23}, m_{23}) respectively, we can write:

$$\varphi_{12}(\bar{r}_{12}) = \varphi_{12}(r_{12})\, Y_{l_{12}, m_{12}}(\hat{r}_{12})$$

$$\varphi_{23}(\bar{r}_{23}) = \varphi_{23}(r_{23})\, Y_{l_{23}, m_{23}}(\hat{r}_{23}) \qquad (27)$$

Using:

$$e^{ik\cdot r} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l\, j_l(kr)\, Y_{lm}(\hat{r})\, Y^*_{lm}(\hat{k})$$

$$\int e^{ik\cdot r}\, Y^*_{lm}(\hat{r})\, d\bar{r} = 4\pi\, i^l\, j_l(kr)\, Y^*_{lm}(\hat{k}) \qquad (28)$$

(26)(27)(28) $\rightarrow$

$$I_1(q_1) = 4\pi(-i)^{l_{12}}\, Y_{l_{12}, m_{12}}(\hat{q}_1) \int r_{12}^2\, \varphi_{12}(r_{12})\, j_{l_{12}}(q_1 r_{12})\, dr_{12}$$

$$I_3(q_3) = 4\pi(+i)^{l_{23}}\, Y^*_{l_{23}, m_{23}}(\hat{q}_3) \int r_{23}^2\, \varphi^*_{23}(r_{23})\, j_{l_{23}}(q_3 r_{23})\, dr_{23}$$

(29)

Substituting (29) in (25) and using the usual procedure of evaluating the cross-section by averaging over the initial magnetic quantum number m_{12} and summing over the final magnetic quantum number m_{23}, we obtain,

$$f_{(1+2)3 \to 1(2+3)}(\theta,\varphi) = 8\pi (i)^{\ell_{23}-\ell_{12}} \mu_f \left(\epsilon_{12} + \frac{q_1^2}{2\mu_{12}}\right) Y_{\ell_{12},m_{12}}(\hat{q}_1)\, Y^*_{\ell_{23},m_{23}}(\hat{q}_3) \cdot R_1(q_1) R_3(q_3) \quad (30)$$

$$\sigma_{(1+2)3 \to 1(2+3)}(\theta,\varphi) = 4(4\pi)^2 \mu_i \mu_f \frac{P_f}{P_i} \left(\epsilon_{12} + \frac{q_1^2}{2\mu_{12}}\right)^2 \frac{1}{2\ell_{12}+1} \cdot \sum_{m_{12},m_{23}} \left| Y_{\ell_{12},m_{12}}(\hat{q}_1) Y^*_{\ell_{23},m_{23}}(\hat{q}_3)\right|^2 R_1^2(q_1) R_3^2(q_3)$$

$$= 4 \mu_i \mu_f \frac{P_f}{P_i} \left(\epsilon_{12} + \frac{q_1^2}{2\mu_{12}}\right)^2 (2\ell_{23}+1) R_1^2(q_1) R_3^2(q_3) \quad (31)$$

Where $\sum_{m=-\ell}^{\ell} |Y_{\ell m}|^2 = \frac{2\ell+1}{4\pi}$

$$R_1(q_1) = \int_0^\infty r_{12}^2\, \varphi_{12}(r_{12})\, j_{\ell_{12}}(q_1 r_{12})\, dr_{12}$$

$$R_3(q_3) = \int_0^\infty r_{23}^2\, \varphi_{23}(r_{23})\, j_{\ell_{23}}(q_3 r_{23})\, dr_{23}$$

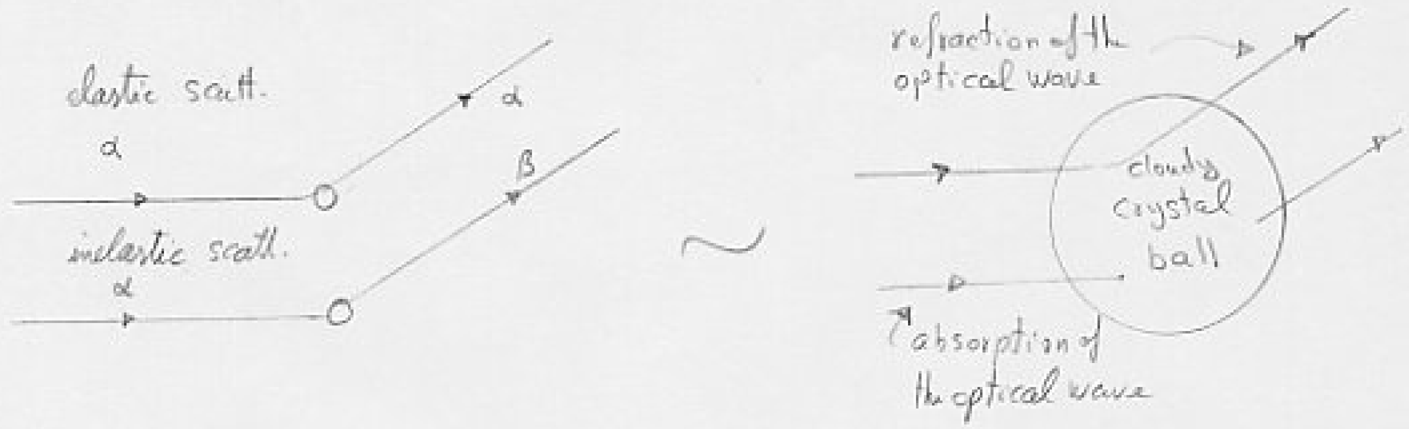
Note: $\theta : (\bar{P}_i \wedge \bar{P}_f)$, (31) depends on θ through q_1 and q_3 but not φ.
observable quantities: $- P_i$ of lighter incident (1+2)
P_f of lighter outgoing particle 1.

7-4 The Optical Model

Besides $\begin{cases} \text{1- Compound nucleus formation} \\ \text{2- Direct reaction} \end{cases}$, we may also be

interested in the <u>average</u> result of a reaction at a <u>given</u> bombarding energy.

→ For such <u>purposes</u>, we often invoke the <u>analogy</u> of an <u>optical</u> wave through a <u>cloudy crystal</u> ball.



The aim of the optical model is to find a potential <u>V</u> to describe the <u>smooth</u> variations of <u>σ</u> as a func. of $\begin{cases} E \\ A \text{ (target nucleon number)} \end{cases}$

The <u>general</u> situation may be quite <u>complex</u>.

However, great simplification may be obtained if we are interested only in the average properties, away from

$\left\{ \begin{array}{l} \text{resonances} \\ \text{and states strongly excited by direct reactions} \end{array} \right.$

→ The basic idea $\xrightarrow{\text{similar}}$ to the mean field approach.

We will involved here mainly with the elastic scatt..

There are two main sources of contribution to σ.

1- The potential scatt.

$$\sigma^{el} = \frac{\pi}{k^2} \Big| \underbrace{e^{2ikR_c} - 1}_{\text{Pot. scatt}} - \frac{i\Gamma_\alpha}{(E - E_c) - \frac{i}{2}\Gamma} \Big|^2$$

↳ gives smooth back ground

2- Multiple scattering with intermediate states involving the excited states of the nuclei

Not all such multiple scatts. return the system back to the incident channel.

→ As a result, some of the incident channel flux is lost to other reaction channels.

Rather than trying to calculate the σ of each inelastic channel $\xrightarrow{\text{we attempt}}$ to represent the sum of their contributions by the imaginary part of an optical pot.

Formal derivation of the optical model Potential:

We make a connection between optical model pot. and averaging over contributions from a large number of intermediate states.

Consider a free nucleon scattering off a nucleus made of A-nucleons.

r_0: the coord. of the projectile

r_i: " " " " A-nucleons in the target $(i=1,\ldots A)$

r represent all deg. of freedom (spin ...).

Our aim is to solve the schrödinger equ.:

$$H(r_0; r_1, r_2 \cdots r_A)\, \Psi(r_0; r_1, r_2 \cdots r_A) = E\, \Psi(r_0; r_1, r_2 \cdots r_A)$$

under boundary conds. appropriate for scatt..

For the time being we ignore the antisymmetrization between the projectile and the nucleons in the target.

$$H(r_0; r_1, r_2 \cdots r_A) = T_0 + \sum_{i=1}^{A} V(r_{0i}) + H_A(r_1, r_2 \cdots r_A)$$

where $\bar{r}_{0i} = \bar{r}_0 - \bar{r}_i \quad i \neq 0$

T_0 : Kinetic energy of the projectile

H_A : the Hamiltonian operating only among the nucleons in the target.

$V(r_{0i})$: the int. between the projectile and the target nucleons

We assume the eigenvalue prob.

$$H_A(r_1, r_2 \cdots r_A)\, \Phi_i(r_1, r_2 \cdots r_A) = \epsilon_i\, \Phi_i(r_1, r_2 \cdots r_A)$$

has already been solved.

$\{\Phi_i\}$: a complete set of sols.

Also we assume φ_i is normalized to unity and the set is orthogonal.

$$\psi(r_0, r_1, r_2 \dots r_A) = \sum_{ij} \chi_i(r_0)\,\varphi_j(r_1, r_2 \dots r_A) \qquad (1)$$

the general sol. for the complete system

$\chi_i(r_0)$: the wave func. of the projectile

Since we are interested to the elastic scatt., the only part of ψ that is of interest to us here, is $\chi_0 \varphi_0$.

$\chi_0 \varphi_0$: Both projectile and target in their respective lowest energy states.

Our prob here is to obtain χ_0 (φ_0 is assumed to be already known).

We shall find an equ. for χ_0.

Let:

$P = |\varphi_0\rangle\langle\varphi_0|$ projection op. for the ground state of the target

(any integration is carried over the coords. of target nucleons only.)

$(1) \rightarrow P\Psi = \chi_0 \varphi_0$

$$Q = 1 - P$$

$$P^2\Psi = P\Psi \quad , \quad Q^2\Psi = Q\Psi \qquad PQ\Psi = QP\Psi = 0 \quad (2)$$

Since $P + Q = 1 \xrightarrow[\text{equ.}]{\text{the Schrödinger}} (E - H)(P + Q)\Psi = 0$

$$\rightarrow P(E-H)(P+Q)\Psi = 0 \rightarrow (EP^2 - PHP)\Psi + 0 - PHQ\Psi = 0$$

$$\rightarrow (E - PHP)P\Psi = (PHQ)Q\Psi \qquad (3)$$

similarly: $Q(E-H)(P+Q)\Psi = 0$

$$\rightarrow (E - QHQ)Q\Psi = (QHP)P\Psi \qquad (4)$$

$$(4) \rightarrow Q\Psi = \frac{1}{E - QHQ} QHPP\Psi \qquad (5)$$

$$(5) \text{ in } (3) \rightarrow \left\{ E - PHP - PHQ \frac{1}{E - QHQ} QHP \right\} P\Psi = 0$$

Multiplying by $\langle \varphi_0 |$ from the left, and integrating over the coords of the target and remembering that:

$$\Psi = \sum_{ij} \chi_i \varphi_j \qquad , P\Psi = \chi_0 \varphi_0$$

$$P\varphi_0 = \varphi_0$$

$$\left\{ E - \langle \varphi_0 | H | \varphi_0 \rangle - \langle \varphi_0 | H Q \frac{1}{E - QHQ} Q H | \varphi_0 \rangle \right\} \chi_0 = 0 \qquad (6)$$

This equ. must be solved in order to obtain χ_0.

The zero-point of the energy scale is still arbitrary at this point $\longrightarrow$ We may set it at the ground state of the target nucleus $\xrightarrow{\text{That is}}$ we let $\epsilon_0 = 0$

$$H_A' \varphi_0 = \epsilon_0 \varphi_0 = 0$$

i.e. $H_A \longrightarrow H_A'$ with the mentioned property

$$\rightarrow \left\{ E - T_0 - \langle \varphi_0 | V | \varphi_0 \rangle - \langle \varphi_0 | V Q \frac{1}{E - QHQ} Q V | \varphi_0 \rangle \right\} \chi_0 = 0$$

$V \equiv V(r_{0i})$ $\qquad (Q T_0 | \varphi_0 \rangle = T_0 Q | \varphi_0 \rangle = 0)$

$(6')$

Since $(1 \pm x)^{-n} = 1 \mp \frac{nx}{1!} + \frac{n(n+1)x^2}{2!} + \cdots \qquad |x| < 1$

$$\frac{1}{E - QHQ} = \frac{1}{E} \left[1 + \frac{1}{E} QHQ + \frac{1}{E} QHQ \frac{1}{E} QHQ + \cdots + \left(\frac{1}{E} QHQ \right)^k + \cdots \right]$$

multiple int of order k

$QHQ \longrightarrow$ implies that the int. takes place with the target nucleons in one of its excited states (due to Q).

$(6') \rightarrow \quad (E - T_0 - \mathcal{V}(r_0))\, \chi_0 = 0 \qquad (7)$

where $\mathcal{V}(r_0) = \langle \varphi_0 | V | \varphi_0 \rangle + \langle \varphi_0 | V Q \dfrac{1}{E - QHQ} Q V | \varphi_0 \rangle$

Since we have not yet made approx. in arriving (7), we don't have any better chance of solving it than the original equ. $H\Psi = E\Psi$

The aim of an optical model is to replace the equivalent pot. $\mathcal{V}(r_0)$ by an optical pot. U_{opt}, such that the equ.

$$(E - T_0 - U_{opt})\, \chi_0 = 0$$

can be solved.

In general $\mathcal{V}(r_0)$ is nonlocal (that is, the pot. acting at one point of space may depend on the value of the wave func. at a different point.

(7) takes on the form;

$$(E - T_0)\, \chi_0(r_0) = \mathcal{V}_0\, \chi_0(r_0) + \int \mathcal{V}'(r_0, r_0')\, \chi_0(r_0')\, dr_0'$$

This greatly complicates the prob. and one may wish to approximate with a local pot., often done in practice

Furthermore, the derivation here may have given the impression that all the scatt. into the Q-space eventually returns the target to the ground state.

This is certainly not true in general.
In order to represent the loss of flux from the incident channel by scatt. that ends up in other exit channels,
$\longrightarrow$ the optical pot. is usually complex.

Phenomenological Optical Model Potential:

Since nucl. force is short $\longrightarrow$ Radial dependence of an optical model pot. $U_{opt}(r)$ follows closely the nucleon dist. in a nucleus.

For simplicity a 2-parameter Fermi form may be used to describe such a dist.,

$$f(r, r_0, a) = \frac{1}{1 + \exp[(r - r_0 A^{1/3})/a]} \qquad (1)$$

In optical model studies this is also known as the Woods-Saxon form.

$$U_{Vol}(r) = -\left\{ V_0 f(r, r_v, a_v) + i W_0 f(r, r_w, a_w) \right\} \qquad (2)$$

(in general complex)

V_0, W_0: depths of the real and imaginary parts of the pot. well.
They may be taken free parameters to be determined from the experimental data.

If $f(r, r_v, a_v)$ and $f(r, r_w, a_w)$ obey (1) $\longrightarrow$

we may take the radii $\begin{cases} r_v \\ r_w \end{cases}$ and the surface diffuseness

$\begin{cases} a_v \\ a_w \end{cases}$ also as adjustable parameters.

U_{vol} is reffered to volume term of U_{opt}, since the int.

follows the volume dist. of nucleons.

In addition to this, optical potentials are also known to

have a spin dependence.

A spin-orbit term may be used of the following form;

$$U_{s.o.}(r) = \sigma \cdot l \left(\frac{\hbar}{m_n c}\right)^2 \frac{1}{r} \left\{ V_s \frac{d}{dr} f(r, r_{sv}, a_{sv}) + i W_s \frac{d}{dr} f(r, r_{sw}, a_{sw}) \right\}$$

$\begin{cases} V_s \\ W_s \end{cases}$, $\begin{cases} r_{sv} \\ r_{sw} \end{cases}$ and $\begin{cases} a_{sv} \\ a_{sw} \end{cases}$ may be adjusted to fit

scatt. data.

$\left(\frac{\hbar}{m_n c}\right)^2 \approx 2$

$$\sigma(\theta,\varphi) = \frac{I_{outgoing}}{I_{incident}} = \lim \left[\frac{1}{2i\mu_f}\left(\chi_f^* \frac{\partial\chi_f}{\partial r_f} - \frac{\partial\chi_f^*}{\partial r_f}\chi_f\right)R^2\right]\frac{1}{P_i/\mu_i}$$

$$= \frac{\mu_i P_f}{\mu_f P_i}\,|f(\theta,\varphi)|^2$$

$$= \frac{\mu_i \mu_f}{(2\pi)^2}\frac{P_f}{P_i}\left|\int \varphi_{23}^*(r_{23})\, e^{i(q_3\cdot r_{23} - q_1\cdot r_{12})}\left[V_{12}+V_{13}\right]\varphi_{12}(r_{12})\, d^3r_{12}\, d^3r_{23}\right|^2 \tag{21}$$

Equ. (20) (21) can be simplified further by neglecting V_{13}.

This can be done because in the reaction:

$$(1+2)+3 \longrightarrow 1+(2+3)$$

particle 1 and 3 never appear in a bound state.

For example: in stripping reaction

$${}^{A}X\,(d,P)\,{}^{A+1}X \quad ; \quad V_{13} = V_{P,A}$$

and in the pick up reaction

$${}^{A}X\,(P,d)\,{}^{A-1}X \quad , \quad V_{13} = V_{P,A-1}$$

$V_{12} = V_{n,p}$ for both reaction

Remark:

We may understand the reason for such a radial-dependence from the analogy of the Thomas spin-orbit pot. for an atomic electron in the Coulomb field of a nucleus.

$$W(r) = -\bar{\mu}_e \cdot \bar{B}(r)$$

$\bar{B}(r)$: that the electron feels due to its orbital motion.

$$\bar{B}(r) = -\frac{1}{c[c]} \bar{v} \times \bar{E}(r)$$

$\bar{E}(r)$: the field of nucleus

$\bar{B}(r)$ may be found by a Lorentz tr. of $\bar{E}(r)$, stationary in lab. into a frame of ref. at rest with the orbiting electron

$$\bar{l}\hbar = \bar{r} \times \bar{p} = m_e(\bar{r} \times \bar{v})$$

$$\bar{E}(r) = -\frac{dV}{dr}\frac{\bar{r}}{r}$$

$$\bar{B}(r) = -\frac{\hbar}{m_e c r [c]} \frac{dV}{dr} \bar{l}$$

$$W(r) = -\mu_e \cdot B(r) = \frac{e\hbar^2}{m_e^2 c^2 r} \frac{dV}{dr} \sigma \cdot l$$

when $\mu_e = \frac{e\hbar[c]}{m_e c} \sigma$

For charged particle the optical pot. has also a Coulomb term;

$$U_c(r) = \begin{cases} \left[\frac{1}{4\pi\epsilon_0}\right] \frac{3Ze^2}{2R_c}\left(3 - \frac{r^2}{R_c^2}\right) & r < R_c \\ \left[\frac{1}{4\pi\epsilon_0}\right] \frac{3Ze^2}{r} & r \geq R_c \end{cases}$$

R_c: Coulomb radius (Free parameter)

where we have assumed a uniformly charged sphere.

In practice, the scatt. results are not sensitive to the details of the Coulomb pot. $\longrightarrow$ $R_c = 1.2\, A^{1/3}$ fm may be used.

$$U_{opt.}(r) = U_{Vol.}(r) + U_{S.O.}(r) + U_c(r)$$

Total number of adjustable parameters $= 12$

(R_c, not included)

They can be obtained by fitting the calculated results to the measured quantities.

A large amount of information has been accumulated for proton scatt. off nuclei up to a lab. energy of 200 MeV.

However, for other projectiles like neutron there are far fewer experimental data.

→ The optical pots are not well known for these cases.

Also because of the phenomenological nature of the pot., the approach does not lend itself easily to extrapolation to regions where experimental data are scarce.

Also the forms of the mentioned radial dependence for $U_{vol}(r)$ and $U_{s.o.}(r)$ are found to be inadequate in higher bombarding energies.