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Nuclear Physics - II
Useful Book : Nuclear Physics,
Samuel S.M. Wong

Chapter 5

Nuclear Excitation and Decay:

$$H = H_{nucl.} + \underbrace{H_{Elec.} + H_{weak}}_{\text{perturbation}}$$

$$H_{nucl.} \Psi = E \Psi \qquad \Psi: \text{nuclear state}$$

Transitions between nuclear states can take place by: $(\Psi \longrightarrow \Psi')$

- i) γ-ray emission
- ii) α-particle decay
- iii) β- " "
- iv) fission

The perturbation part of H is responsible for the mentioned decays.

Other reactions like nucleon emission are sufficiently fast and must be treated differently (chapter 7).

5-1) Nuclear Transition Matrix Element:

Transition Probability:

Consider a sample of N radioactive nuclei:

The decay probability of any nucleus P at a given time (dt) is:

- i) independent of the status of other nuclei in the sample
- ii) other external effects

$P(dt) \sim dt$ (if $dt \ll 1 \longrightarrow P(dt) \ll 1$)

$\longrightarrow P(dt) = W\,dt$

$1 - P(dt)$ surviving probability of a certain nucleon after a time of dt

$\longrightarrow (1-P(dt))^n$ The same after $t = n\,dt$

$$(1 - W dt)^n = \left(1 - \frac{Wt}{n}\right)^n \xrightarrow[\substack{n \to \infty \\ dt \to 0}]{} e^{-Wt}$$

For N_0 similar nuclei at $t=0$;

$$N(t) = N_0\, e^{-Wt} \quad \text{(decay law)}$$

$$\longrightarrow \frac{dN}{dt} = -W\,N(t)$$

W : transition probability (decay const.)

Half-life: At $t = T_{1/2}$ $N(T_{1/2}) = \frac{N_0}{2}$

$$\longrightarrow T_{1/2} = \frac{\ln 2}{W} = \frac{0.693}{W}$$

$T_{1/2}$ is the amount of time it takes for the activity of a sample to be reduced by half.

The life-time (mean life): the average amount of time it takes for <u>a</u> radioactive nucleus to decay (or average surviving time for a radioactive nucleus).

$$\tau = \overline{T} = \frac{\int t\,dN(t)}{\int dN(t)} = \frac{N_0\int_0^\infty t\, e^{-wt}\,dt}{N_0\int_0^\infty e^{-wt}\,dt} = \frac{1}{w} = \frac{T_{1/2}}{0.693}$$

$$\underset{\substack{\uparrow\\ \text{The number of}\\ \text{emitted particles}}}{dN_p} = \underset{\substack{\uparrow\\ \text{The number of}\\ \text{decayed nuclei}}}{-dN} = w\,N\,dt$$

$$dN_1 + dN_2 + \dots = -dN = w_1 N dt + w_2 N dt + \dots$$

$$\rightarrow N = N_0\, e^{-(w_1+w_2+\dots)t} = N_0\, e^{-w_{tot}t}$$

$$T_{1/2} = \frac{\ln 2}{w_{tot}}$$

$$\frac{dN_i}{dt} = +w_i N = w_i N_0\, e^{-w_{tot}t}$$

Width: $\Delta E \Delta t \approx \hbar$

$$\langle E \rangle = \frac{1}{N} \sum_{i=1}^{N} E_i$$

N: number of excited nuclei (large)

E_i: i-th excited nucleus energy

Note: Variance;

$$(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2 = \langle A^2 \rangle - \langle A \rangle^2$$

$$\Gamma = \left\{ \frac{1}{N} \sum_{i=1}^{N} (E_i^2 - \langle E \rangle^2) \right\}^{\frac{1}{2}}$$

(standard deviation)

natural line width of a state

(not related to the instrumental uncertainty)

$$\begin{cases} \Delta E = \Gamma \\ \Delta t = T \end{cases} \longrightarrow \Gamma = \frac{\hbar}{T} = \hbar w$$

In terms of wave funcs.;

$$|\psi(\bar{r}, t)|^2 = |\psi(\bar{r}, t=0)|^2 e^{-wt} \qquad (1)$$

For a stationary state; $\psi(\bar{r}, t) = \psi(\bar{r}) e^{-iEt/\hbar}$ (2)

For a decaying (excited) state;

$$E \longrightarrow \langle E \rangle - \frac{i}{2} \hbar w$$

$$\psi(\bar{r}, t) = \psi(r) e^{-i\langle E \rangle t/\hbar - wt/2} \qquad (3)$$

consistant with (1)

Alternatively: since an excited state has <u>no</u> definite energy:

$$\psi(\bar{r},t) = \psi(\bar{r}) \int a(E)\, e^{-iEt/\hbar}\, dE \qquad (4)$$

$a(E)$: the probability amp. for finding the state at energy E.

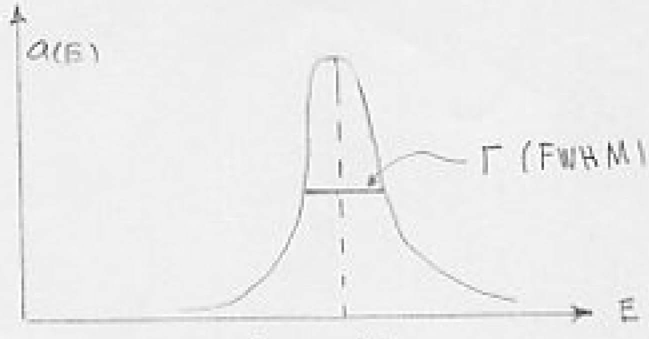
$$(3)(4) \longrightarrow \quad e^{-Wt/2} = \int a(E)\, e^{-i(E-\langle E\rangle)t/\hbar}\, dE$$

That is $e^{-Wt/2}$ is the Fourier tr. of $a(E)$

$$\longrightarrow a(E) = \frac{1}{2\pi\hbar}\int_0^\infty e^{[i(E-\langle E\rangle)/\hbar - \frac{W}{2}]t}\, dt = \frac{i}{2\pi}\,\frac{1}{(E-\langle E\rangle)+i\frac{\hbar W}{2}}$$

$$|a(E)|^2 = \frac{1}{4\pi^2}\,\frac{1}{(E-\langle E\rangle)^2+(\frac{\Gamma}{2})^2}$$ The probability for finding the excited state with energy E

where $\Gamma = \hbar W$



Lorentzian dist.

Remark: $\displaystyle\int_0^\infty e^{(-\alpha+i\beta)r}\, dr = \frac{-1}{-\alpha+i\beta}$

Remark: Fourier tr.:

$$\begin{cases} \psi(\bar{r},t) = \dfrac{1}{(2\pi\hbar)^{3/2}}\displaystyle\int \varphi(\bar{p},t)\, e^{\frac{i}{\hbar}p\cdot r}\, d^3p \\ \varphi(\bar{p},t) = \dfrac{1}{(2\pi\hbar)^{3/2}}\displaystyle\int \psi(\bar{r},t)\, e^{-\frac{i}{\hbar}p\cdot r}\, d^3r \end{cases}$$

FWHM (the full width at the half max)

$$\begin{cases} f(t) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega \\ F(\omega) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \end{cases}$$

Branching ratio:

An excited state $\xrightarrow{\text{may decay}}$ several final states

$$\rightarrow \quad W = \sum_i W_i \qquad (1)$$

$\rightarrow$ Similarly $\quad \Gamma = \sum_i \Gamma_i \qquad \left(\Gamma = \frac{\hbar}{\tau} = \hbar W \right)$

$$(1) \rightarrow \quad \frac{1}{T_{1/2}} = \sum_i \frac{1}{T_{1/2}(i)}$$

where $T_{1/2}(i) = \dfrac{\ln 2}{W_i}$ partial half-life

The branching ratio gives the partial transition probability to a particular final state as a fraction of the total transition probability from a specific initial state.

Ex.:

$$\pi^0 \begin{cases} \xrightarrow{98.8\%} \gamma + \gamma \\ \xrightarrow{1.17\%} \gamma + (e^+ + e^-) \\ \xrightarrow{2\times10^{-7}\%} e^+ + e^- \end{cases} \qquad (\tau = 8.4\times10^{-17}\ s)$$

$\uparrow$
branching ratios

$$\frac{W_i}{W}$$

Transition matrix element:

$$W \sim |M_{fi}(M_f, M_i)|^2$$

where $M_{fi}(M_f, M_i) = \langle J_f M_f, \xi | O_{\lambda\mu} | J_i M_i \zeta \rangle$

nucl. matrix element

$O_{\lambda\mu}$: nucl. part of transition op. with spherical tensor rank (λ, μ).

The transition may involve different particles:

$$| J_f, M_f, \zeta \rangle = | \rangle | \rangle \ldots$$

Using Wigner-Eckart Theorem;

$$M_{fi}(M_f, M_i) = (-)^{J_f - M_f} \begin{pmatrix} J_f & \lambda & J_i \\ -M_f & \mu & M_i \end{pmatrix} \langle J_f \xi \| O_\lambda \| J_i \zeta \rangle$$

Now,

i) If the measurement is not sensitive to the orientation of the spin in the final state, the transition includes all the possible final states differing only by the value of M_f.

ii) Furthermore, if the transition op. is not restricted to any specific dir. in space, all the allowed values of μ must be included.

$$|M_{fi}|^2 = \sum_{\mu M_f} \left| (-)^{J_f - M_f} \begin{pmatrix} J_f & \lambda & J_i \\ -M_f & \mu & M_i \end{pmatrix} < J_f \xi \| O_\lambda \| J_i \xi > \right.$$

$$= |< J_f \xi \| O_\lambda \| J_i \xi >|^2 \sum_{\mu M_f} \left| \begin{pmatrix} J_f & \lambda & J_i \\ -M_f & \mu & M_i \end{pmatrix} \right|^2$$

Using the orthogonality relation;

$$\begin{cases} \sum_{m_1 m_2} \begin{pmatrix} J_1 & J_2 & J_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \dfrac{\Delta(J_1, J_2, J_3)}{2J_3+1} \delta_{J_3 J_3'} \delta_{m_3 m_3'} \\ \Delta(J_1, J_2, J_3) = \begin{cases} 1 & \text{for } J_3 = J_1 + J_2 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

$$\rightarrow \quad |M_{fi}|^2 = \frac{\Delta(J_f, \lambda, J_i)}{2J_i+1} |< J_f \xi \| O_\lambda \| J_i, \xi >|^2$$

Note: $|M_{fi}|^2$ is <u>indep.</u> of M_i.

5-2) Electromagnetic Transition

We shall deal mainly the decay of the form;

$$\text{nucleus} \xrightarrow[\uparrow]{} \text{nucleus}' + \gamma$$

int. between nucleus and external electromagnetic field.

Assumption: nucleus is made of <u>Point</u> nucleons.

n has $\left\{\right.$ intrinsic mag. dipole moment

p has $\left\{\begin{array}{l}\text{i- intrinsic mag dipole moment} \\ \text{ii- mag. dipole moment due to orbital motion} \\ \text{iii - Charge}\end{array}\right.$

(Charge dist. <u>interaction</u> external field) $\xrightarrow{\text{Causes}}$ Electric transitions

((intrinsic + that one due to orbital motion) mag. moments) $\xrightarrow{\text{Causes}}$ Magnetic "

El-Mag. transitions for a <u>dominant</u> mode of decay for <u>low-lying</u> excited states in nuclei, particulary for <u>light</u> ones.

Main reason: Despite nucleon emission $\underset{\text{faster}}{\gg}$ γ-decay

it is forbidden until $E_{\text{excitation}} > E_{\text{nucleon separation}}$

$E_{\text{neutron separation}} \sim$ 8-10 Mev $\qquad E_{\text{proton separation}} < E_{\text{neutron separation}}$ due to Coulomb repulsion

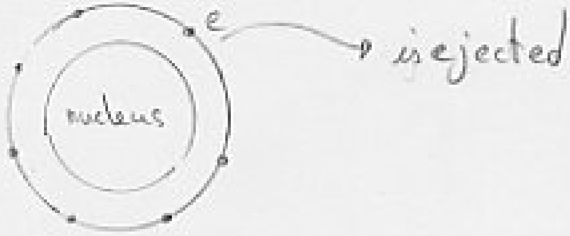
Other possible decay modes;

$\left\{\begin{array}{l}\beta\text{ - decay} \\ \alpha\text{-particle emission} \\ \text{fission}\end{array}\right. \underset{\text{generally slower}}{<} \gamma\text{- decay}$

Other modes:

Electromag. transitions (de-excitation process)
- γ-decay
- internal conversion
  - 1- inter. direct conv. in the Coulomb field
  - 2- inter. photoelectric conv.
- Pair production

Internal conversion;



In this process one of the <u>atomic</u> electrons is ejected.

This is usually more important for <u>heavy</u> nuclei, where
- i- the electromag. fields are <u>strong</u>
- ii- the inner shell electrons are <u>close</u> to the nucleus

The probability of pair production process $\underset{\text{smaller}}{<<}$ γ-decay

It becomes important γ-decay process is <u>forbidden</u>; by angular momentum consideration.

This happens in the transitions; $J_i^\pi = 0^+ \longrightarrow J_f^\pi = 0^+$

(γ-ray carries one <u>unit</u> of angular momentum)

Using the first-order time-dep. perturbation theory;

$$W = \frac{2\pi}{\hbar} \left| \langle \varphi_k(\bar{r}) | H' | \varphi_0(\bar{r}) \rangle \right|^2 \rho(E_f)$$

Fermi's Golden rule

H': perturbation due to nuclear and electromag. fields

$$\rho(E_f) = \rho_{nucl.} \, \rho_{el\text{-}mag}$$

where $\begin{cases} \rho_{nucl.}: \text{the number nucl. states/energy interval at } E_f \\ \rho_{el\text{-}mag}: \text{" " el-mag " "} \end{cases}$

$\varphi_0(\bar{r}) = |nucl.\rangle_0 \, |el\text{-}mag\rangle_0$

$\varphi_k(\bar{r}) = |\ \text{"}\ \rangle_k \, |\ \text{"}\ \rangle_k$

Coupling to electromagnetic field:

$$H' = H'_{nucl.} \, H'_{el\text{-}mag}$$

$H'_{nucl.}$: acting on nuclear wave func

$H'_{el\text{-}mag}$: " " the wave func. of external el-mag. field

(Ex.: γ-ray)

Our primary interest: $\langle nucl. | H'_{nucl.} | nucl \rangle$

Similar to the quantized nuclear wave funcs (with definite angular momentum), the external electromagnetic field is also quantized and decomposed by a multipole expansion into components with definite spherical tensor ranks.

(often the lowest order multipole dominates the nuclear transition)

Consider a point particle carrying a charge q, but no magnetic moment.

$$H_0 = \frac{P^2}{2m}$$ in the absence of any external el-mag field.

$$P \longrightarrow P + \frac{q}{c} A$$ in the presence of an el-mag. field (momentum conjugate to r)

$$H_0 \longrightarrow H = \frac{1}{2m}\left(P - \frac{q}{c} A\right)^2$$ (for a charged particle)

In general $H_{total} = H + H_{\substack{ext.\\ el\text{-}mag\ field}} + H_{electrostatic}$

$$H = H_0 + H'$$

$$H' = -\frac{q}{2mc}\left(P \cdot A + A \cdot P\right) + \frac{q^2}{2mc^2} A \cdot A = -\frac{q}{mc} A \cdot P + \frac{q^2}{2mc^2} A \cdot A$$

coupling with the ext. el-mag. field

Where we have used $P \cdot A = A \cdot P$

Using $\nabla \cdot A = 0$ (an el-mag. field can have only trasverse) components (transversality cond.)

and $p = -i\hbar \nabla$

$$p \cdot A\psi = -i\hbar \nabla \cdot A\psi = -i\hbar A \cdot \nabla\psi - i\hbar \psi (\nabla \cdot A)$$

$$\longrightarrow \quad p \cdot A = A \cdot p$$

The quadratic term $A \cdot A$ involves two photones at the same time and may be ignored in the lowest order consideration.

$$\longrightarrow \quad H' \simeq -\frac{q}{mc} A \cdot p$$

using $\mathcal{J} = q\bar{v} = q\frac{\bar{p}}{m}$ current density

$$H' = -\frac{1}{c} A \cdot \mathcal{J}$$

In addition to the electric charge, intrinsic magnetic dipole moment of the nucleus can also interact with the external electromagnetic field.

$$\mathcal{J}'(r) = \frac{c}{[c]} \nabla \times \underset{\uparrow}{M(r)} \qquad \left(\rho_m(r) = -\nabla \cdot M(r) \right)$$

magnetization density — magnetic charge density

$$\mathcal{J}'(r) = \sum_{i=1}^{A} \left\{ e g_\ell(i) \underset{\uparrow}{\frac{p(i)}{M_N}} + \frac{e\hbar}{2M_N} g_s(i) \underset{\uparrow}{\nabla \times S(i)} \right\} \delta(\bar{r} - \bar{r}(i))$$

convective part — magnetization part

Consequently, instead of changing the form of H' to include the nucleon magnetic moments, we may generalize the definition of $\mathcal{J}$.

Most general form of H' must include the possible interaction of the charge dist. with the external electrostatic fields;

$$\longrightarrow \quad H' = -\frac{1}{c}\sum_{\mu=1}^{4} A_\mu J_\mu$$

where $A_\mu = (A, iV)$, $J_\mu = (\mathcal{J}, i\rho c)$

The 4-th component is usually not important in nuclear transitions.

Remark:

The intrinsic (spin) magnetic moment is known to interact with the magnetic field as;

$$H_{int}^{spin} = \sum_i \left(-\frac{e}{mc} S_i\right) \cdot B(x_i, t)$$

$$= -\sum_i \frac{e\hbar}{2mc} \sigma_i \cdot [\nabla \times A(x,t)]_{x=x_i}$$

Remark:

Maxwell's equs. in a matter-free region;

$$\nabla \cdot E(\bar{r},t) = 4\pi \rho(\bar{r},t) \qquad (1) \qquad \text{Coulomb law}$$

$$\nabla \times H(\bar{r},t) = \frac{4\pi}{c} j(\bar{r},t) + \frac{1}{c}\frac{\partial}{\partial t} E(\bar{r},t) \qquad (2) \qquad \text{Ampere's } "$$

$$\nabla \times E(\bar{r},t) + \frac{1}{c}\frac{\partial}{\partial t} H(\bar{r},t) = 0 \qquad (3) \qquad \text{Faraday's } "$$

$$\nabla \cdot H(\bar{r},t) = 0 \qquad (4) \qquad \text{No free mag. monopole}$$

$$\nabla \cdot j(\bar{r},t) + \frac{\partial \rho(\bar{r},t)}{\partial t} = 0 \qquad \text{continuity equ.}$$

$$F = \int \rho(r,t) E(r,t) dr + \frac{1}{c}\int j(r,t) \times H(r,t) dr \qquad \text{Lorentz force}$$

Now,

If $H(\bar{r},t) = \nabla \times A(\bar{r},t)$ (5) $\longrightarrow$ Equ. (4) is satisfied $\forall$ A

$$(5) \text{ in } (3) \longrightarrow \nabla \times \left(E + \frac{1}{c}\frac{\partial A}{\partial t} \right) = 0$$

Since $\nabla \times (\nabla \varphi) = 0$

$$\rightarrow \nabla \varphi(\bar{r},t) = - E(\bar{r},t) - \frac{1}{c}\frac{\partial A(\bar{r},t)}{\partial t} \qquad (6)$$

$$(1),(6) \rightarrow \nabla^2 \varphi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot A) = -4\pi\rho \qquad (7)$$

$$(5)(6) \text{ in } (2) \longrightarrow \nabla \times (\nabla \times A) = \frac{4\pi}{c} j - \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla \varphi + \frac{1}{c}\frac{\partial A}{\partial t}\right)$$

$$\rightarrow \quad \nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c} j + \nabla\left(\nabla\cdot A + \frac{1}{c}\frac{\partial \varphi}{\partial t}\right) \qquad (8)$$

Equs. (7) and (8) with the conds. (5) and (6) are equivalent to maxwell's equs.

(A, φ) behave like the components of of a 4-vector under Lorentz tr. (φ: time-like component).

Equs (7) and (8) couple φ and A. This awkwardness can be removed by gauge tr.;

$$A'(r,t) = A(r,t) + \nabla \Lambda(r,t) \qquad (9)$$

Equ. (9) satisfies (5).

$$\rightarrow \quad \nabla \times A' = \nabla \times A = H \qquad \text{(invariant)}$$

Furthermore if at the same time we make the transformation:

$$\varphi'(r,t) = \varphi(r,t) - \frac{1}{c}\frac{\partial \Lambda(r,t)}{\partial t} \qquad (10)$$

$$(6) \rightarrow \quad -\nabla\varphi' - \frac{1}{c}\frac{\partial A'}{\partial t} = -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t} = E \qquad \text{(invariant)} \qquad (11)$$

We shall now exploit the freedom inherent in gauge tr. by imposing a special cond.:

$$\nabla\cdot A + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0$$ Lorentz cond. (12)

(Lorentz covariant statement)

We can always find potentials satisfying this cond.,

In the event ; $\nabla\cdot A + \frac{1}{c}\frac{\partial\varphi}{\partial t} = f(r,t) \neq 0$ (13)

we can always carryout the trs. (9) and (10) to a new set of A' and φ' which do satisfy (12).

$$(13) \rightarrow \nabla\cdot A + \frac{1}{c}\frac{\partial\varphi}{\partial t} = \nabla\cdot A' - \nabla^2\Lambda + \frac{1}{c}\frac{\partial\varphi'}{\partial t} + \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = f(r,t)$$

$$\rightarrow \Box\Lambda = \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -f(r,t)$$

where $\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ d'Alembertian op

We can solve this equ. for Λ, imposing the necessary boundary conds. for an appropriate A' and φ'.

The new potentials will then satisfy the Lorentz cond..

$$(7)(8) \rightarrow \begin{cases} \Box\varphi = -4\pi\rho \\ \Box\vec{A} = -\frac{4\pi}{c}\vec{J} \end{cases} \quad (14)$$

Lorentz cond. does not yet fix the gauge of our potentials.

That is if the old A' and φ' satisfy the Lorentz cond., we can still make a new tr. to A'' and φ'' which are satisfying the Lorentz cond. Such trs. are generated by the funcs., satisfying

$$\Box \Lambda = 0$$

(14) $\longrightarrow$ $\Box A_\mu = -\frac{4\pi}{c} J_\mu$ Covariant expression

(the same in every coord. sys.)

where $\Box \equiv \partial_\nu \partial_\nu$ $\quad \partial_\nu \equiv (\nabla, \frac{1}{ic}\frac{\partial}{\partial t})$

$$A_\mu \equiv (A, i\varphi) \equiv (A, iA_0) \equiv (A, iV)$$

$J_\mu \equiv (J, iJ_0)$ $\quad J_0 = c\rho$

If $J_\mu = 0$ (region outside any charge and current dists.)

we may make the gauge trs. (9) and (10), such that:

$$\varphi(r,t) = \frac{1}{c}\frac{\partial \Lambda(r,t)}{\partial t} \longrightarrow \varphi'(r,t) = 0$$

$\longrightarrow \nabla \cdot A' + \frac{1}{c}\frac{\partial \varphi'}{\partial t} = 0 \longrightarrow \nabla \cdot A' = 0$ $\begin{cases} \text{radiation gauge} \\ \text{Coulomb "} \\ \text{Transversality Cond.} \end{cases}$

Classical Lagrangian for relativistic charged particle:

$$\frac{dP}{dt} = F \quad \longrightarrow \frac{d}{dt}\frac{mV}{\sqrt{1-\frac{V^2}{c^2}}} = q\left(E + \frac{V}{c}\times H\right) \qquad (1)$$

$$\begin{matrix}(5)(6)\\(P15)\end{matrix} \longrightarrow \begin{cases} H = \nabla\times A \\ E = -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}\end{cases}$$

$$(1) \longrightarrow \frac{d}{dt}\frac{mV}{\sqrt{1-V^2/c^2}} = q\left(-\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t} + \frac{V}{c}\times(\nabla\times A)\right)$$

$$= \nabla\left(-q\varphi + q\frac{V}{c}\cdot A\right) - \frac{q}{c}\left(\frac{\partial}{\partial t} + v\cdot\nabla\right)A$$

$$= \nabla\left(-q\varphi + q\frac{V}{c}\cdot A\right) - \frac{q}{c}\frac{dA}{dt} \qquad (2)$$

where we have used;

$$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$

and $\frac{d}{dt}A(r(t),t) = \left(\frac{\partial}{\partial t} + v\cdot\nabla\right)A(r(t),t)$

$$(2) \longrightarrow \frac{d}{dt}\left(p + \frac{q}{c}A\right) + \nabla\left(q\varphi - \frac{q}{c}v\cdot A\right) = 0 \qquad \text{equ. of motion}$$

where $p = \frac{mV}{\sqrt{1-V^2/c^2}}$ Kinetic momentum

From $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

it is clear that;

$$L(r,v) = -mc^2\sqrt{1-\frac{v^2}{c^2}} + \frac{q}{c}\,v\cdot A(r,t) - q\varphi(r,t)$$

$$\begin{cases} P_i = \dfrac{\partial L}{\partial \dot{q}_i} \\ \dot{P}_i = \dfrac{\partial L}{\partial q_i} \end{cases} \longrightarrow P = \nabla_v L = \frac{mv}{\sqrt{1-v^2/c^2}} + \frac{q}{c}A(r,t)$$

$$= p + \frac{q}{c}A(r,t) \qquad \text{Canonical mom.} \qquad (3)$$

$$\longrightarrow \quad p \longrightarrow \underbrace{p}_{\text{measured quantity}} + \frac{q}{c}A(r,t)$$

at the presence of an electromag. field

Note: The quantization prescription requires;

$$P \longrightarrow \frac{\hbar}{i}\nabla \qquad \left(\text{not } p \to \frac{\hbar}{i}\nabla\right)$$

$$H = P\cdot v - L = \frac{mc^2}{\sqrt{1-v^2/c^2}} + q\varphi \qquad (4)$$

(3) in (4) $\longrightarrow$ $H = c\left\{\left[P - \frac{q}{c}A(r,t)\right]^2 + (mc)^2\right\}^{1/2} + q\varphi(r,t)$

In <u>nonrelativistic</u> limit; $\quad |p| = |P - \frac{q}{c}A| \ll mc$

$$\longrightarrow H = \underbrace{mc^2}_{\text{const.}} + \frac{1}{2m}\left(P - \frac{q}{c}A\right)^2 + q\varphi$$

Second Quantization:

Boson Ops.:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \qquad \text{One-dim H. Osc.}$$

$$[q, p] = i\hbar$$

Define:
$$b = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}}\, q + i \sqrt{\frac{1}{m\hbar\omega}}\, p \right)$$

$$b^+ = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}}\, q - i \sqrt{\frac{1}{m\hbar\omega}}\, p \right)$$

$$bb^+ = \frac{1}{2} \left(\frac{m\omega}{\hbar} q^2 + \frac{1}{m\hbar\omega} p^2 + \frac{i}{\hbar}(pq - qp) \right) = \frac{1}{2} \left(\frac{m\omega}{\hbar} q^2 + \frac{1}{m\hbar\omega} p^2 + 1 \right)$$

Similarly;
$$b^+ b = \frac{1}{2} \left(\frac{m\omega}{\hbar} q^2 + \frac{1}{m\hbar\omega} p^2 - 1 \right)$$

$$\rightarrow \quad [b, b^+] = 1$$

$$\rightarrow \quad H = \frac{1}{2} (b^+ b + bb^+) \hbar\omega = (b^+ b + \tfrac{1}{2}) \hbar\omega = (N + \tfrac{1}{2}) \hbar\omega$$

$$N|n\rangle = n|n\rangle \quad \rightarrow \quad H|n\rangle = (n + \tfrac{1}{2}) \hbar\omega |n\rangle$$

$$N b^+ |n\rangle = \left([N, b^+] + b^+ N \right) |n\rangle \quad \rightarrow \quad N b^+ |n\rangle = (n+1) b^+ |n\rangle$$

Similarly: $N b |n\rangle = (n-1)|n\rangle \longrightarrow a|n\rangle = c_n |n-1\rangle$

$$\langle n | b^+ b | n \rangle = |c_n|^2 \langle n-1 | n-1 \rangle \quad \rightarrow \quad n \langle n | n \rangle = |c_n|^2 \langle n-1 | n-1 \rangle$$

$$\rightarrow \quad n = |c_n|^2$$

The phase of $|n\rangle$ is arbitrary, because the vector is only defined by $N|n\rangle = n|n\rangle$

→ Thus we are free to choose its phase so that C_n is real and positive

$$\rightarrow \quad b|n\rangle = \sqrt{n}\,|n-1\rangle$$

Similarly; $b^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Note; The squared norm of $b|n\rangle$ is;

$$(\langle n|b^{+})(b|n\rangle) = \langle n|N|n\rangle = n$$

Since the norm must be <u>nonnegative</u> $\Rightarrow n \geq 0$

$$|n\rangle = \frac{1}{\sqrt{n!}}(b^{+})^{n}|0\rangle$$

$$b|0\rangle = 0$$

In general $N = \sum_j N_j = \sum_j b_j^{+} b_j$

$$[b_r, b_{r'}^{+}] = \delta_{rr'}$$

$$[b_r, b_{r'}] = [b_r^{+}, b_{r'}^{+}] = 0$$

Fermion Ops.;

a_r, a_r^+: Second-quantized single-particle fermion ops.

$$\{a_r, a_{r'}^+\} = \delta_{rr'}$$

$$\{a_r, a_{r'}\} = \{a_r^+, a_{r'}^+\} = 0$$

$$|1_r\rangle = a_r^+|0\rangle$$

but $\quad a_r^+ a_r^+ |0\rangle = \frac{1}{2}\{a_r^+, a_r^+\}|0\rangle = 0$

Then we can not put two particles in the same state, (consistent with the Pauli exclusion principle).

If $\quad r \neq r'$

$$a_r^+ a_{r'}^+ |0\rangle = -a_{r'}^+ a_r^+ |0\rangle$$

$\rightarrow$ The two particle state is necessarily antisym. under $\quad r \longleftrightarrow r'$

$N_r = a_r^+ a_r \qquad N_r|0\rangle = a_r^+ a_r|0\rangle = 0$

$$N_r a_r^+|0\rangle = a_r^+(1 - a_r^+ a_r)|0\rangle = a_r^+|0\rangle$$

Note that in general;

$$N_r^2 = a_r^+ a_r a_r^+ a_r = a_r^+(1 - a_r^+ a_r)a_r = a_r^+ a_r - \cancel{a_r^+ a_r^+ a_r a_r}$$
$$= N_r$$

$$N_r^2 = N_r \longrightarrow N_r(N_r - 1) = 0$$

$$N_r = 0\ ,\ 1$$

Ex. $\psi(1,2) = \frac{1}{\sqrt{2}}\left(\varphi_\alpha(1)\,\varphi_\beta(2) - \varphi_\beta(1) - \varphi_\alpha(2)\right)$

if $\alpha = \beta \longrightarrow \psi(1,2) = 0$

$$N = \sum_j N_j = \sum_j a_j^+ a_j$$

$\sum_{k,l} a_k^+ \langle k | T | l \rangle a_l$ one-body op in the second qu. notation

$\sum_{qrst} a_q^+ a_r^+ \langle qr | V | ts \rangle a_s a_t$ two-body " " "

External electromagnetic field;

The form of the external elmag. field is given by the sol. to Maxwell's equs. -

$$\Box A_\mu = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A_\mu(\bar{r},t) = 0 \qquad (1)$$

in a region outside any charge and current dist.

Expanding vector pot. $\bar{A}(\bar{r},t)$ in terms of components with definite <u>wave number</u> $\bar{k}$,

$$\bar{A}(\bar{r},t) = \sum_{\bar{k}} \bar{A}_{\bar{k}}(\bar{r})\, e^{-i\omega t} \quad , \quad \omega = |\bar{k}|c \qquad (2)$$

(2) in (1) $\rightarrow$ $(\nabla^2 + k^2)\bar{A}_{\bar{k}}(r) = 0$ (3)

with the familiar sol.; $\bar{A}_{\bar{k}}(\bar{r}) = B_k e^{ik\cdot r} + C_k e^{-ik\cdot r}$ (4)

(4) in (2) $\rightarrow$ $\bar{A}(\bar{r},t) = \frac{1}{N}\sum_{K}\sum_{\alpha}\left(b_{\bar{k}\alpha}\epsilon_\alpha e^{i(k\cdot r-\omega t)} + b^{+}_{\bar{k}\alpha}\epsilon_\alpha e^{-i(k\cdot r-\omega t)}\right)$

in Cartesian coord.

$\nabla\cdot A = 0 \rightarrow K\cdot\epsilon_\alpha = 0$

Also $\nabla\cdot A = 0 \longrightarrow \nabla^2\varphi = 0$ for source free region

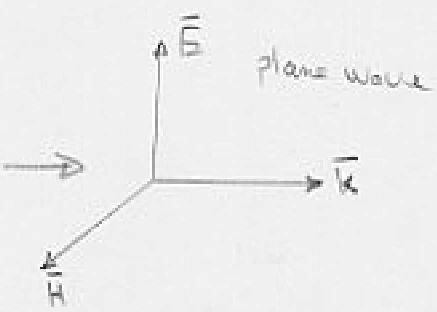
(see equ 7 p 15)

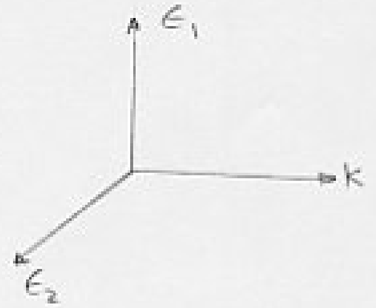
one may take $\rightarrow$ $\varphi = 0$

(Equ 6, P15) $\longrightarrow$ $\left\{ \begin{array}{l} E(\bar{r},t) = -\frac{1}{c}\frac{\partial A(\bar{r},t)}{\partial t} \\ \\ H(\bar{r},t) = \nabla \times A(\bar{r},t) \\ \\ K \cdot \epsilon_\alpha = 0 \end{array} \right.$

and since

together with

$\longrightarrow$ 

Also, 

($\nabla \cdot A = 0 \longrightarrow K \cdot \epsilon_\alpha = 0$ restricts ϵ in two dirs)

(Note: $H \sim \bar{K} \times \bar{E}$)

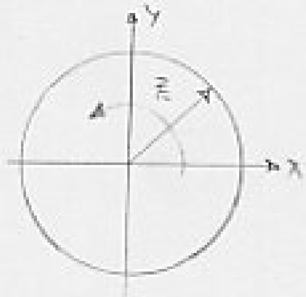
Note; $\quad \xi_{\pm} = \mp \frac{1}{\sqrt{2}}(e_1 \pm i e_2) \quad \xi_0 = e_3 \quad$ spherical basis set

If $\quad K \parallel \xi_0$, then since $K \cdot \epsilon = 0$

$$\rightarrow \epsilon = \sum_{\mu = \pm 1} (-1)^\mu \, \epsilon_{-\mu} \, \xi_\mu = -\sum_{\mu = \pm 1} \epsilon_{-\mu} \, \xi_\mu$$

For $\left\{ \begin{array}{l} \epsilon_1 = 1 \\ \epsilon_{-1} = 0 \end{array} \right.$ right circularly polarized photons

and $\left\{ \begin{array}{l} \epsilon_1 = 0 \\ \epsilon_{-1} = 1 \end{array} \right.$ left " " "



$K \parallel Z$

left circularly polarized light (positive helicity)

If $\quad \epsilon_{\pm 1} = \pm \frac{1}{\sqrt{2}}$ $\longrightarrow$ plane polarization in the x-dir

and $\quad \epsilon_{\pm} = \pm \frac{i}{\sqrt{2}}$ $\longrightarrow$ " " " " y-dir

A photon carries one unit of angular momentum.

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$S^2 = S(S+1)I = 2I \quad \rightarrow S = 1$$

Now, $b_{\bar{k}\alpha}$ and $b^+_{\bar{k}\alpha}$ related to B_k and C_k must be determined using the boundary conds. (restriction on $\bar{k}$).

$b^+_{\bar{k},\alpha}$: Creation op. of photons

$b^+_{\bar{k},\alpha}$: Annihilation op. " "

$\hbar\omega = \hbar ck$: energy of a field quanta for a given $\bar{k}$

The lowest order term, in H' given by $H' = -\frac{1}{c}\sum_{\mu=1}^{4} A_\mu J_\mu$ involves a linear combination of $b^+_{\bar{k}\alpha}$ and $b_{\bar{k},\alpha}$ (single photon creation or annihilation).

J_μ : nuclear field

A_μ : el-mag "

Ex. ; Absorbtion (of a photon by a nucleus);

Using the Time-dep. perturbation theory to first order;

$$\text{Transition matrix} = \langle B | H_I | A \rangle$$

$$|A\rangle = |a\rangle |n_{\bar{k},\alpha}\rangle$$

$$|B\rangle = |b\rangle |n_{\bar{k},\alpha} - 1\rangle$$

nucl. states ↑ photon states ↑

$$A(x,t) = \frac{1}{\sqrt{V}} \sum_{k,\alpha} c \sqrt{\frac{\hbar}{2\omega}} \left[a_{k\alpha}(t) \epsilon_\alpha e^{i\bar{k}\cdot\bar{x}} + a^+(t)\, \epsilon_\alpha e^{-i\bar{k}\cdot\bar{x}} \right]$$

$$(\langle b | n_{k,\alpha} - 1 |)\, H_I\, (|n_{k,\alpha}\rangle |a\rangle) = \langle b; n_{k,\alpha} - 1 | H_I | a; n_{k,\alpha}\rangle$$

$$= -\frac{e}{mc} \langle b; n_{k\alpha} - 1 | \sum_i A(x_i,t)\cdot \bar{P}_i | a; n_{k,\alpha}\rangle +$$

$$+ \frac{e^2}{mc^2} \langle b, n_{k\alpha} - 1 | \sum_i A(x_i,t)\cdot A(x_i,t) | a, n_{k,\alpha}\rangle$$

For the interaction;

$$H_I = -\frac{e}{mc} \sum_i A(x_i,t)\cdot \bar{P}_i + \frac{e^2}{mc^2} \sum_i A(x_i,t)\cdot A(x_i,t)$$

$\sum_{k,\alpha}$ has been dropped (since we are involved with a single photon).

Note: $a(t) = a(0) e^{-i\omega t}$

$$\langle b; n_{k,\alpha}-1 | H_I | a; n_{k,\alpha} \rangle =$$

$$= -\frac{e}{mc} \langle b; n_{k,\alpha}-1 | \sum_i c\sqrt{\frac{\hbar}{2\omega}}\sqrt{\frac{1}{V}} \Big(\underbrace{a_{k\alpha}(0)\, e^{ik\cdot x_i - i\omega t}\, \bar{p}_i \cdot \epsilon_\alpha}_{①} +$$

$$a^{+}_{k\alpha}(0)\, e^{-ik\cdot x_i + i\omega t}\, \bar{p}_i \cdot \epsilon_\alpha \,|a; n_{k,\alpha}\rangle +$$

$$+ \frac{e^2}{mc^2} \langle b; n_{k,\alpha}-1 | \sum_i c^2 \frac{\hbar}{2\omega V} \Big[a_{k,\alpha}(0)\, a_{k,\alpha}(0)\, e^{2i(k\cdot x_i - \omega t)}\, \epsilon_\alpha \cdot \epsilon_\alpha +$$

$$+ a^{+}_{k,\alpha}(0)\, a^{+}_{k,\alpha}(0)\, e^{-2i(k\cdot x_i - \omega t)}\, \epsilon_\alpha \cdot \epsilon_\alpha +$$

$$+ \underbrace{a_{k\alpha}(0)\, a^{+}_{k,\alpha}(0)\, \epsilon_\alpha \cdot \epsilon_\alpha}_{②} + \underbrace{a^{+}_{k,\alpha}(0)\, a_{k,\alpha}(0)\, \epsilon_\alpha \cdot \epsilon_\alpha}_{③} \Big] |a, n_{k,\alpha}\rangle$$

Since; $a^{+}_{k,\alpha}(0) | n_{k,\alpha}\rangle = \sqrt{n_{k,\alpha}+1}\, | n_{k,\alpha}\rangle$

$$a_{k,\alpha}(0) | n_{k,\alpha}\rangle = \sqrt{n_{k,\alpha}}\, | n_{k,\alpha}-1\rangle$$

$\longrightarrow$ The non-vanishing terms are; (1), (2) and (3)

Also $\quad [a_{k,\alpha}, a^{+}_{k,\alpha}] = 1 \quad \rightarrow a_{k,\alpha} a^{+}_{k,\alpha} = 1 + a^{+}_{k,\alpha} a_{k\alpha}$

$$\langle b; n_{k,\alpha}-1 | H_I | a; n_{k,\alpha} \rangle =$$

$$= -\frac{e}{mc} \langle b; n_{k,\alpha}-1 | \sum_i c\sqrt{\frac{\hbar}{2\omega V}}\, a_{k,\alpha}(0)\, e^{i(k\cdot x_i - i\omega t}\, \bar{p}_i \cdot \bar{\epsilon}_\alpha |a, n_{k,\alpha}\rangle$$

$$+ \frac{e^2}{mc^2} \langle b; n_{k\alpha}-1 | \sum_i c^2 \frac{\hbar}{2\omega V} (2N_{k,\alpha}+1)\, \epsilon_\alpha \cdot \epsilon_\alpha |a, n_{k,\alpha}\rangle$$

$$= -\frac{e}{m}\sqrt{\frac{\hbar}{2\omega V}}\ \sqrt{n_{k,\alpha}}\ \sum_i \langle b|\, e^{ik\cdot x_i}\ p_i\cdot\epsilon_\alpha\,|a\rangle\, e^{-i\omega t}$$

$$+\ \frac{e^2}{m}\ \frac{\hbar}{2\omega V}\ \sum_i (2n_{k,\alpha}+1)\ \langle a|b\rangle\ \cancel{\langle n_{k,\alpha}-1|\, n_{k,\alpha}\rangle}^{\,0}$$

$$= -\frac{e}{m}\sqrt{\frac{\hbar n_{k,\alpha}}{2\omega V}}\ \sum_i \langle a|\, e^{ik\cdot x_i}\ \bar{p}_i\cdot\epsilon_\alpha\,|a\rangle\, e^{-i\omega t}$$

$$w \sim |\sqrt{n_{k,\alpha}}|^2 = n_{k,\alpha}$$

Multipole expansion of the electromagnetic field;

For applications to problems with rotational symmetry, for example, as in the case of nuclear transitions, it is convenient to expand $A(r,t)$ as;

$$\bar{A}(\bar{r},t) = \sum_{\lambda\mu} \bar{A}_{\lambda\mu}(\bar{r},t)$$

where the vector funcs. of spherical tensor ranks (λ,μ) satisfy the relations;

$$J^2 A_{\lambda\mu}(\bar{r},t) = \lambda(\lambda+1) A_{\lambda\mu}(\bar{r},t), \quad J_z A_{\lambda\mu}(\bar{r},t) = \mu A_{\lambda\mu}(\bar{r},t)$$

Since the time-dep. is simple (equ. 2 P25), we shall drop it from now on in order to simplify the notation.

Despite $A_{\lambda\mu}(\bar{r},t)$ are eigenvalues of angular momentum ops., they are different from spherical harmonics ($Y_{\lambda\mu}$).

$\bar{A}_{\lambda\mu}$; Vector funcs. having spherical tensor ranks (λ,μ).

They may be expressed in terms of vector spherical harmonics which are vector funcs. constructed from (scalar) spherical harmonics $Y_{\ell m}(\theta,\varphi)$.

Instead two polarization dirs. (ϵ_1, ϵ_2) allowed for $A(r,t)$ we have now two different types of multipole fields satisfying (equ. 3, p 25);

$$\bar{A}_{\lambda\mu}(E\lambda,\bar{r}) = -\frac{i}{k}\,\bar{\nabla}\times(\bar{r}\times\bar{\nabla})\left(j_\lambda(kr)\,Y_{\lambda\mu}(\theta,\varphi)\right)$$

Electric multipole trs.

$$\bar{A}_{\lambda\mu}(M\lambda,\bar{r}) = (\bar{r}\times\bar{\nabla})\left(j_\lambda(kr)\,Y_{\lambda\mu}(\theta,\varphi)\right) \qquad (1)$$

Magnetic " "

$$\text{General sol.} = \sum_k \sum_{\lambda\mu}\left(a\,A_{\lambda\mu}(E\lambda,\bar{r}) + b\,A_{\lambda\mu}(M\lambda,\bar{r})\right)e^{-i\omega t}$$

Electromagnetic multipole transition ops.:

$$H' = -\frac{1}{c}\sum_{\mu=1}^{4} A_\mu J_\mu \rightarrow$$

$$O_{\lambda\mu}(E\lambda) = -\frac{i(2\lambda+1)!!}{c\,k^{\lambda+1}(\lambda+1)}\,J(\bar{r})\cdot\nabla\times(r\times\nabla)\left(j_\lambda(kr)\,Y_{\lambda\mu}(\theta,\varphi)\right)$$

$$O_{\lambda\mu}(M\lambda) = -\frac{(2\lambda+1)!!}{c\,k^{\lambda}(\lambda+1)}\,J(\bar{r})\cdot(r\times\nabla)\left(j_\lambda(kr)\,Y_{\lambda\mu}(\theta,\varphi)\right) \qquad (2)$$

The normalization used are such that they reduce to static moment ops. in the limit $k\rightarrow 0$

Since H' and hence $O_{\lambda\mu}{}^{(E\lambda)}$ and $O_{\lambda\mu}{}^{(M\lambda)}$ are <u>scalar</u> ops. $\longrightarrow$ only the $(\lambda,-\mu)$ multipol part of $J(r)$ can make a nonvanishing contribution in the tr. -

Expanding the spherical Bessel func.:

$$j_\lambda(kr) \simeq \frac{(kr)^\lambda}{(2\lambda+1)!!}\left(1 - \frac{1}{2}\frac{(kr)^2}{2\lambda+3} + \cdots\right)$$

Since $E_\gamma < 10$ MeV tipically in nucl. trs.

Corresponding to $\longrightarrow$ $k = E_\gamma/\hbar c \simeq \frac{1}{20}\ fm^{-1}$ or less

and since $R < 10$ fm ($R = r_0 A^{1/3} \sim 7$ fm for ^{208}Pb)

(i.e. $R_{nucleus} << \lambda_{radiation}$)

$\longrightarrow$ $kr << 1$

(i.e. : despite $\int_{r=0}^{\infty}$, but $|\psi|^2 \xrightarrow[\text{as } r > 10 fm]{} 0$)

$\longrightarrow$ this series is a fast <u>convergent</u> one.

$\longrightarrow$ It can be approximated by its <u>first</u> term (<u>long wave limit approx.</u>)

$$\lambda = \frac{2\pi}{k} = \frac{2\pi\hbar c}{E_\gamma} \sim 10^2\ fm >> R \quad (\text{for tipical } \gamma\text{-rays})$$

$\longrightarrow$ These γ-rays can <u>not</u> be sensetive to the <u>details</u> of radial nucl. wave funcs.

$\longrightarrow j_\lambda(kr) \sim (kr)^\lambda$ ($\sim$ leading order term)

Including the <u>density</u> of final states ;

(Golden rule) (equ. P32) $\longrightarrow$

$$\begin{cases} K_n = \dfrac{n\pi}{R_{\text{nucleus}}} \rightarrow dn = \dfrac{R}{\pi} dk \\ \rho = \dfrac{dn}{dE} = \dfrac{R}{\pi \hbar c} \\ \hbar\omega = \hbar k c \text{ the energy of a photon confined to a sphere of radius } R. \end{cases}$$

$$W(\lambda; J_i \varsigma \to J_f \xi) = \frac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2} \frac{k^{2\lambda+1}}{\hbar} B(\lambda; J_i \varsigma \to J_f \xi) \qquad (1)$$

for each multipole λ

when $B(\lambda; J_i \varsigma \to J_f \xi) = \sum_{\mu M_f} |\langle J_f M_f \xi | O_{\lambda\mu} | J_i \varsigma \rangle|^2$

$= \dfrac{1}{2J_i+1} |\langle J_f \xi \| O_\lambda \| J_i \varsigma \rangle|^2$ ($\Delta(J_f, \lambda, J_i)$ is implicit in the reduced matrix element)

The reduced tr. probabilities are dimensioned quantities:

$B(E\lambda)$: in units of $e^2 \, fm^{2\lambda}$

$B(M\lambda)$: " " " $\mu_N^2 \, fm^{2\lambda-2}$

W : $\dfrac{\text{number}}{\text{time}}$

Since $E = pc = \hbar kc$

$$\rightarrow W = \begin{cases} \alpha\hbar c \dfrac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2} \dfrac{1}{\hbar} \left(\dfrac{1}{\hbar c}\right)^{2\lambda+1} E_\gamma^{2\lambda+1} B(\lambda \text{ in } e^2 fm^{2\lambda}) \\ \alpha\hbar c \left(\dfrac{\hbar c}{2M_p c^2}\right)^2 \dfrac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2} \dfrac{1}{\hbar} \left(\dfrac{1}{\hbar c}\right)^{2\lambda+1} E_\gamma^{2\lambda+1} B(\lambda \text{ in } \mu_N^2 fm^{2\lambda-2}) \end{cases}$$

where $e^2 = \alpha\hbar c$ in cgs

$W(E1) = 1.59\times10^{15}\, E_\gamma^3\, B(E1)$	$W(M1) = 1.76\times10^{13}\, E_\gamma^3\, B(M1)$
$W(E2) = 1.23\times10^{9}\, E_\gamma^5\, B(E2)$	$W(M2) = 1.35\times10^{7}\, E_\gamma^5\, B(M2)$
$W(E3) = 5.71\times10^{2}\, E_\gamma^7\, B(E3)$	$W(M3) = 6.31\times10^{0}\, E_\gamma^7\, B(M3)$
$W(E4) = 1.70\times10^{-4}\, E_\gamma^9\, B(E4)$	$W(M4) = 1.88\times10^{-6}\, E_\gamma^9\, B(M4)$

E_γ: in Mev, $B(E\lambda)$: in $e^2 fm^{2\lambda}$, $B(M\lambda)$: in $\mu_N^2\, fm^{(2\lambda-2)}$

Assuming: { The electric charge of the nucleus: point charge of protons
The magnetization currents: Due to intrinsic mag. dipole moments of nucleons + the orbital motion of the protons

$$\rightarrow O_{\lambda\mu}(E\lambda) = \sum_{i=1}^{A} e(i)\, r_i^\lambda\, Y_{\lambda\mu}(\theta_i, \varphi_i) \qquad (1)$$

and; $$\mathcal{O}_{\lambda\mu}(M\lambda) = \sum_{i=1}^{A}\left\{g_s(i)\,\bar{s}_i + g_\ell(i)\,\frac{2\bar{\ell}_i}{\lambda+1}\right\}\cdot\nabla_i\left(r_i^{\lambda}\,Y_{\lambda\mu}(\theta_i,\varphi_i)\right)$$

$$= \sqrt{\lambda(2\lambda+1)}\sum_{i=1}^{A} r_i^{(\lambda-1)}\left\{\left(g_s(i) - \frac{2g_\ell(i)}{\lambda+1}\right)\left(Y_{\lambda-1}(\theta_i,\varphi_i)\times\bar{s}_i\right)_{\lambda\mu} + \frac{2g_\ell(i)}{\lambda+1}\left(Y_{\lambda-1}(\theta_i,\varphi_i)\times J_i\right)_{\lambda\mu}\right\}$$

where $\bar{J}_i = \bar{\ell}_i + \bar{s}_i$ (2)

$$e(i) = \begin{cases} 1e \\ 0 \end{cases} \quad g_\ell(i) = \begin{cases} 1\mu_N \\ 0 \end{cases} \quad g_s(i) = \begin{cases} 5.586\,\mu_N & \text{for } P \\ -3.826\,\mu_N & \text{" } n \end{cases}$$

Remark: $$(T_{j_1}\times U_{j_2})_{J_3 m_3} = \sum_{m_1,m_2}(J_1 m_1 J_2 m_2 / J_3 m_3)\, T_{J_1 m_1}\, U_{J_2 m_2}$$

Ref. (for derivations): Blatt and Weisskopf (Theoretical Nuclear Physics, Wiley, New York, 1952).

Dimensional Check:

$$O_{\lambda\mu}(E\lambda) \sim e r^{\lambda} \longrightarrow B(E\lambda) \sim (\text{charge})^2 (\text{length})^{2\lambda}$$

$$O_{\lambda\mu}(M\lambda) \sim \mu_N r^{\lambda-1} \longrightarrow B(M\lambda) \sim (\text{nucl. magneton})^2 (\text{length})^{2\lambda-2}$$

El. multipole λ $B(E\lambda)$: $e^2 \, fm^{2\lambda}$

Mag. " " $B(M\lambda)$: $\mu_N^2 \, fm^{2\lambda-2}$

$$(\text{Eqn. 1 P31}) \rightarrow W(\lambda) \sim \frac{K^{2\lambda+1}}{\hbar} B(\lambda)$$

Since $K: \frac{1}{\text{length}}$, $\hbar$: energy . time

$$W(E\lambda): \frac{(fm^{-1})^{2\lambda+1}}{(Mev.\,S)} e^2 \, fm^{2\lambda}$$

and since $e^2 = \alpha \hbar c \rightarrow e^2$: Mev.-fm

α: dim.-less, $\hbar c$: Mev.-fm

$$W(E\lambda): \frac{(fm^{-1})^{2\lambda+1}}{(Mev.-S)} (Mev.-fm)(fm)^{2\lambda} = S^{-1} \qquad (1)$$

Similarly;

$$\mu_N = \frac{e\hbar}{2M_p c} \rightarrow \mu_N^2 = e^2 \frac{\hbar^2 c^2}{4(M_p c^2)^2}$$

$$\mu_N^2: e^2 \frac{(Mev.S)^2 (fm/s)^2}{(Mev)^2} = e^2 (fm)^2 \rightarrow W(M\lambda): S^{-1}$$

Selection Rules:

(Equ. 1 P34) $\rightarrow$ $R(E) = \dfrac{W(E,\lambda+1)}{W(E,\lambda)} \sim (kr)^2$ (r is included to make it dimensionless)

Similarly $R(M) = \dfrac{W(M,\lambda+1)}{W(M,\lambda)} \sim (kr)^2$

$\longrightarrow R = \dfrac{W(\lambda+1)}{W(\lambda)} \sim (kr)^2$ Ratio between two multipole trs. λ and $(\lambda+1)$

Since $R_0 \sim 1$ fm nucl. size

for $E_\gamma = pc = \hbar kc \sim 1$ MeV $\longrightarrow R \sim 3\times10^{-5}$

Because of this large reduction in W with increasing multipolarity λ

the tr. $J_i^\pi \longrightarrow J_f^\pi$ is usually dominated by the

lowest order multipoles (allowed by $\begin{cases}\text{i- ang. mom. selection rules} \\ \text{ii- parity " "}\end{cases}$)

$$\bar{J}_f = \bar{\lambda} + \bar{J}_i \quad \rightarrow \quad |J_f - J_i| \leq \lambda \leq J_f + J_i$$

Also, for allowed values of λ;

$$W(E,\lambda) \gg W(E,\lambda+1) \quad ; \quad W(M,\lambda) \gg W(M,\lambda+1)$$

Parity selection rules;

(Equ. 1, P35) $\rightarrow$ $O_{\lambda\mu}(E\lambda) \sim Y_{\lambda\mu}(\theta,\varphi)$

$$O_{\lambda\mu}(E\lambda) \xrightarrow{\Pi} (-1)^{\lambda}\, O_{\lambda\mu}(E\lambda)$$

$$(\text{Equ 2, P36}) \longrightarrow O_{\lambda\mu}(M\lambda) \sim \nabla(r^{\lambda} Y_{\lambda\mu}(\theta,\varphi))$$

$$O_{\lambda\mu}(M\lambda) \xrightarrow{\Pi} (-1)^{\lambda+1}\, O_{\lambda\mu}(M\lambda)$$

$$\rightarrow \begin{cases} \Pi_i \Pi_f = (-)^{\lambda} & \text{for } E\lambda \\ \Pi_i \Pi_f = (-)^{\lambda+1} & \text{" } M\lambda \end{cases}$$

parity selection rules
(for non-vanishing matrix elements)

<u>Ex.</u> J^{Π} : $2^+ \longrightarrow 0^+$ only E2 tr. is allowed

$2^- \longrightarrow 0^+$ " M2 " "

However, the selection rule is <u>not an exact</u> one.

In general Mag. tr. $\underset{\text{weaker}}{<}$ Elect. tr.

Since <u>electric</u> and magnetic trs. of the <u>same</u> multipolarity (λ) <u>can not</u> occur between <u>the same pair of the</u> nucl. states a direct comparison <u>is not</u> possible (as a result of parity selection rule).

If $E\lambda$ and $M(\lambda+1)$ trs. are allowed by angular momentum and parity selection rules, $\longrightarrow$ $E\lambda$ mode usually dominates the tr. by a large factor.

On the other hand, if both $M\lambda$ and $E(\lambda+1)$ trs. are allowed $\longrightarrow$ $E(\lambda+1)$ may be competitive with $M\lambda$.

Reason:

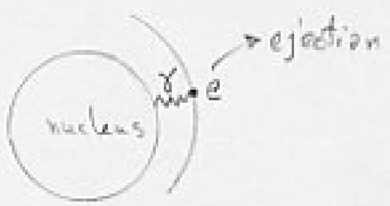
The gradiant (∇) term in mag. tr. op. reduces the size of the matrix.

$$\delta^2 = \frac{W(E(\lambda+1);\, J_i \xi \to J_f \xi)}{W(M\lambda;\, J_i \xi \to J_f \xi)}$$

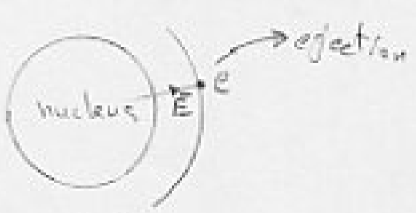
mixing ratio when $E(\lambda+1)$ and $M(\lambda)$ are allowed.

Internal conversion and internal pair production:

El.-Mag. decay $\begin{cases} \gamma\text{-ray emission} \\ \text{Internal conversion} \begin{cases} \text{i- Internal photoelectric conversion} \\ \text{ii- Direct internal conversion} \end{cases} \\ \text{pair production} \end{cases}$



(i)



(ii)

In γ emission: $E_\gamma = E_i - E_f$

Internal conversion: (i) $K_e = E_\gamma = E_i - E_f - B_e$

(ii) $K_e = E_i - E_f - B_e$

→ The electrons have discrete energies (distinguishable from continuous spectrum of electrons emitted in β-decay)

Note: Both types of decay (β-decay and internal conversion) take place in medium and heavy nuclei

Justification:

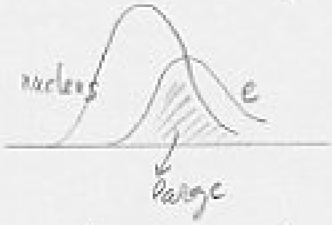
When a nucleus de-excites, say $\begin{cases} \text{i- by a nucleon jumping from higher to lower level} \\ \text{ii- by a change of total J of the nucleus as a whole} \end{cases}$

→ a sudden disturbance is sent to the surrounding el-mag. field.

→ The electrons in the innermost orbits (such as K, L) are affected by this change in field.

Note: Auger effect (Auger electron) is a similar effect to internal conversion but in the atomic level.

Internal conversion is important in heavy nuclei for two reasons:

i) 

ii) Coulom field is large (large Z)

→ importance of the internal conversion $\sim Z^3$

Internal pair production:

Through el-mag processes, → An $e^- e^+$ pair may be emitted (in the place of γ-decay)

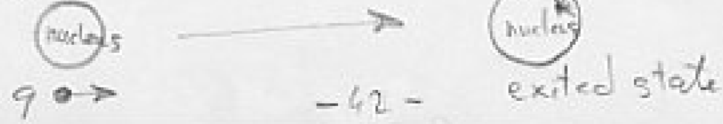
Cond: $E_i - E_f \gtrsim 2 m_e c^2 \approx 1.02$ MeV

However it is not an efficient process: (several orders of magnitude retarded compared with allowed γ-emission)

Therefore → it becomes important only when γ-ray emission is forbidden. (like J^π: $0^+ \to 0^+$)

Note: The inverse of γ-ray emission is Coulomb excitation:

The nucleus is excited to higher states as a result of changes in the surrounding el.-mag field.

Ex. – 

We will see later (chap. 7-1);

γ-ray tr. matrix elements $\underset{\text{similar}}{\sim}$ Coulomb excitation matrix elements

5-3) Single Particle Values;

Assumption; to simplify the cal. we adopt an extreme <u>indep-particle</u> picture, and consider nucl. trs. to be taking place by a nucleon moving from <u>one single-particle</u> orbit to another <u>without</u> affecting the <u>rest</u> of the nucleons.

$E\lambda$-transitions;

In the case of $E\lambda$-tr, this means that a <u>proton</u> moves from an initial single-particle state $|j_i, m_i\rangle$ to a final one $|j_f, m_f\rangle$.

(Equ-P35) $\longrightarrow$ $\langle J_f M_f, \xi | \sum_{i=1}^{A} e(i)\, r_i^{\lambda} Y_{\lambda\mu}(\theta_i, \varphi_i) | J_i, M_i, \xi \rangle$

$$= \langle j_f m_f | e\, r^{\lambda} Y(\theta,\varphi) | j_i, m_i \rangle$$

In cases where the wave func. of each nucleon has a definite $\underline{\ell}$

$$|j m\rangle = R_{n\ell}(r)\, \left(Y_{\ell}(\theta,\varphi) \times \chi_{\frac{1}{2}} \right)_{jm}$$

$$= R_{n\ell}(r) \sum (\tfrac{1}{2} m_s, \ell, m_\ell | j m)\, Y_{\ell m}(\theta,\varphi)\, \chi_{\frac{1}{2}}^{m_s}$$

$$\langle J_f m_f | e r^\lambda Y_{\lambda\mu}(\theta,\varphi) | J_i m_i \rangle = \int_0^\infty R^*_{n_f \ell_f}(r)\, r^\lambda R_{n_i \ell_i}(r)\, r^2 dr$$

$$\cdot \langle (Y_{\ell_f}(\theta,\varphi) \times \chi_{1/2})_{j_f m_f} | Y_{\lambda\mu}(\theta,\varphi) | (Y_{\ell_i}(\theta,\varphi) \times \chi_{1/2})_{j_i m_i} \rangle$$

$$\langle r^\lambda \rangle = \int R^*_{n_f \ell_f}(r)\, r^\lambda R_{n_i \ell_i}(r)\, r^2 dr$$

Further simplifying assumption; Nucleus is an <u>sphere</u> of uniform density with the radius given by $R = r_0 A^{1/3}$

$$\rightarrow \langle r^\lambda \rangle = \frac{3}{\lambda+3} r_0^\lambda A^{\lambda/3} \qquad r_0 \approx 1.2\ fm \qquad (1)$$

$$\rightarrow B(E\lambda) = \sum_{\mu m_f} |\langle J_f M_f \xi | O_{\lambda\mu}(E\lambda) | J_i M_i \xi \rangle|^2$$

$$\approx e^2 \langle r^\lambda \rangle^2 \sum_{m_f \mu} \langle (Y_{\ell_f}(\theta,\varphi) \times \chi_{1/2})_{j_f m_f} | Y_{\lambda\mu}(\theta,\varphi) | (Y_{\ell_i}(\theta,\varphi) \times \chi_{1/2})_{j_i m_i} \rangle^2 \qquad (2)$$

Since the total <u>solid angle</u> about a point is 4π <u>steradian</u>

$\rightarrow$ an average of any angular-dep. must be $\approx \frac{1}{4\pi}$

$$\rightarrow B(E\lambda) \approx e^2 \langle r^\lambda \rangle^2 \frac{1}{4\pi}$$

$$\rightarrow B_W(E\lambda) = \frac{1}{4\pi} \left(\frac{3}{\lambda+3}\right)^2 (1.2)^{2\lambda} A^{2\lambda/3} \quad e^2 fm^{2\lambda} \qquad (3)$$

Weisskopf single-particle estimate

$(e = 1)$

$M\lambda$-transitions:

In a similar manner; using (Equ.2. P36);

$$B(M\lambda) = \sum_{\mu M_f} |\langle J_f M_f, \xi | O_{\lambda\mu}(M\lambda) | J_i, M_i, \xi \rangle|^2$$

$$\approx \lambda(2\lambda+1)\langle r^{\lambda-1}\rangle^2 \sum_{m_f\ \mu} \Big\{ \left(g_s - \frac{2g_\ell}{\lambda+1}\right) \cdot$$

$$\cdot \langle (Y_{\ell_f}(\theta,\varphi) \times \chi_{1/2})_{j_f m_f} | (Y_{(\lambda-1)}(\theta,\varphi) \times S)_{\lambda\mu} | (Y_{\ell_i}(\theta,\varphi) \times \chi_{1/2})_{j_i m_i} \rangle$$

$$+ \frac{2g_\ell}{\lambda+1} \langle (Y_{\ell_f}(\theta,\varphi) \times \chi_{1/2})_{j_f m_f} | (Y_{(\lambda-1)}(\theta,\varphi) \times j)_{\lambda\mu} | (Y_{\ell_i}(\theta,\varphi) \times \chi_{1/2})_{j_i m_i} \rangle \Big\}^2 \quad (1)$$

$j = l + s$

Again assuming the nucleus to be a const. density sphere of radius R

$$\langle r^{\lambda-1}\rangle = \frac{3}{\lambda+2} r_0^{\lambda-1} A^{(\lambda-1)/3} \approx \frac{3}{\underset{\uparrow}{\underline{\lambda+3}}} r_0^{\lambda-1} A^{(\lambda-1)/3} \quad (2)$$

to conform with that for $E\lambda$-trs.

For the purpose of an estimate:
we evaluate one of the two terms inside the curly brackets in (1) and multiply the result by <u>two</u>.

The numerical value for the coefs. is:

$$\lambda(2\lambda+1)\left(g_s - \frac{2g_\ell}{\lambda+1}\right)^2 \simeq 10$$

The average of square of the angular part $\simeq \frac{1}{4\pi}$ (as before)

$$B_W(M\lambda) \simeq \frac{10}{\pi}\left(\frac{3}{\lambda+3}\right)^2 (1.2)^{2\lambda-2} \; A^{(2\lambda-2)/3} \qquad \mu_N^2 \, fm^{2\lambda-2} \qquad (3)$$

(Equ.1 P31)
(Equ 3 P41)
(Equ 3 P43)

$$\longrightarrow \begin{cases} W_W(E\lambda) = \alpha\hbar c \dfrac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2} \dfrac{K^{2\lambda+1}}{\hbar} \dfrac{1}{4\pi}\left(\dfrac{3}{\lambda+3}\right)^2 (1.2)^{2\lambda} A^{2\lambda/3} \\[2ex] W_\gamma(M\lambda) = \alpha\hbar c \left(\dfrac{\hbar}{2M_p c}\right)^2 \dfrac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2} \dfrac{K^{2\lambda+1}}{\hbar} \dfrac{10}{\pi}\left(\dfrac{3}{\lambda+3}\right)^2 (1.2)^{2\lambda-2} \cdot A^{(2\lambda-2)/3} \end{cases}$$

Weisskopf single particle estimates for $E\lambda$ and $M\lambda$ - tr probabilities and widths

Multipole λ	$E\lambda$ $W(s^{-1})$	Γ (MeV)		$M\lambda$ $W(s^{-1})$	Γ (MeV)	
1	1.02×10^{14}	6.75×10^{-8}	$A^{2/3} E_\gamma^3$	3.15×10^{13}	2.07×10^{-8}	$A^0 E_\gamma^3$
2	7.28×10^{7}	4.79×10^{-14}	$A^{4/3} E_\gamma^5$	2.24×10^{7}	1.67×10^{-14}	$A^{2/3} E_\gamma^5$
3	3.39×10	2.23×10^{-20}	$A^2 E_\gamma^7$	1.04×10	6.85×10^{-21}	$A^{4/3} E_\gamma^7$
4	1.07×10^{-5}	7.02×10^{-27}	$A^{8/3} E_\gamma^9$	3.27×10^{-6}	2.16×10^{-27}	$A^2 E_\gamma^9$
5	2.40×10^{-12}	1.58×10^{-33}	$A^{10/3} E_\gamma^{11}$	7.36×10^{-13}	4.84×10^{-34}	$A^{8/3} E_\gamma^{11}$

(E_γ in MeV). The E_γ and A-dep. factors are common to both W and Γ.

5-4 Weak Interaction and B-decay.

Nuclear B-decay is a facet of the weak interaction.

weak int. is observed in

$\begin{cases} \text{i) In trs. between nucl. states.} \\ \text{ii) Other phenomena involving both hadrons and leptons.} \end{cases}$

In most cases, these weak processes can not be observed since they are slower by several orders of magnitude compared with competing reactions induced by $\begin{cases} \text{el.-mag int.} \\ \text{strong "} \end{cases}$

as a result $\Rightarrow$ Weak ints can be studies in case where the faster processes are $\begin{cases} \text{either forbidden} \\ \text{or hindered} \end{cases}$ by selection rules.

The basic weak reaction;

$$\begin{aligned} n &\longrightarrow p + e^- + \bar{\nu}_e \\ p_{\text{bound}} &\longrightarrow n + e^+ + \nu_e \end{aligned} \qquad (1)$$

Examples of general class of decay;

$$\pi^+ \longrightarrow \mu^+ + \nu_\mu$$
$$\longrightarrow e^+ + \nu_e$$

$$\pi^- \longrightarrow \mu^- + \bar{\nu}_\mu \qquad (2)$$
$$\longrightarrow e^- + \bar{\nu}_e$$

$$\Sigma^- \longrightarrow n + \pi^-$$
$$K^+ \longrightarrow \pi^+ + \pi^0 \qquad (3)$$
$$\longrightarrow \pi^+ + \pi^- + \pi^+$$

Equ. (1, 2) $\xrightarrow{\text{are referred to}}$ semileptonic processes (since both leptons and hadrons are involved)

Equ (3) $\xrightarrow{\text{" "}}$ non-leptonic processes

There are also purely leptonic processes, such as;

$$\mu^+ \longrightarrow e^+ + \nu_e + \bar{\nu}_\mu \qquad (4)$$

Universal weak interaction;

Weak int. processes are often said to be <u>universal</u>, since, the strength of the basic int. is the <u>same</u> for all <u>three</u> types of reactions.

$\rightarrow$ $G_F = 1.43584(3) \times 10^{-62}\ J.m^3 = 1.16637(2) \times 10^{-11} (\hbar c)^3\ Mev^{-2}$

coupling const. (the same for all weak int.)

The field quanta for weak ints. are the vector bosons

$W^\pm$ and Z°

$$m_W c^2 = 80.9 \pm 1.4\ Gev \qquad m_Z c^2 = 91.9 \pm 1.8\ Gev$$

large mass $\longrightarrow$ short range $\left(r_0 = \frac{\hbar}{mc} \sim 10^{-3}\ fm\right)$

$\rightarrow$ range of weak int. < long-range part of nucl. force (by 3-orders of mag.)

For this reason weak int. is called Zero-range or contact ints. .

Most of the weak-decay processes are mediated by the charge bosons, (Equ. 1,2,3 P44,45);

(Equ 1. P44)

The neutral boson Z^0 is the source of neutral weak current.

As an example; $\nu + e^- \longrightarrow \nu + e^-$ neutrino-electron scatt.

On a more fundamental level;

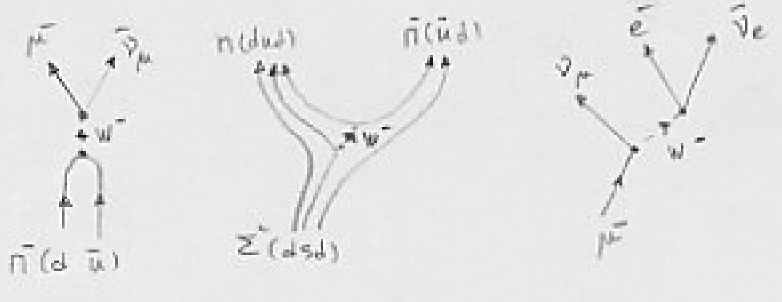
$$d \xrightarrow[\text{current}]{\text{charged weak}} u + e^- + \bar{\nu}_e \qquad (1)$$

$$\underset{n}{(udd)} \xrightarrow{\beta^- \text{-decay}} \underset{p}{(uud)} + e^- + \bar{\nu}_e \qquad (2)$$

Note: In weak int. flavor of quarks is not conserved.

similarly; $$u \longrightarrow d + e^+ + \nu_e \qquad (3)$$

$$(uud) \longrightarrow (udd) + e^+ + \nu_e \qquad (4)$$



(Equ. 2,3,4, P45)

When a quark decays, it does not necessarily have to result in a quark of definite flavor.

Among 4-quarks, u, d, s and c,

$$u \longrightarrow d' = d \cos\theta_c + s \sin\theta_c$$
$$c \longrightarrow s' = -d \sin\theta_c + s \cos\theta_c \qquad (1)$$

(flavor mixing)
Symmetry mixing

θ_c : Cabibbo angle

The observed weak trs. are however, between quarks of definite flavor.

Equ (1) $\xrightarrow{\text{implies}}$ Observed β-decay strength in reactions is not the fundamental weak int coupling const. G_F itself but a value modified by the mixing angle θ_c.

It is customary to express the tr (1) in the form of a charged current:

$$\mathcal{J}^{+}_{\text{weak}} = (\bar{u}, \bar{c}) \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ u \end{pmatrix}$$

More general case involving 6-quarks;

$$\mathcal{J}^{+}_{\text{weak}} = (\bar{u}\,\bar{c}\,\bar{t}) \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

Kobayashi-Maskawa matrix

For nucl. β-decay, we are solely concerned with tr.- between u and d-quarks. ($\theta \approx 0$ small).

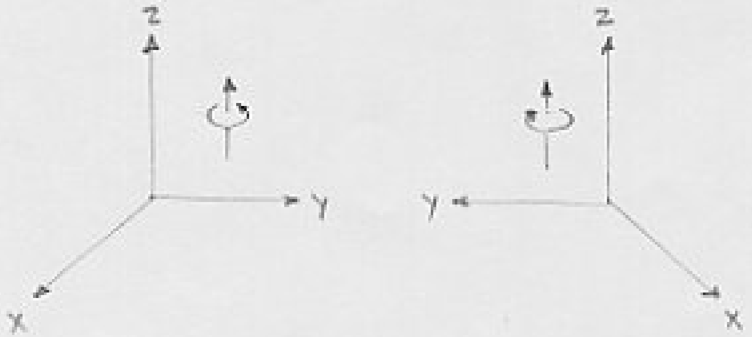
$\longrightarrow$ $G_F \cos\theta_c$ enters into the process

$$G_V = G_F \cos\theta_c \qquad \text{vector coupling const.}$$

Parity Non-Conservation:

Parity <u>is not</u> conserved in weak-decay:

$$(x, y, z) \xrightarrow{\Pi} (-x, -y, -z)$$



The change in the coord axes is accomplished by the operation $(x, y, z) \to (-x, -y, -z)$ followed by a rotation of 180° around the y-axis.

A usual scalar (S) $\xrightarrow{\Pi}$ unchanged

" " vector (V) $\xrightarrow{\Pi}$ changes sign (Polar vectors)

Ex. $\bar{P}$, $\bar{r}$ are polar vectors.

An axial vector (A) $\xrightarrow{\Pi}$ unchanged

Ex. $\bar{\ell}\hbar = \bar{r} \times \bar{P}$ axial vector

$\bar{J}$, $\bar{S}$ " "

pseudoscalar (P) $\xrightarrow{\Pi}$ changes sign

Ex. Axial vector · Polar vector

Quantum Mechanically;

$\{\Pi, \bar{r}\} = 0 \quad \{\Pi, \bar{P}\} = 0 \quad \longrightarrow [\Pi, \bar{L}] = 0$

$[\Pi, \bar{S}] = 0 \quad [\Pi, \bar{J}] = 0$

$\{\Pi, \bar{S} \cdot \bar{r}\} = 0 \qquad [\Pi, \bar{L} \cdot \bar{S}] = 0$

still scalars and pseudoscalars transfor in the same manner under rotations.

An operator made of a <u>mixture</u> of scalars and pseudoscalars, or a mixture of vectors and axial vectors <u>does not</u> have a <u>definite</u> parity $\longrightarrow$ Parity is not conserved under its action.

$$O = a\, O_{odd} + b\, O_{even}$$

In $\left\{\begin{array}{l}\text{strong int.}\\ \text{el.-mag "}\end{array}\right.$ the parity is conserved.

This <u>is not</u> true in <u>weak</u> int.

Evidences:

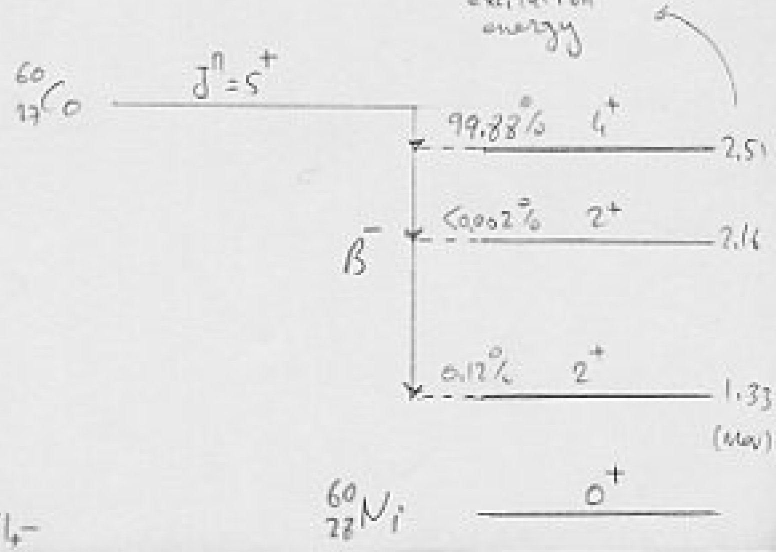
i) $K^+ \xrightarrow{\text{weak}} \pi^+ + \pi^0 \quad \rightarrow$ Parity of $K^+ = (-)^2$
$ \xrightarrow{\text{"}} \pi^+ + \pi^- + \pi^+ \quad \rightarrow$ " " $= (-)^3$ } contradiction

Two decay modes have <u>different</u> parities.

$K^+, \pi^\pm, \pi^0$: spinless particles

ii) ${}^{60}_{27}Co \xrightarrow{\text{weak}} {}^{60}_{28}Ni + e^- + \bar{\nu}_e$

↑ purely Gamow-Teller type



A $\underline{\text{non-zero}}$ spin $\xrightarrow{\text{makes it possible}}$ for the nucleus to be polarized along a mag. field (external)

Suppose the spins of all the ${}^{60}_{27}Co$ nuclei are aligned along $\bar{\alpha}$ (unit vector along $\bar{B}$).

$\longrightarrow \quad \bar{J} \parallel \bar{\alpha}$ $\qquad$ $\bar{J}$: groundstate spin of ${}^{60}Co$

It can be shown; $\quad W(\theta) = 1 + a \dfrac{\alpha . p}{E} c \quad$ the angular dist. of electrons

$E = \sqrt{m_0^2 c^4 + p^2 c^2} \qquad (m, p) \in$ electron

$E \simeq m_0 c^2 \quad p \simeq m_0 v \quad \rightarrow W(\theta) = 1 + a \dfrac{v}{c} \cos\theta$

(Ref: M. Morita, β-decay and Muon Capture Benjamin, Reading, Massachusets, 1973, P. 67; J.M Eisenberg and W. Greiner, Excitation Mechanisms of the Nucleus, North Holland, Amesterdam, 1970, P. 290)

$\left\{ \begin{array}{l} \alpha: \text{axial vector} \\ p: \text{polar} \quad '' \end{array} \right. \longrightarrow \alpha . p \quad$ pseudoscalar

$$\longrightarrow \quad W(\theta) = \underset{\substack{\uparrow \\ \text{scalar}}}{1} + a \underset{\substack{\uparrow \\ \text{pseudoscalar}}}{\dfrac{\alpha . p}{E}} c$$

$\alpha . p \xrightarrow{\Pi} -\alpha . p$

If parity is conserved in the decay $\xrightarrow{\text{must}}$ $a = 0$

(to be $W(\theta)$ invariant under parity op)

If $a = 0$ $\longrightarrow$ angular dist. of electrons is isotropic

However, experimentally a turned out to be $\underline{-1}$

<u>indicating</u> $\rightarrow$ Max. parity violation

Fermi and Gamow-Teller operators;

The weak int Hamiltonian is given by

$$H' = \frac{G}{\sqrt{2}} \int J^{\mu +}(r)\, G(\bar{r},\bar{r}')\, J_\mu(r')\, d\bar{r}\, d\bar{r}'$$

$J_\mu(r)$: total weak current

$G(\bar{r},\bar{r}')$: Propagator (carries the int. between terms of the total weak current)

$J^{\mu +} = J_\mu^{+} \quad \mu = 1,2,3 \qquad J^{4+} = -J_4^{+}$

For weak int. $\quad G(r,r') = \delta(\bar{r}-\bar{r}')$ contact int.

$$\rightarrow \quad H' = \frac{G}{\sqrt{2}} \int J^{\mu +}(r)\, J_\mu(r)\, d\bar{r}$$

For nucl. weak process;

$$J_\lambda(r) = i\left[\,\bar{\Psi}_\mu(r)\Gamma_\lambda \Psi_{\nu_\mu}(r) + \bar{\Psi}_e(r)\Gamma_\lambda \Psi_{\nu_e}(r) + \bar{\Psi}_n(r)\Gamma_\lambda \Psi_p(r)\right] \qquad (1)$$

Ψ : spinor field for particles (to be expanded in terms of <u>creation</u> and <u>annihilation</u> ops. for the relevant complete set of single <u>fermion</u> states)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \psi^{+} = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \qquad \bar{\psi} = \psi^{+}\gamma^{0}$$

$$\gamma^{0} = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\begin{cases} \psi: & \text{annihilate a particle or create an antiparticle} \\ \bar{\psi}: & \text{" an antiparticle " " a particle} \end{cases}$$

$$n \longrightarrow p + e^{-} + \bar{\nu}_e$$

$$H'_{\beta} = \frac{G}{\sqrt{2}} \int [\bar{\psi}_n \Gamma^{\lambda} \psi_p]^{+} [\bar{\psi}_e \Gamma_{\lambda} \psi_{\nu_e}] \, dr \qquad (2)$$

destroys n creates p creates e^{-} creates $\bar{\nu}_e$ (antiparticle)

$$p + \mu^{-} \longrightarrow n + \nu_{\mu} \qquad \text{(muon capture)}$$

$$H'_{cap} = \frac{G}{\sqrt{2}} \int [\bar{\psi}_{\mu} \Gamma^{\lambda} \psi_{\nu_\mu}]^{+} [\bar{\psi}_n \Gamma_{\lambda} \psi_p] \, dr \qquad (3)$$

$$\mu^{-} \longrightarrow e^{-} + \bar{\nu}_e + \nu_{\mu}$$

$$H'_{\mu^-} = \frac{G}{\sqrt{2}} \int [\bar{\psi}_{\mu} \Gamma^{\lambda} \psi_{\nu_\mu}]^{+} [\bar{\psi}_e \Gamma_{\lambda} \psi_{\nu_e}] \, dr \qquad (4)$$

$\mu^+ \longrightarrow e^+ + \nu_e + \bar{\nu}_\mu$

$$H'_{\mu^+} = \frac{G}{\sqrt{2}} \int [\bar{\psi}_e \Gamma^\lambda \psi_{\nu_e}]^+ [\bar{\psi}_\mu \Gamma_\lambda \psi_{\nu_\mu}] d\tau \qquad (5)$$

Three forms of Γ_λ in (equ. 1957) are presumed all to be the same.

leading $\longrightarrow$ G to be same for (2)(3)(4)(5).

This is the assumption of a Universal Fermi Int. (UFI).

Of course, for the processes (2) and (3), the weak ints are accompanied by strong int. effects

$\longrightarrow$ To some extent simple description under UFI is broken.

Since the parity in weak int. is not conserved $\longrightarrow$

$\longrightarrow$ H' must contain a mixture of scalar and pseudoscalar terms

$\longrightarrow$ The current must contain vector and axial-vector terms.

The assumption of $\Gamma_\lambda = \gamma_\lambda(1+\gamma_5)$

decomposes the current into two parts:

$$J_\lambda(x) = J_\lambda^V(x) + J_\lambda^A(x)$$

$$J_\lambda^V(x) = i\left[\bar{\Psi}_\mu(x)\gamma_\lambda\Psi_{\nu_\mu}(x) + \bar{\Psi}_e(x)\gamma_\lambda\Psi_{\nu_e}(x) + \bar{\Psi}_n(x)\gamma_\lambda\Psi_p(x)\right]$$

$$J_\lambda^A(x) = i\left[\bar{\Psi}_\mu(x)\gamma_\lambda\gamma_5\Psi_{\nu_\mu}(x) + \bar{\Psi}_e(x)\gamma_\lambda\gamma_5\Psi_{\nu_e}(x) + \bar{\Psi}_n(x)\gamma_\lambda\gamma_5\Psi_p(x)\right]$$

This is referred to V−A int. (V minus A).

$\Gamma_\lambda = \gamma_\lambda(1+\gamma_5)$ ⟶ Vector and axial vector copling consts enter with <u>equal</u> magnitude but <u>opposite</u> sign.

In <u>reality</u> this simple assumption of the UFI int with V−A coupling <u>is not</u> directly applicable for the processes of interest (due to the effect of <u>strong</u> int.)

⟶ i) A more general Dirac op than $\Gamma_\lambda = \gamma_\lambda(1+\gamma_5)$ is involved.

ii) Other terms enter

Illustrative exaple;

The effects of strong ints. on the nucleon electromagnetic currents due to mesonic currents introduces a change in the mag. moment coupling of the nucleon to the el-mag. field.

Under the UFI hypothesis with V–A coupling the form factors of vector part (f) and axial-vector part (g) are both equal unity and all others due to other currets (like tensor op. terms, or induced pseudoscalar part largely comming from pseudoscalar pion cloud surroding the nucleons, ..etc.)

Note: The form factors f and g are introduced to account for the effect of strong ints.

Bilinear Covariants:

1- $\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x)$ scalar

2- $\bar{\psi}'(x')\gamma_5\psi'(x') = \det|a|\,\bar{\psi}(x)\gamma_5\psi(x)$ pseudoscalar

3- $\bar{\psi}'(x')\gamma^\mu\psi'(x') = a^\mu{}_\nu\,\bar{\psi}(x)\gamma^\nu\psi(x)$ vector

4- $\bar{\psi}'(x')\gamma_5\gamma^\mu\psi'(x') = \det|a|\,a^\mu{}_\nu\,\bar{\psi}(x)\gamma_5\gamma^\nu\psi(x)$ pseudo-vector

5- $\bar{\psi}'(x')\sigma_{\mu\nu}\psi'(x') = a^\mu{}_\alpha a^\nu{}_\beta\,\bar{\psi}(x)\sigma_{\alpha\beta}\psi(x)$ tensor

Fermi and Gamow-Teller operators:

$$H_{weak} = G \left\{ f_1 H_V + g_1 H_A + f_2 H_T + g_2 H_{IP} + f_3 H_S + g_3 H_{PT} \right\}$$

f_1: Form factor for vector term

g_1: " " " axial-vector "

f_2: " " " Tensor "

g_2: " " " Induced pseudoscalar term

f_3: " " " scalar term

g_3: " " " pseudoscalar "

Ex.

The term $g_2 H_{IP}$ is the induced pseudoscalar part and comes about largely because the nucleon is surrounded by a cloud of pseudoscalar pions which can undergo weak decays such as:

$$\pi \longrightarrow \mu + \nu_\mu$$

$$\pi \longrightarrow e + \nu_e$$

Under the UFI hypothesis with V-A coupling $f_1 = g_1 = 1$ and all others would vanish.

But due to strong ints. $f_1 \neq 1$, $g_1 \neq 1$

and all the others $\neq 0$.

If time-reversal invariance holds in both strong and weak ints. $\longrightarrow$ Six form factors will be real.

However violations of CP operations have been observed in strangeness-changing weak ints.

This implies $\rightarrow$ noninvariance in time-reversal operation (T) provided the CPT theorem is accepted.

charge conjugation   parity   time-reversal

Note:

particle $\xrightarrow{C}$ antiparticle

$(e, \bar{P}, E, \bar{S}) \xrightarrow{C} (-e, -\bar{P}, -E, \bar{S})$

Conserved Vector Current (CVC) hypothesis;

$$G_\beta = G_F \underset{\substack{\uparrow \\ \text{zero-momentum transfer}}}{f_1(0)} = (1.4029 \pm 0.0022) \times 10^{-62}\ \mathrm{J \cdot m^3} \quad (\text{from } \beta\text{-decay})$$

$$G = G_F = (1.4350 \pm 0.0011) \times 10^{-62}\ \mathrm{J \cdot m^3} \quad (\text{from the decay rate in } \mu^- \longrightarrow e^- + \bar{\nu}_e + \nu_\mu)$$

The difference between these two is about 2% within the relevant uncertainities.

$\longrightarrow$ They are tentatively equal.

$\longrightarrow f_1(0) = 1$

This result is rather surprizing!
It states; the effects of strong int. leave the weak int coupling unmodified at zero-momentum transfer;

i.e. $$\underset{\substack{\downarrow \\ \text{strong int.} \\ \text{present}}}{G_\beta} = \underset{\substack{\downarrow \\ \text{strong int.} \\ \text{absent}}}{G}$$

We can gain some guidance in thinking about possible reasons for this situation by referring to electromagnetic theory.

The effects of strong ints. apparently do not modify the electromagnetic coupling const., or charge, e which is the same for the proton as for the electron or moun.

Indeed the relevant form factor has the property; $F_1(0)=1$

The strong ints. do actually bring about changes in the charge dist., but because the electromagnetic current is conserved
$\longrightarrow$ the total charge in the interaction region is const., in spite of the strong coupling effects.

The dimensions of the int. region are governed by the Compton wave lengths of the various particles participating in the strong int.

The lightest of these particles (the pion) has $m_\pi \simeq 140$ MeV

$$\lambda \simeq \frac{\hbar c}{140} = 1.41 \text{ fm}$$

The very long wavelength photon corresponding to zero-momentum transfer will sample the total charge in the region.

Thus $\longrightarrow$ The electromagnetic coupling at low momentum transfer is unmodified by the strong ints. as a consequence of current conservation.

A similar situation will pertain for the weak ints., if we require that the vector current involved in the weak ints. be conserved.

$$\sum_{\mu=1}^{4} \frac{\partial J_\mu^V}{\partial x_\mu} = 0$$

The hypothesis of a partially conserved axial vector current (PCAC);

$g_1(0) = 1.23 \longrightarrow$ Strong ints modify the axial-vector coupling const.

$\xrightarrow{\text{would seem to rule out}}$ the possibility of the axial-vector current conservation.

Much stronger evidence;

If the axial-vector current were conserved $\xrightarrow{\text{then}}$ the decay

$\pi \longrightarrow \mu + \nu$ would not be possible

Calculation shows $\longrightarrow$ $W = 0$ (tr. rate)

(infinite life time)

Since J_μ^A is an axial-vector $\longrightarrow \sum_{\mu=1}^{4} \frac{\partial J_\mu^A}{\partial x_\mu}$ = pseudoscalar

Also the pion is a pseudoscalar particle and $\xrightarrow{\text{Therefore}}$ is described by a pseudoscalar field.

$$\longrightarrow \underbrace{\sum_{\mu=1}^{4} \frac{\partial J_\mu^A}{\partial x_\mu}}_{\text{weak axial current}} = a \underbrace{\varphi_\pi}_{\text{strong int. field}}$$

The connection between the two coupling consts, G_A (Gamow-Teller part) and G_V (vector part) is given by the Goldberger-Trieman relation;

$$g_A \equiv \frac{G_A}{G_V} = \frac{f_\pi g_{\pi N}}{M_N c^2}$$

M_N: nucleon mass

$f_\pi = \frac{F_\pi}{\sqrt{2}} \simeq 93\,\text{MeV}$ $\qquad$ F_π: Pion decay const.

$g_{\pi N}$: Pion-nucleon coupling const.

$$\frac{|g_{\pi N}|^2}{4\pi} \simeq 14 \quad \text{empirically}$$

$$\rightarrow \; |g_A| \simeq 1.31$$

The measured value from nuclear β-decay $\rightarrow g_A = -1.259 \pm 0.004$ confirming the PCAC hypothesis.

5-5 Nuclear β-decay:

β^--decay: $\quad A(Z,N) \longrightarrow A(Z+1, N-1) + e^- + \bar{\nu}_e$

β^+-decay: $\quad A(Z,N) \longrightarrow A(Z-1, N+1) + e^+ + \nu_e$

Analogous to internal conversion in electromagnetic decay, an atomic electron may be captured by the nucleus instead of emitting a positron in β^+-decay:

$$e^- + A(Z,N) \longrightarrow A(Z-1, N+1) + \nu_e$$

Except for a small difference in the energies involved, such an electron capture process has the same selection rule as β^+-decay and is usually in competition with it.

$P \sim Z^3$ the probability of electron capture

$P \sim$ the decrease in the radii of electronic orbits

Q-Values:

Def.: $Q = T_f - T_i$ (the difference in the total kinetic energy of the system)

For B-decay the parent nucleus may be assumed to be at rest (in lab.) $\rightarrow$ $T_i = 0$

In order for a B-decay to take place: $Q > 0$

Since there are e^- or e^+ in the final states $\longrightarrow$ Q <u>is not</u> simply the difference between the energies of the initial and final <u>nuclear</u> states (m_ν = too small to play a significant role here).

$$\Delta M_{\beta^\pm} = M_P(Z,N) - M_D(Z \mp 1, N \pm 1)$$

M includes the me and the Be

$$Q_{\beta^-} = T_f - T_i = T_f - 0 = T_D(Z-1, N+1) + T_{e^-} + T_{\bar\nu} \simeq 0 + T_{e^-} + T_{\bar\nu}$$

$$= T_{\bar{e}(max)}$$

Note that; $mc^2 = W = m_0c^2 + T$

$W = (p^2c^2 + m_0^2c^4)^{\frac{1}{2}}$

$\rightarrow W_\nu = T_\nu = pc \qquad (m_0 = 0)$

$$\begin{cases} M_P(Z,N)c^2 = M_D(Z-1,N+1)c^2 + T_D(Z-1,N+1) + m_ec^2 + T_e + W_\nu \\ 0 = \bar{P}_{D(Z-1,N+1)} + \bar{P}_e + \bar{P}_\nu \end{cases}$$

$$T_D = \frac{P_D^2}{2M_D} = \frac{(\bar{P}_e + \bar{P}_\nu)^2}{2M_D} = \frac{P_e^2 + P_\nu^2 + 2P_eP_\nu\cos\theta}{2M_D}$$

$$T_{D(max)} = \frac{P_e^2 + P_\nu^2 + 2P_eP_\nu}{2M_D}$$

If we neglect the term $2P_eP_\nu$, the max effect in $T_{D(max)}$ will be 100% -

On the other hand if we take $m_\nu = m_e \rightarrow T_{D(max)}$ will be considerably increased;

$$T_{D(max)} = \frac{2m_e(T_e + T_\nu)}{2M_D} = \frac{m_e}{M_D}(T_e + W_\nu)$$

and this is still very low, then $T_D \approx 0$

$$Q_{\beta^-} = \left\{ M_p(Z,N) - \left[M_D(Z+1, N-1) + m_e \right] \right\} c^2 +$$

$$- \left[Z\, m_e - (Z+1) m_e \right] c^2$$

Remark: $M_{nucl} c^2 = (Z M_p + N M_n) c^2 - E_B$ $\qquad \left(\Delta \equiv \frac{E_B}{c^2} \text{ mass effect} \right)$

$$M_{at} c^2 = (Z M_H + N M_n) c^2 - E_B$$

$M_H c^2 = (M_p + m_e) c^2 - E_B'$ $\qquad \begin{cases} E_B : \text{nucl. binding energy} \\ E_B' : \text{electron} \ \text{"} \quad \text{"} \end{cases}$

$$\rightarrow Q_{\beta^-} = \left(M_p(Z,N) - M_D(Z+1, N-1) \right) c^2$$

$$Q_{\beta^+} = T_{e^+} + T_D = T_{e^+(max)}$$

$$Q_{\beta^+} = \left\{ M_p(Z,N) - \left[M_D(Z-1, N+1) + m_e \right] \right\} c^2 +$$

$$- \left[Z m_e - (Z-1) m_e \right] c^2$$

$$Q_{\beta^+} = \left(M_p(Z,N) - M_D(Z-1, N+1) \right) c^2 - 2 m_e c^2$$

$$Q_{EC} = \left\{ \left[M_p(Z,N) + m_e \right] - M_D(Z-1, N+1) \right\}$$

$$- \left[Z m_e - (Z-1) m_e \right] c^2 - E_{B_e}$$

$$Q_{EC} = \left(M_p(Z,N) - M_D(Z-1, N+1) \right) c^2 - \underbrace{E_{B_e}}_{\approx 10\ eV}$$

We may ignore E_{B_e} unless we are concerned with high accuracy measurements (like neutrino mass measurements).

$$A(Z,N) + e^- \longrightarrow A(Z-1, N+1) + \nu \quad \text{(electron capture)}$$

Because of the difficulty in detecting neutrinos, the most prominent signature of electron capture processes is the x-ray emitted when atomic electrons in higher orbits decay to the lower ones left empty when an inner shell electron is absorbed by the nucleus. (It can be detected also by emitting the Auger electrons).

In terms of binding energy;

$$Q_{\beta^-} = E_B(Z+1, N-1) - E_B(Z,N) + 0.782 \text{ MeV}$$

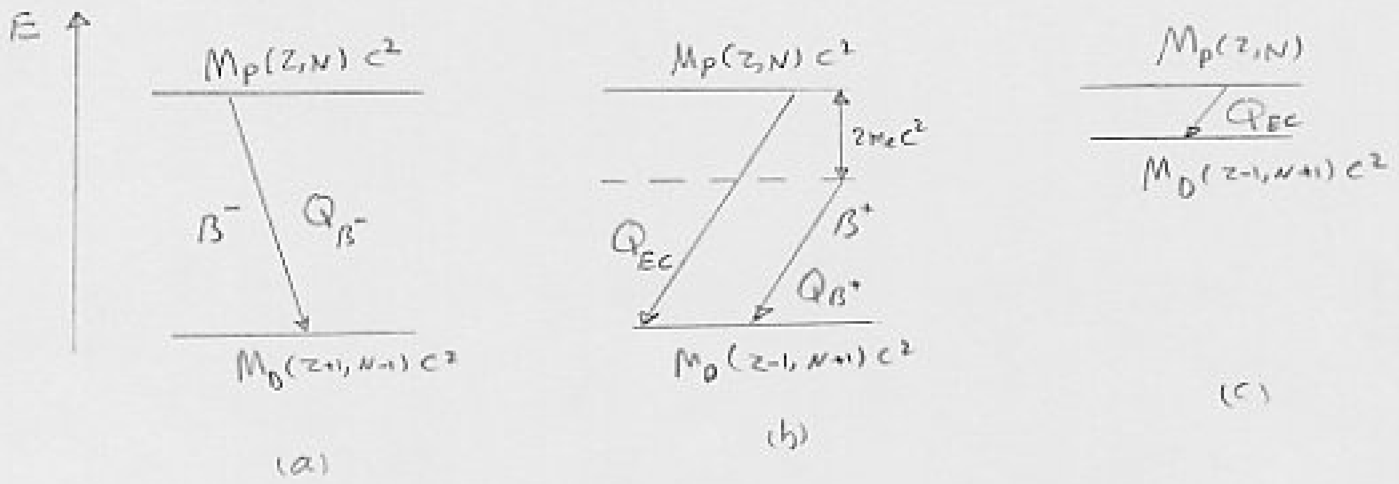
$$Q_{\beta^+} = E_B(Z-1, N+1) - E_B(Z,N) - 2m_ec^2 - 0.782 \text{ MeV}$$

$$Q_{EC} = E_B(Z-1, N+1) - E_B(Z,N) - E_{Be} - 0.782 \text{ MeV}$$

$$0.782 = M_n - M_H$$

Note: Auger effect;

Instead of emitting a photon when an atomic electron de-excites from a higher to a lower energy orbit, one of the atomic electrons is ejected.



a) β^--decay

b) β^+-decay and electron capture. β^+-decay is possible if

$$M_p - M_D > 2m_e$$

c) Electron capture. When $0 < M_p(Z,N) - M_D(Z-1,N+1) < 2m_e$

only electron capture takes place.

Mass parabola:

$$E_B(Z,N) = \left(Z M_H + N M_n - M(Z,N)\right) c^2$$

On the other hand

$$E_B(Z,N) = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z(Z-1)}{A^{1/3}} - \alpha_4 \frac{(N-Z)^2}{A} + \Delta + \eta$$

α_1: Volume energy parameter
α_2: Surface 〃 〃
α_3: Coulomb 〃 〃
α_4: Symmetry 〃 〃

Δ: pairing force term
η: Shell effect 〃

$$\Delta = \begin{cases} \delta & \text{for even-even nuclei} \\ 0 & \text{" odd mass nuclei} \\ -\delta & \text{" odd-odd nuclei} \end{cases}$$

$$M(Z,N)\,c^2 = aA + bZ + cZ^2 \mp \delta - \eta$$

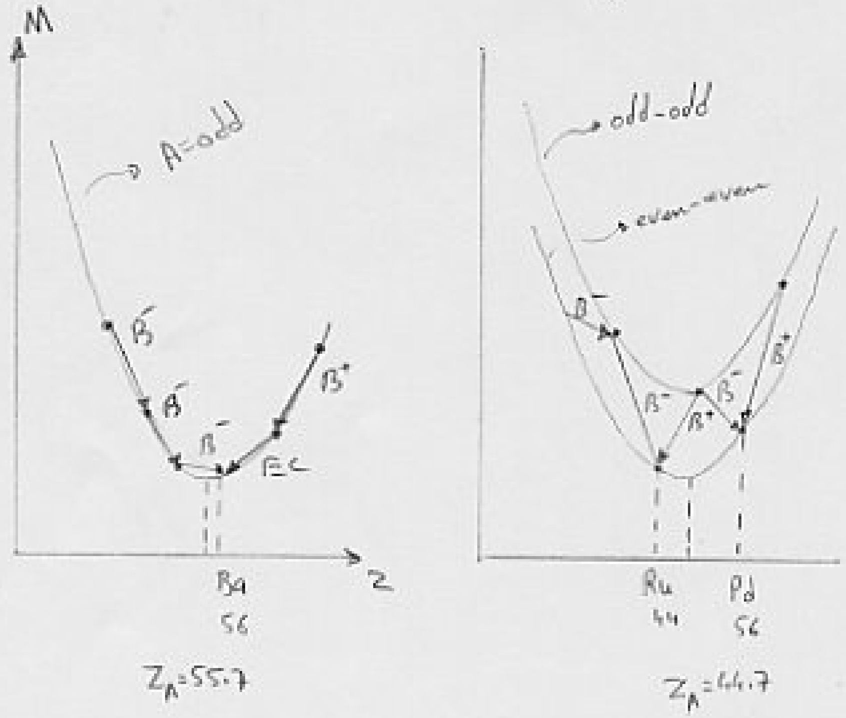
$$a = M_n c^2 - \alpha_1 + \alpha_4 + \frac{\alpha_2}{A^{1/3}}$$

$$b = -4\alpha_4 - (M_n - M_H)\,c^2 \simeq -4\alpha_4$$

$$c = 4\frac{\alpha_4}{A} + \frac{\alpha_3}{A^{1/3}}$$

For a const. A, this is the equ. of a parabola.

$$\frac{\partial (Mc^2)}{\partial Z} = 0 \quad \rightarrow \quad Z_A = -\frac{b}{2c} \simeq \frac{\frac{A}{2}}{1 + \frac{1}{4}\left(\frac{\alpha_3}{\alpha_4}\right) A^{2/3}}$$



Transition rates for β-decay;

$$W = \frac{2\pi}{\hbar} \left| \langle \varphi_k(\bar r) | H' | \varphi_0(\bar r) \rangle \right|^2 \rho(E_f)$$

$|\varphi_0(\bar r)\rangle = |\underset{\text{Parent}}{J_i M_i}, \xi\rangle$ initial state (stationary in lab.)

$|\varphi_k(\bar r)\rangle = \frac{1}{\sqrt{V}} e^{i k_e \cdot r} \frac{1}{\sqrt{V}} e^{i k_\nu \cdot r} \underset{\text{Daughter}}{|J_f M_f \xi\rangle}$ final states

where we have ignored any Coulomb effect betwen charged lepton and daughter nucleus (resulting to a plane wave-func. for the charged lepton).

We may expand the plane-waves in terms of spherical harmonics and spherical Bessel funcs.;

$$e^{i k \cdot r} = \sum_{\lambda=0}^{\infty} \sqrt{4\pi(2\lambda+1)}\; i^\lambda j_\lambda(kr)\, Y_{\lambda 0}(\theta, 0)$$

where $K = |\bar K| = |\bar K_e + \bar K_\nu|$

Since Q-value of the tr. is typically of the order of a few MeV,

$\longrightarrow$ we make use of the long wave length approx.

In this limit we need to retain the first term in $j_\lambda(kr)$.

$$j_\lambda(kr) \approx \frac{(kr)^\lambda}{(2\lambda+1)!!}$$

$$|\varphi_{\bar{k}}(\bar{r})\rangle = \frac{1}{V}\left\{1 + i\sqrt{\frac{4\pi}{3}}\,(kr)\,Y_{10}(\theta,0) + O((kr)^2)\right\}|J_f M_f S\rangle \quad (1)$$

Nuclear transition matrix elements;

In β^--decay $\quad n \longrightarrow p$

and in β^+- " $\quad p \longrightarrow n$

$\longrightarrow$ nuclear operator must be one-body op.

It should contain $\tau_\pm$

In non-relativistic limit;

Vector part $\sim I\,\tau_\pm$

Axial vector " $\sim \sigma\,\tau_\pm$

A proper derivation of this result requires manipulations of Dirac wave funcs. and γ-matrices.

(Ref. – Morita, β-decay and Muon capture, Benjamin, Reading Massachusets, 1973) -

$$\langle \varphi_{k}(\vec{r}) | H' | \varphi_0(\vec{r}) \rangle = \frac{1}{V} \langle J_f M_f \xi | \sum_{j=1}^{A} \{ G_V \tau_{\mp}(j) + G_A \sigma(j) \tau_{\mp}(j) \}$$

$$\cdot \Big[\underbrace{1 - i\sqrt{\tfrac{4\pi}{3}}\,(kr)\, Y_{10}(\theta,0)}_{\text{Two leading terms}} + O((kr)^2) \Big] | J_i M_i , \xi \rangle \qquad (1)$$

We are mainly concerned with the two leading order terms (operators for allowed trs.)

The higher order terms involve spherical harmonics of order greater than zero (forbidden decays).

For the allowed decays, the nuclear part of the $\beta^{\mp}$-decay tr. op. has the form;

$$\sum O_\lambda = \underbrace{G_V \sum_{j=1}^{A} \tau_{\mp}(j)}_{O_\lambda = O_0} + \underbrace{G_A \sum_{j=1}^{A} \vec{\sigma}(j)\, \tau_{\mp}(j)}_{O_\lambda = O_1 \text{ (angular momentum)}}$$

$$\langle \varphi_k(\vec{r}) | H' | \varphi_0(\vec{r}) \rangle \approx \frac{G_V}{V} \sum_{\mu M_f} \Big\{ \langle J_f M_f \xi | \sum_{j=1}^{A} \tau_{\mp}(j) | J_i M_i \xi \rangle$$

$$+ g_A \langle J_f M_f \xi | \sum_{j=1}^{A} \vec{\sigma}(j)\, \tau_{\mp}(j) | J_i M_i \xi \rangle \Big\}$$

where $g_A = \dfrac{G_A}{G_V}$ (2)

First term : Fermi decay

Second " : Gamow-Teller decay

The matrix elements of ops. with $\lambda > 1$ are usually much smaller in values (since they come from the higher order terms in (equ 1 P77))

Their contributions are important only in trs. when the two lowest order terms are forbidden by angular momentum and parity selection rules.

Density of final states:

In β-decay there are 3-body final states.

In 2-body case E and $\bar{P}$ of a particle is restricted by the values taken up by the other particle due to conservation laws - resulting to equivalent one-body one.

$$M_P c^2 - M_D c^2 = E_D + E_{e^-} + E_{\bar{\nu}} \quad (E:\text{total})$$

The kinetic part of $E_{f(\text{nucl.})}$ = small

$$\bar{P}_{i(\text{nucl.})} = \underbrace{\bar{P}_{f(\text{nucl.})}}_{\text{small}} + \bar{P}_{e^-} + \bar{P}_{\bar{\nu}}$$

Another point;

The charged lepton is emitted in the Coulomb field of the daughter nucleus $\longrightarrow$ its wave func. is distorted due to the el-mag int.

$\longrightarrow$ This also has an effect on the density of final states available to the charged lepton.

Since a neutrino hardly interacts with its surrounding

$\longrightarrow$ It may be considered as a free particle after the creation.

$$K_x = \frac{2\pi}{L} n_x , \; \cdots \qquad P_x \equiv \hbar K_x = \frac{2\pi\hbar}{L} n_x \cdots$$

$$\Delta \bar{K} = \Delta K_x \, \Delta K_y \, \Delta K_z = \left(\frac{2\pi}{L}\right)^3 \Delta n_x \, \Delta n_y \, \Delta n_z$$

$$\omega = |\bar{K}| c = \frac{2\pi c}{L} (n_x^2 + n_y^2 + n_z^2)^{1/2}$$ for Photons

$$d^3 n = \frac{L^3}{(2\pi\hbar)^3} d^3 p = \frac{V}{(2\pi\hbar)^3} d^3 p = \frac{V}{(2\pi\hbar)^3} d\Omega_p \, p^2 dp$$

$$= \frac{4\pi V}{(2\pi\hbar)^3} p^2 dp = \frac{V}{2\pi^2 \hbar^3} p^2 dp$$ the number of states with momentum p without any regard to the dir. of the particle

$$\rightarrow d^3 n_\nu = \frac{V}{2\pi^2 \hbar^3} p_\nu^2 dp_\nu$$

$$E_\nu^2 = (m_\nu c^2)^2 + p_\nu^2 c^2$$

Instead of Q-value, it is customary to express energies in β-decay in terms of kinetic energy of the charged lepton emitted. (because the energy of the electron or positron is a quantity that can be observed directly).

$$E_\nu = E_0 - E_e$$

E_0: Max. kinetic energy of $e^\pm$ (end-point energy)

There is a variation in E_0 due small differences in the recoil energy of the daughter nucleus (we have ignored).

Since $M_D >> m_e$ and m_ν

→ The recoil of nucleus is considered only when high precision is required.

$$d^3 n_\nu = \frac{V}{2\pi^2 \hbar^3} \frac{(E_0 - E_e)}{c^3} \left[(E_0 - E_e)^2 - (m_\nu c^2)^2 \right]^{\frac{1}{2}} dE_e \qquad (1)$$

where $p_\nu c = \sqrt{E_\nu^2 - (m_\nu c^2)^2}$

In order for account the distortion of the plane wave of the charged lepton (due to Coulomb int.), a distortion factor $F(Z, E_e)$ is introduced;

$$d^3 n_e = \frac{V}{2\pi^2 \hbar^3} F(Z, E_e)\, P_e^2\, dP_e \qquad (1)$$

Fermi func.

$$F(Z, T_e) = \frac{x}{1-e^{-x}} \qquad \text{for } V \lll c$$

$x = \mp 2\pi \alpha Z \frac{c}{V}$ for $\beta^{\pm}$-decay $\quad \alpha$: structure const.

(Equ.2 P77), (Equ.1, P80), (Equ.1, P81) $\rightarrow$

$$W(P_e) = \frac{1}{2\pi^3 \hbar^7 c^3} \sum_{\mu M_f} |\langle J_f M_f \xi | O_{\lambda\mu}(\beta) | J_i M_i, \xi \rangle|^2$$

certain P_e

$$\cdot\, F(Z, E_e)\, P_e^2 (E_0 - E_e) \left[(E_0 - E_e)^2 - (m_\nu c^2)^2 \right]^{\frac{1}{2}}$$

$dP_e\, dE_e$ is removed (because we are intrested in certain P_e)

If we take $m_\nu \approx 0$

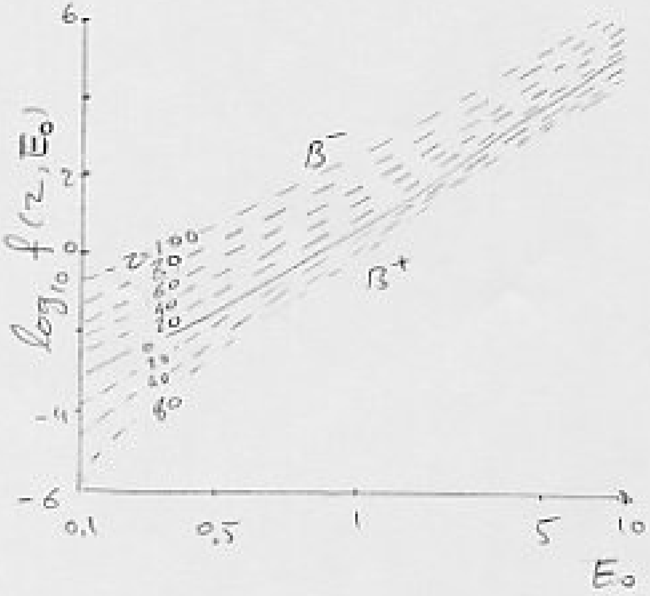
$$W(P_e) = \frac{1}{2\pi^3 \hbar^7 c^3} \sum_{\mu M_f} |\langle J_f M_f \xi | O_{\lambda\mu}(\beta) | J_i M_i, \xi \rangle|^2$$

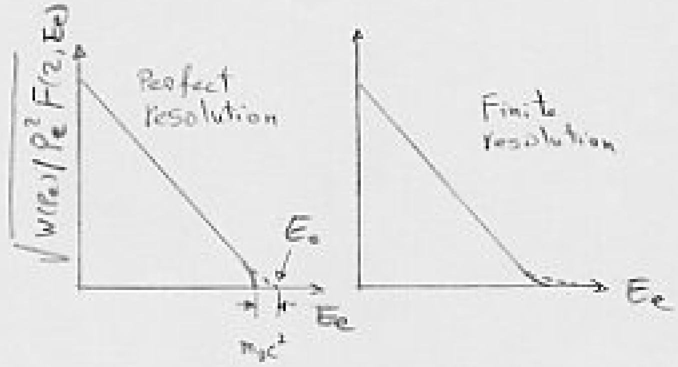
$$\cdot\, F(Z, E_e)\, P_e^2\, (E_0 - E_e)^2$$

This approx. is not good when $E_e \approx E_0$

At this region the effect of m_ν is most evident.

$$\rightarrow \sqrt{\frac{W(p_e)}{p_e^2 F(Z,E_e)}} \sim (E_0 - E_e)$$





Kurie Plot

---- $m_\nu = 0$

slope ~ nucl. matrix element

Neutrino mass measurement;

$m_{\nu_e} \simeq 30$ eV

Still the possibility of $m_{\nu_e} = 0$ is not ruled out.

$m_{\nu_\mu} \simeq 0.5$ MeV $\quad m_{\nu_\tau} \simeq 70$ MeV

Direct measurement;

$$\underset{\text{(tritium)}}{t} \longrightarrow {}^3He + \bar{e} + \bar{\nu}_e$$

$\downarrow$

Ground state (its excited state is very high in E)

The only possible final state

$T_{1/2} = 12.3$ yr

$Q = 18.6$ Kev (low)

→ The influence of a small m_ν stands out more prominently than otherwise, for example in a Kurie plot.

There are, however several difficulties;

1- low counting rate near the end-point (a common prob. in all β-decays.) -

2- The rest mass of the neutrino comparable to the energies encountered in atoms.

→ Atomic effects, which are seldom in nucl. measurements become important.

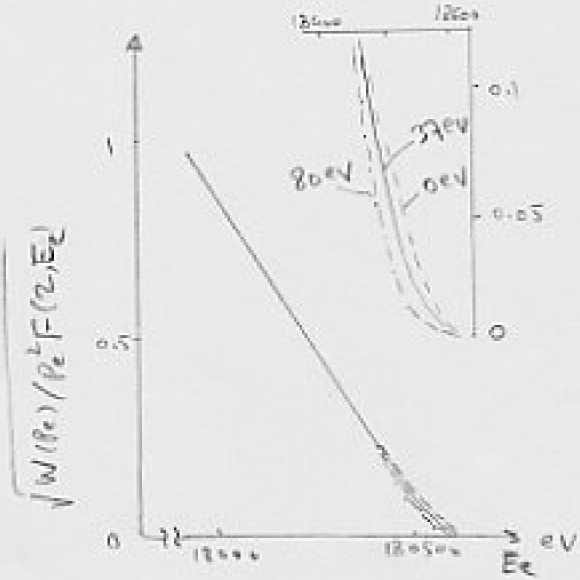
For example there are two possible final atomic states for ^3He in the decay, and the relative probability of forming them must be known.

Still, there is no agreement between the measured values from different laboratories.

$m_{\nu_e} \approx 30$ ev the most often quoted value

Total transition probability:

If we are not interested in the dist. of the charged leptons emitted as a func. of E_e, we can obtain the total W by integrating $W(P_e)$ over all possible P_e:



$$W_{i\to f} = \frac{2\pi}{\hbar}\int \prod_K V \frac{d^3\bar{P}_K}{(2\pi\hbar)^3} |M_{fi}|^2 \delta(E_i^0 - E_f^0 - \sum E_K)\,\delta(\bar{P}_i - \bar{P}_f - \sum \bar{P}_K)$$ Kurie plot for tritium

- - - - $m_{\nu_e} = 0$

— · — · — $m_{\nu_e} = 80/c^2$

—— $m_{\nu_e} = 37/c^2$ (best fit)

$$W = \int W(P_e)\,dP_e =$$

$$= \frac{m_e^5 c^4}{2\pi^3\hbar^7} f(Z,E_0) \left|\sum_{\mu M_f} \langle J_f M_f \xi | O_{\lambda\mu}^{(\beta)} | J_i M_i \xi \rangle\right|^2$$

where

$$f(Z,E_0) = \int F(Z,E_e)\left(\frac{P_e}{m_e c}\right)^2 \left(\frac{E_0 - E_e}{m_e c^2}\right)^2 \frac{dP_e}{m_e c}$$

$$= \frac{1}{m_e^5 c^7}\int F(Z,E_e)\, P_e^2 (E_0 - E_e)^2\, dP_e$$ Fermi integral (Fig. P82) (dim.-less)

Except in $Z=1$, $f(Z,E_0)$ must be evaluated numerically.

$$T_{1/2} = \frac{\ln 2}{W} = \frac{1}{f(Z,E_0)} \frac{2\pi^3\hbar^7}{m_e^5 c^4} \frac{\ln 2}{\left|\sum_{\mu M_f} \langle J_f M_f \xi | O_{\lambda\mu}^{(\beta)} | J_i M_i \xi \rangle\right|^2}$$

Instead of half-lives, nucl. β-decay rates are often quoted in terms of ft values:

$$ft \equiv f(Z, E_0)\, T_{1/2} = \frac{2\pi^3 \hbar^7}{m_e^5 c^4} \frac{\ln 2}{|\sum_{\mu M_f} \langle J_f M_f \xi | O_{\lambda\mu}(\beta) | J_i M_i \zeta \rangle|^2}$$

$$ft \sim \frac{1}{|\text{nucl. matrix element}|^2}$$

on the other hand;

$T_{1/2}$ involves $f(Z, E_0)$ (complicated func.)

$\longrightarrow$ ft : more meaningful physical quantity.

ft value in β-decay plays a similar role as $B(\lambda)$ in electromagnetic trs..

ft-values vary over many orders of magnitude if we include both allowed and forbidden decays.

$\longrightarrow$ to use $\log_{10} ft$ will be convenient.

Allowed β-decay;

The Fermi term involves only:

$$\sum_{j=1}^{A} \tau_{\mp}(j) = T_{\mp}$$

$$\langle J_f M_f T_f T_{zf} | \sum_{j=1}^{A} \tau_{\mp}(j) | J_i M_i T_i T_{zi} \rangle =$$

$$= \sqrt{T_i(T_i+1) - T_{zi}(T_{zi} \mp 1)}\ \delta_{J_f J_i}\ \delta_{M_f M_i}\ \delta_{T_f T_i}\ \delta_{T_{zf}(T_{zi} \mp 1)}$$

(without having to know explicitly the wave fuc.)

This result is derived under the assumption that isospin is an *exact quantum* number.

Selection rules for Fermi-type of $\beta^{\pm}$-decay

$J_f = J_i$ $(\Delta J = 0)$

$T_f = T_i \neq 0$ $(\Delta T = 0$ but $T_i = 0 \to T_f = 0$ forbidden$)$

$T_{zf} = T_{zi} \mp 1$ $(\Delta T_z = 1)$

$\Delta \Pi = 0$ (no parity change

The Gamow-Teller Term involves

$$\sum_{j=1}^{A} \underbrace{\bar{\sigma}(j)}_{\text{spherical tensor rank}=1}\ \underbrace{\tau_{\mp}(j)}_{\ldots=1}$$

$\bar{\sigma}$: axial vector

$\rightarrow \Delta J = 0, 1$ but $J_i = 0 \rightarrow J_f = 0$ forbidden

$\Delta T = 0, 1$ " $T_i = 0 \rightarrow T_f = 0$ "

$T_{z_f} = T_{z_i} \mp 1$ $(\Delta T_z = 1)$

$\Delta \Pi = 0$

when we have used the selection rules;

$\Delta(J_i \, J \, J_f)$ $\Delta(T_i \, T, T_f)$, $\Pi_i \Pi \Pi_f = 1$

In general;

$|\langle \; | \text{Gamow-Teller} | \; \rangle| < |\langle \; | \text{Fermi} | \; \rangle|$

For allowed β-decay, the square of nucl. matrix element;

$$\sum_{\mu M_f} |\langle J_f M_f \xi | O_{\lambda\mu}(\beta) | J_i M_i \xi \rangle|^2 =$$

$$= G_V^2 \left[\sum_{M_f} |\langle J_f M_f \xi | T_\mp | J_i M_i \xi \rangle|^2 + g_A^2 \sum_{\mu M_f} |\langle J_f M_f \xi | \sum_{j=1}^{A} \sigma_\mu(j) \, \tau_\mp(j) | J_i M_i \xi \rangle|^2 \right]$$

$$\equiv G_V^2 \left\{ \langle F \rangle^2 + g_A^2 \langle GT \rangle^2 \right\}$$

There is no cross section term between F and GT matrix elements, since the later vanishes on summing over all possible projections on the quantization axis.

For allowed B-decay;

$$ft = \frac{K}{\langle F\rangle^2 + g_A^2 \langle GT\rangle^2} \qquad K = \frac{2\pi^3 \hbar^7 \ln 2}{m_e^5 c^4 G_V^2} = 6141.2 \pm 3.25$$

Among the consts, G_V is perhaps the least well known one.

The best measured value of K is 6141.2 ± 3.25

$\longrightarrow G_V = 1.41549 \times 10^{-49}$ erg. cm^3

or in more commonly quoted form; $G_V/(\hbar c)^3 = 1.1493 \times 10^{-11}$ MeV^{-2}

$G_F/(\hbar c)^3 = 1.16637(2) \times 10^{-11}$ MeV^{-2}

Superallowed B-decay;

$J_i^\pi = 0^+ \longrightarrow J_f^\pi = 0^+$ trs. form a special class of B-decay, since the GT term does not contribute anything here.

Trs. are purely Fermi $\longrightarrow$ least sensetive to the details of nucl. wave funcs.

$\longrightarrow$ They are useful in determining K and hence G_V

Light nuclei are prefered, since isospin breaking effects are at a minimum.

Superallowed β^--decay is often forbidden by Q-value considerations, since Coulomb energy is higher for nuclei with one more proton.

Most of the examples found are positron emitters;

$$^{14}O \longrightarrow {}^{14}N + e^+ + \nu_e$$

$$J_i^\pi = 0^+ \longrightarrow J_f^\pi = 0^+$$

↑ first excited state of ^{14}N at 2.311 Mev

$T_{1/2}(^{14}O) = 74$ s

$Q = 1.12$ Mev $\qquad ft = 3109$ s

If the initial and final nucl. states are truly isobaric analogue states of each other, the Fermi matrix element may be obtained without referring explicitly to the nucl. wave funcs..

$$g_A = \frac{G_A}{G_V} = -1.259 \pm 0.004$$

G_A: obtained from Gamow-Teller decay

β^--decay of neutrons is ideal reaction in determination of G_A, since only the intrinsic spin wave func of a free neutron enters into the calculations.

In this case, the Gamow-Teller matrix element can be evaluated using the relation,

$$\sum_{\mu m_f} |\langle \chi_{m_f} | \sigma_\mu | \chi_{m_i} \rangle|^2 = 3$$

However, our knowledge is limited on the half-life of neutrons. $T_{1/2}(n) = 621 \pm 7.5$

Forbidden decays:

From the selection rules we saw that for the allowed decays;

$\Delta J = 0, 1 \qquad \Delta \Pi = 0$

The trs. with $\Delta J > 1$ are known as forbidden decays. Their ft-values are much larger (smaller probabilities)

(Eqn 1, P76) $\rightarrow$ spherical harmonics of order greater than zero are involved in forbidden decays.

Forbidden decays are classified into different groups by l-value of the spherical harmonics.

For a given order l, the possible ops. with definite spherical tensor ranks are,

$$Y_{lm}(\theta, \varphi) \quad , \quad (Y_l(\theta,\varphi) \times \bar{\sigma})_{\lambda\mu}$$

The selection rules for the l-th order forbidden tr. are,

$$\Delta J = l \text{ or } (l \pm 1) \qquad \Delta \Pi = (-1)^l$$

$$\Delta T_z = 1 \qquad \Delta T = 0, 1 \qquad (\text{but } T_i = 0 \rightarrow T_f = 0 \text{ forbidden})$$

For <u>first</u>-order forbidden trs., the operators are;

$$r Y_{1\mu}(\theta, \varphi) \sim \bar{r}$$

and $\quad (\bar{\sigma} \times r Y_1(\theta,\varphi))_{\lambda\mu} \qquad$ with $\lambda = 0, 1, 2$

Since the parity of $Y_{1\mu}(\theta,\varphi)$ is <u>-1</u> $\longrightarrow \Pi_f = -\Pi_i$;

The reason for <u>large</u> ft-values;

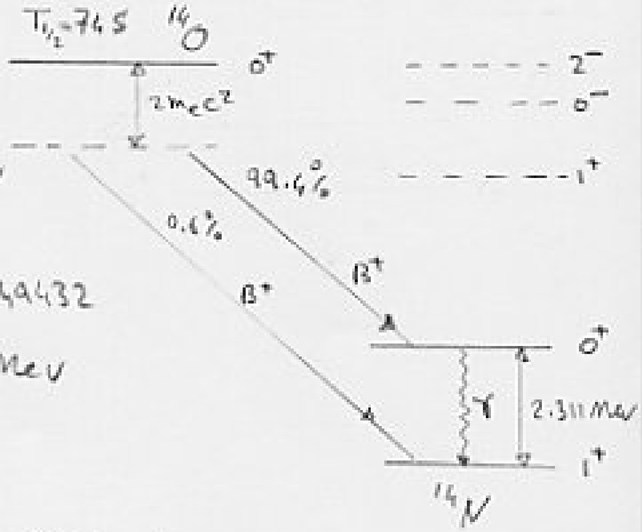
For $l > 0$ there is an angular momentum barrier, <u>inhibiting</u> the emission of leptons

In general, it is quite <u>difficult</u> to calculate the nuclear matrix elements for forbidden β-decays with any <u>reliability</u>.

Decay type	$\log_{10} ft$
Superallowed	2.9 – 3.7
Allowed	4.4 – 6.0
First forbidden	6 – 10
Second "	10 – 13
Third "	> 15

Ex. — Consider the superallowed β^+-decay:

$$^{14}O \longrightarrow {}^{14}N^* + e^+ + \nu_e$$



$$\Delta(Z,N) = [M(Z,N) \text{ in } u - A] \times 931.49432 \text{ MeV}$$

$$M(Z,N) \times 931.49432 = \Delta(Z,N) + A \times 931.49432 \text{ MeV}$$

$$u = \frac{\text{Mass of } ^{12}C \text{ atom}}{12} = \frac{1\,kg}{N_A}$$

$$= 1.6605402(10) \times 10^{-27}\,kg = 931.49432(28) \text{ MeV}/c^2$$

$$N_A = 6.0221367(36) \times 10^{26}\,(kg\ mole)^{-1}$$

$$M(^{14}O) = \Delta(8,6) + 14\,(931.49432) = 13048.927 \text{ MeV}$$
$$M(^{14}N) = \Delta(7,7) + 14\,(931.49432) = 13043.784 \quad "$$

$$E_B(Z,N) = [Z M_H + N M_n - M(Z,N)]\,c^2$$

$$E_B(^{14}O) = (8 M_H + 6 M_n - M(^{14}O)]\,c^2$$

$$E_B(^{14}N) = (7 M_H + 7 M_n - M(^{14}N)]\,c^2$$

$$E_B(^{14}N) - E_B(^{14}O) = -M_H + M_n - M(^{14}N) + M(^{14}O)$$

$$= 0.782 + 5.143 = 5.925 \text{ MeV}$$

$$E_B(^{14}N^*) - E_B(^{14}O) = 0.782 + 5.143 - 2.311 = 3.61 \text{ MeV}$$

$$Q_{\beta^+} = E_B(Z-1, N+1) - E_B(Z,N) - 2m_ec^2 - 0.782 \text{ MeV}$$

$$Q_{\beta^+} = 3.61 - 2(0.5) - 0.782 = 1.828 \text{ MeV}$$

$Q_{\beta^+} = 1.12$ MeV experimentally

charge exchange reactions:

$$n \xrightarrow{\text{changes to}} p \qquad \text{or } p \xrightarrow{\text{changes to}} n \quad \text{in a nucleus}$$

Although the process primarily involves nuclear interactions, the matrix elements that enter into the reaction rates are essentially the same as β-decay.

A typical charge exchange reaction;

(p,n) or (n,p)

$$A(Z,N) + p \longrightarrow B(Z+1, N-1) + n \qquad (1)$$

or $A(p,n)B$

$A(n,p)B$ process is experimentally difficult (because of scarcity of energic neutrons). This is the complement of β^+-decay

The nuclear structure part of (1) bears strong resemblance to β^- decay $\quad A(Z,N) \longrightarrow A(Z+1, N-1) + e^- + \bar{\nu}_e \quad (2)$

(1) is a scattering process, while (2) is a decay (excitation) process

Process (1) is not restricted by Q-value consideration (the required energy can be supplied by the kinetic energy of the incident)

Other charge exchange reactions:

$({}^3He, t)$, $({}^6Li, {}^6He)$

Analysis of reactions involving ${}^3He$ and heavier nuclei is complicated due the probable <u>excitation</u> of incident, target and scattered particles.

(π^+, π^0) , (π^-, π^0)

(π^+, π^-) , (π^-, π^+) double charge exchange reactions

At the moment pion reactions <u>lack</u> the good <u>energy</u> resolutions that can be achieved.

Double β-decay:

$$A(Z,N) \longrightarrow A(Z+2, N-2) + 2e^- + 2\bar{\nu}_e$$

$$A(Z,N) \longrightarrow A(Z-2, N+2) + 2e^+ + 2\nu_e$$

Such process are caused by <u>second-order</u> perturbations in weak interactions $\longrightarrow$ much slower than β-decay

$\longrightarrow$ long life time $T_{1/2} \sim 10^{20}$ yr

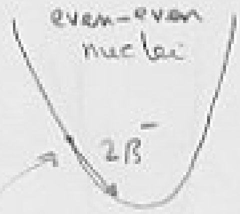
→ Process with such long life time may be observed where ordinary β-decay and faster process are forbidden by Q-value considerations.

Ex.

$^{82}_{34}Se$ is stable against β^- decay to $^{82}_{35}Br$ $(Q = -0.90\,\text{MeV})$

However $^{82}_{34}Se \longrightarrow {}^{82}_{36}Kr + 2e^- + 2\bar{\nu}_e$ $(Q = +3.00\,\text{MeV})$

A large number of nuclei (even-even) undergo double β-decay



near, but not at, the bottom of the valley of stability.

Reason:

Even-even nuclei are more tightly bound compared with neighboring odd-odd nuclei

On the other hand neighboring even-even nucleus with two more protons or two less neutrons may be even more tightly bound because of a larger symmetry energy. $(\sim (N-Z)^2)$

The number of nuclei capable for double β^--decay > for the case of double β^+-decay

Ex. $^{106}_{48}Cd \longrightarrow {}^{106}_{46}Pd + 2e^+ + 2\nu_e$

However $Q = 0.7\,\text{MeV}$ < typical value for double β^--decay (3. MeV)

Question:

Neutrinoless double β-decay is possible?

If neutrinos are Majorna fermions (particle = antiparticle)

we can imagine → neutrino from first β-decay in a double β-decay is absorbed in the intermediate state

→ absorbtion induces the emission of second charged lepton

On the other had such a neutrinoless double β-decay is forbidden if the neutrinos are Dirac particles

$$\text{neutrino} + {}^{37}Cl \longrightarrow e^- + {}^{37}Ar$$

confirms the neutrinos and antineutrinos are different particles

The source of neutrino in this experiment was a reactor which produces mainly $\bar{\nu}_e$. ($\sigma_{observed} <<$ than one expected for Majorna neutrinos)

However in a neutrionless double β-decay, the neutrinos are virtual particles and may be different from real neutrinos. (i.e. they may be Majorna particles).

→ double β-decay can proceed on a much faster scale (~ 6 orders of magnitude).

How to distinguish between the two types of double β-decay?

In neutrinoless double β-decay $T_{e_1} + T_{e_2} = Q$

In the other one $T_{e_1} + T_{e_2}$ = continuous spectrum of energy given by energy-momentum conservation of the 5-body final state.

Another processes; (for double β-decay in addition to two-neutrino and neutrinoless modes.)

$\Delta \xrightarrow{\text{weak}}$ nucleon + 2 charged leptons

Note: Normal decay of Δ: $\Delta \xrightarrow{\text{strong}}$ nucleon + pion

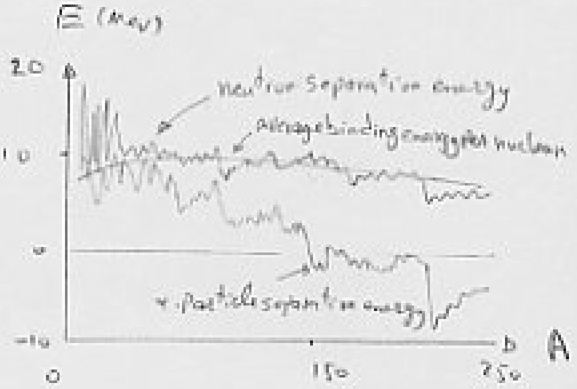
The other possibility

$\Delta \xrightarrow{\text{weak}}$ nucleon + 2 charged leptons + Majoron

boson

5-6 Alpha-particle Decay

Barrier for α-particle emission;



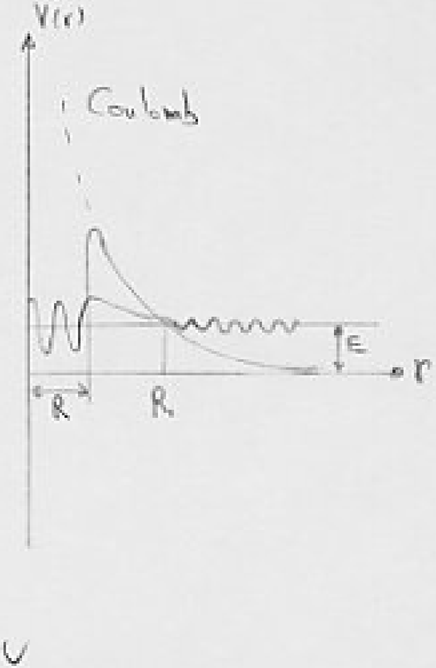
In light nuclei, the threshold for α-particle emission $\sim$ nucleon emission (Fig.)

$\longrightarrow$ α-decay is not energetically possible for $A < 150$ or 50

However, even for heavy nuclei $T_{1/2}(\alpha) =$ long by strong int. time scales.

Furthermore $K_\alpha \sim 4$ to 9 MeV while $T_{1/2}(\alpha)$ differ by a wide range for these energies.

$\longrightarrow$ α-decay wass a puzzle in the beginning.



Consider the inverse process;

An α-particle approaching a heavy nucleus from large distances

Outside the barrier:

Interaction = purely Coulomb repulsive

$$V_c(r) = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{2Ze^2}{r} = \frac{2Z\alpha\hbar c}{r} = \frac{2.88Z}{r} \text{ MeV}$$

$\longrightarrow$ fm

Inside the nuclear surface;

Interaction = Coulomb + short range nuclear forces

At $r<R$, the two int. pot. must produce a minimum

(for a bound system)

For simplicity we take a square attractive well (Fig.)

In this approx. the height of the barrier may be estimated from the amount of work required to overcome Coulomb repulsion in bringing an α-particle to the surface of a heavy nucleus such as $^{238}_{92}U$

$$E_c = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{2Ze^2}{R} = \frac{2Z\alpha\hbar c}{r_0 A^{1/3}} \simeq 35 \text{ MeV}$$

(for this mass number)

Better estimates shows, $E_c \sim 30$ MeV

Classicaly, in order for an α-particle to be emitted from a nucleus it must acquire enough energy to reach the top of the pot. barrier.

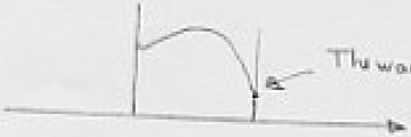
Once it is there, it can leave the nucleus carrying with it all the energy it acquired.

→ when the α-particle is far away from the nucleus, it should have a kinetic energy at least equal to the barrier height.

However, the observed K_α << this barrier height

⟶ Different mechanism is operating here

⟶ Tunneling



i) Saturation of nucl. force forms α-clusters into tightly bound groups of 4-nucleons inside a nucleus.

⟶ $E_B/\text{nucleon}$ (between α-clusters) << that between nucleons inside such a cluster

ii) Coulomb repulsion in heavy nuclei is higher than that in light nuclei

⟶ (i)(ii) → $Q(\alpha) > 0$ for $A > 150$

or separation energy < 0

Decay Probability;

$$\log_{10} W = C - \frac{D}{\sqrt{E_\alpha}}$$ empirically (from μs to 10^{17} yrs)

(Geiger-Nuttall law)

C, D weakly z-dep.

Theoretically;

$$W = P_\alpha \nu T$$

P_α: Probability of finding α-cluster inside a heavy nucleus

ν: The frequency of the α-cluster appearing at the inside edge of the pot. barrier

T: Transmission coef. for the α-cluster to tunnel

$$\nu \sim V_\alpha, \text{ and } \nu \sim \frac{1}{\text{size of the pot.}} \rightarrow \nu = \frac{V_\alpha}{2R} = \frac{\sqrt{2K_\alpha/M_\alpha}}{2R}$$

K_α depends on the depth of the pot. well and is not well known.

It is reasonable to take $\underbrace{K_\alpha}_{\text{Kinetic energy of α-cluster}} = \underbrace{E_\alpha}_{\text{Kinetic energy of α-particle}}$

$$\nu = \frac{\sqrt{2E_\alpha/M_\alpha}}{2R} = \frac{\sqrt{E_\alpha}}{A^{1/3}} \times 2.9 \times 10^{21}\ s^{-1}$$ E_α: in MeV $R = r_0 A^{1/3}$

$r_0 = 1.2$ fm

$\rightarrow \nu = 10^{21}\ s^{-1}$ for ^{238}U with $E_\alpha = 5.6$ MeV

an order of magnitude larger than the best values deduced from measurements.

Some reasons for this <u>poor</u> agreement come from;

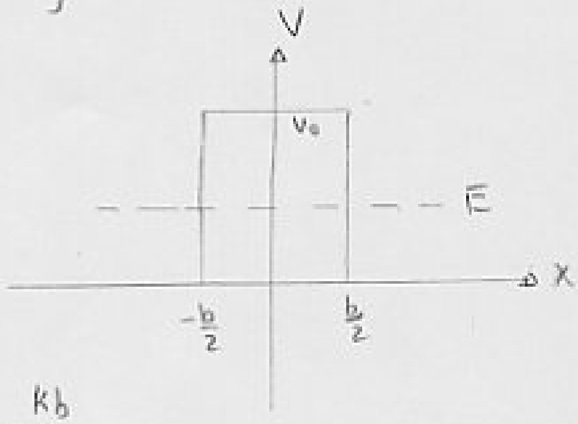
i) Heavy nuclei are not spherical in shape

ii) $K_\alpha = E_\alpha$ assumption

Transmission Coef. ;

The transmission coef. T for a particle to tunnel through a one-dim. square pot. barrier of height V_0 and width b is given by;

$$T = \left[1 + \frac{V_0^2}{4E(V_0-E)} \sinh^2 kb \right]^{-1}$$



$$k = \frac{1}{\hbar}\sqrt{2m(V_0-E)}$$

In the limit $V_0 \gg E$

$\rightarrow kb \rightarrow \infty$ and $\sinh kb \rightarrow e^{kb}$

$$T \rightarrow e^{-2kb}$$

e^{-kb} : attenuation of the amplitude of the wave func. in going through the barrier

$\rightarrow$ It is reasonable $T \sim (e^{-kb})^2$

For our case;

$V_0 \simeq 30$ MeV $\qquad E_\alpha = 4-9$ MeV $\rightarrow V_0 >> E_\alpha$

True Pot. barrier in heavy nuclei is more complicated.
However, the results of one-dim treatment remains valid on the whole as long as the Pot. well is spherically symmetric.

The radial Part of Schrödinger equ.;

$$\frac{d^2 u(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[(E_\alpha - V(r)) - \frac{\ell(\ell+1)}{2\mu r^2}\right]u(r) = 0$$

μ: reduced mass of α-particle $\qquad u(r) = \frac{R(r)}{r}$

$V(r) = V_c(r)$ $\qquad$ Since we are interested in the region just outside the range of nuclear force

$$V_{eff} \equiv V_b(r) = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{2Ze^2}{r} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} = \frac{2Z\alpha\hbar c}{r} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}$$

$$\rightarrow \quad \frac{d^2 u(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E_\alpha - V_b(r)\right]u(r) = 0$$

This result is essentially the same form as a square well used for our simple model above; the major difference is that the barrier height is now a func. of r.

The equ. must be solved by WKB method.

The form of the sol., however remains very similar to our sol.

if we make the replacement;

$$kb \longrightarrow \int_R^{R_1} \sqrt{\frac{2\mu}{\hbar^2}} \left[V_b(r) - E_\alpha \right]^{\frac{1}{2}} dr \quad \left\{ \begin{array}{l} kb = b\sqrt{\frac{2\mu}{\hbar^2}} \left[V_0 - E \right]^{\frac{1}{2}} \\ \text{if } V = V(r) \\ \Rightarrow kb = \int_R^{R_1} \sqrt{\frac{2\mu}{\hbar^2}} \left[V(r) - E \right]^{\frac{1}{2}} \end{array} \right.$$

$r = R$ nuclear surface

$r = R_1$ given by the relation $V_b(R_1) - E_\alpha = 0$

For $l = 0 \longrightarrow R_1 = \left[\frac{1}{4\pi\epsilon_0} \right] \frac{2Ze^2}{E_\alpha} = \frac{2Z\alpha\hbar c}{E_\alpha}$ (1)

In this case; $\left(V_b(r) = \frac{2Z\alpha\hbar c}{r} \right)$,

$$\ln T = -2kb = -\frac{2R_1}{\hbar}\sqrt{2\mu E_\alpha} \left[\cos^{-1}\sqrt{\frac{R}{R_1}} - \sqrt{\frac{R}{R_1}\left(1 - \frac{R}{R_1}\right)} \right]$$

For $R << R_1$ $\left\{ \begin{array}{l} \cos^{-1}\sqrt{\frac{R}{R_1}} \simeq \pi/2 - \sqrt{R/R_1} \\ \sqrt{\frac{R}{R_1}\left(1 - \frac{R}{R_1}\right)} \simeq \sqrt{R/R_1} \end{array} \right.$

$$\xrightarrow{(1)} \ln T = -\frac{2R_1}{\hbar}\sqrt{2\mu E_\alpha}\left(\pi/2 - 2\sqrt{R/R_1}\right) = 3.26\sqrt{ZA^{1/3}} - 3.97\frac{Z}{\sqrt{E_\alpha}}$$

$(R \sim A^{1/3})$ $\quad$ E_α: MeV

Estimation of P_α is much <u>harder</u>.

$P_\alpha \sim \Psi_{\text{nucl. state}}$ $\quad$ Ψ : different from nucleus to nucleus

We assume P_α is the same in all heavy nuclei.

$P_\alpha = 0.1$ in heavy nuclei

Energy and mass dependence:

$$\log_{10} W = \log_{10} P_\alpha + \log_{10} \nu + \log_{10} T$$

$$= 20.46 + \log_{10} \frac{\sqrt{E_\alpha}}{A^{1/3}} + 1.42\sqrt{Z A^{1/3}} - 1.72 \frac{Z}{\sqrt{E_\alpha}}$$

The dominant energy dependence comes from the last term in agreement with the empirical result of Geiger-Nuttall law.

In the above discussion we have assumed that there is only a single E_α for all α-particles emerging from a nucleus.

In fact there are several groups of α-particles emitted by the same parent.

For example ^{212}Bi has more than 14 different decays known and each one leaves the residual nucleus in a different state.

5-7 Nuclear Fission ;

Heavy nuclei can increase their $E_B \xrightarrow{\text{by}}$ fission

There are two types of fission $\begin{cases} \text{Spontaneous fission (rare)} \\ \text{Induced} \quad " \end{cases}$

Fission involves the movement of many nucleons at the same time ⟶ an example of the collective degrees of freedom in nuclei

$$E_B(N,Z) = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z(Z-1)}{A^{1/3}} - \alpha_4 \frac{(N-Z)^2}{A} + \Delta$$

Weizacker semi-empirical mass formula

Due to the deformation ;

$\begin{cases} \text{Small increase in surface area} \\ \text{Repulsive Coulomb energy is decreased (due to the increament of distance between two groups of nucleons)} \end{cases}$

⟶ E_B is increased .

Large deformations occurs in nuclei with $A > 240$

⟶ Fission becomes important in nuclei with $A > 240$

Ex.

$^{232}U \qquad T_{1/2}(\text{fission}) \simeq 10^{14}\,yr \qquad T_{1/2}(\alpha) = 2.85\,yr$

partial   partial

$^{254}_{98}Cf \qquad T_{1/2}(\,\text{"}\,) \simeq 60\,\text{days} \qquad \text{Branching ratio} = 99\%$

Induced fission:

$A \longrightarrow B_1 + B_2 + B_3 + \cdots$ Spontaneous fission

$A + a \longrightarrow B + b$ Induced "

small mass   of comparable mass

$$^{235}_{92}U + n \rightarrow {}^{236}U^{*} \longrightarrow {}^{92}_{36}Kr + {}^{142}_{56}Ba + 2n \qquad (1)$$

thermal neutrons   prompted neutrons

$K \approx 0$

The excitation energy of $^{236}U^{*}$ is 6.5 MeV

The ground state of ^{236}U is unstable toward α-emission with $T_{1/2} = 2.4 \times 10^{7}\,yr$

The energy liberated in (1) is around 180 MeV.

^{92}Kr and ^{142}Ba are neutron unstable.

$^{92}Kr \longrightarrow {}^{86}Kr + 6n$

$^{142}Ba \longrightarrow {}^{138}Ba + 4n$ ⟶ delayed neutron

directly or via other unstable nuclei formed

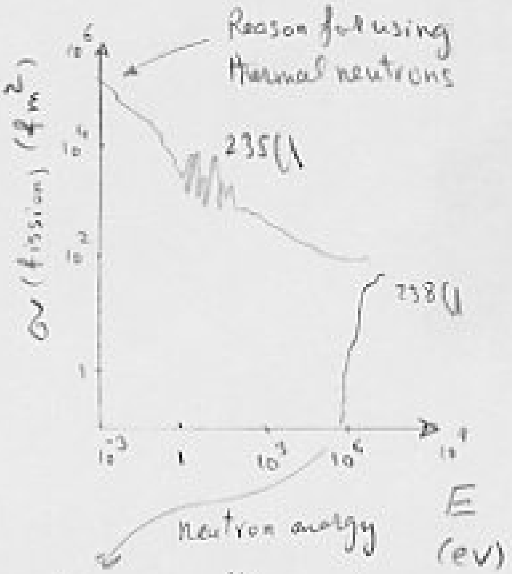
If neutron excess is large, fission into two medium weight nuclei with nucleon number $\simeq \frac{1}{2}A$ each, will result the fragments to be very far from the stability valley.

⟶ Prompt neutrons would have to be emitted

⟶ This in turn increases the number of final states.

On the other hand Phase space considerations strongly favor exit channels with the minimum number of products

⟶ Asymmetric fission solves the problem.



1 MeV ~ 10^{10} K neutrons

Fission barrier;

Fission barrier tries to prevent the spontaneous fission.
A crude model of the fission barrier that inhibits fission may be constructed from a liquid drop model.

Consider a hypothetical nucleus, $A = 300$ $Z = 100$

$\xrightarrow{\text{fission}}$ to two equal fragments $A = 150$, $Z = 50$

$$E_B(Z,N) = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z(Z-1)}{A^{1/3}} - \alpha_4 \frac{(N-Z)^2}{A} + \Delta$$

The volume energy term is unchanged under fission;

$$\Delta E_{Voul.} = \alpha_1 \left[2\left(\tfrac{1}{2}A\right) - A \right] = 0$$

Similarly

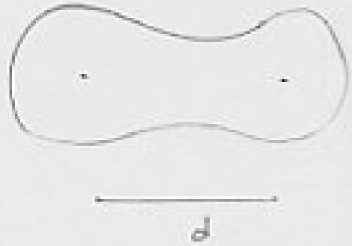
$\Delta E_S \simeq 0$ $\quad\quad \Delta E_\Delta \simeq 0$

$$\Delta E_{surface} = \alpha_2 \left[2\left(\tfrac{1}{2}A\right)^{2/3} - A^{2/3} \right] = 0.26\, \alpha_2 A^{2/3} \simeq 200 \text{ Mev}$$

(the loss in binding energy) $\alpha_2 = 17$ Mev

Before the fission $d = 0$

After " " $d > 2R$



$R_f = r_0 (150)^{1/3} \simeq 6.4$ fm each fragment

The contribution from surface energy at distances between $d=0$ and $d \approx 13\,fm$ is given in the Fig.

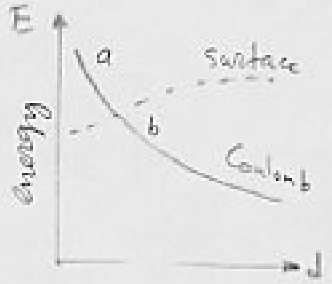
$$\Delta E_c = \alpha_3 \left[2\frac{\frac{Z}{2}(\frac{Z}{2}-1)}{(\frac{A}{2})^{1/3}} - \frac{Z(Z-1)}{A^{1/3}} \right] = 2(275) - 900 = -350\,\text{MeV}$$

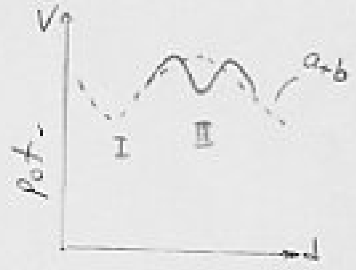
$\alpha_3 = 0.6$ MeV

Such a gain in the Coulomb energy assumes that the two fragments are at infinite distance apart at the end.

Coulomb decreases as $\frac{1}{r}$ (as shown in the Fig.) with increasing d.

The net gain $= -350 + 200 = -150$ MeV





fission barrier

— Realistic pot. barrier