Chapter 4

Bulk Properties of Nuclei

Most of the data (experimental) which give us information about the nuclei are divided into 4-Categories:

$\begin{cases} \text{1- energies} \\ \text{2- static moments} \\ \text{3- transition probabilities} \\ \text{4- reaction rates} \end{cases}$ energy $\begin{cases} \text{ground state} \\ \text{excited} \end{cases}$

4-1 Nuclear Size:

Befor 1911, The model of atom

$R \sim 10^{-10}$ m

uniform charge (+, -) distribution

The Coulomb scattering differential cross section is given by Rutherford formula:

$$\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} = \left\{\left[\frac{1}{4\pi\epsilon_0}\right]\frac{zZe^2}{4T\sin^2\left(\frac{\theta}{2}\right)}\right\}^2 = \left\{\frac{\alpha\hbar c z Z}{4T}\,\frac{1}{\sin^2\left(\frac{\theta}{2}\right)}\right\}^2$$

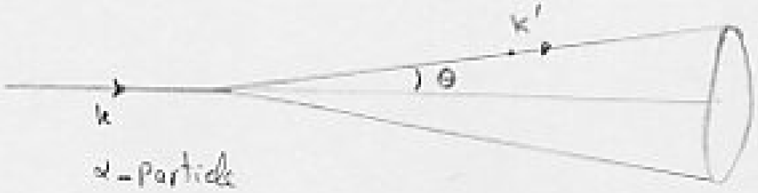
for the mentioned charge distribution.

The incident particles are α-particles ($z=2$) from radioactive materials (low energy)

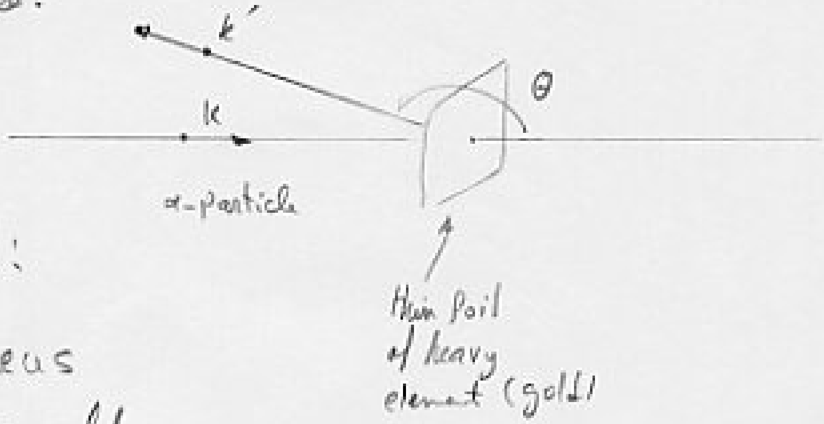
T: Kinetic energy of α-particles in CM.

With this partial cross section $\sim \dfrac{1}{\sin^4(\frac{\theta}{2})}$ we expect most of the α-particles to emerge in a small forward cone. (a).

The experimental results were quite different (b)



Large angle scattering of α-particles was observed.



Explanation of Rutherford:

Positively charged nucleus is concentrated in a very small part of the volume occupied by the atom.

The energies of the α-particles at that time were limited.

For example, $E_\alpha \simeq 7.68$ MeV from naturally occuring radioactive material ^{214}Po.

$$E_\alpha \simeq 7.68 \text{ MeV} \xrightarrow{\text{corresponds}} \lambda_d \simeq 5 \text{ fm} \quad \text{(de Broglie wave length)}$$

$$\lambda_d \sim R_{\substack{\text{heavy}\\ \text{nuclei}}} \qquad \lambda_d \gg R_{\alpha\text{-particle}}$$

As E_2 increases $\longrightarrow$ λ_d decreases

$\longrightarrow$ The scatt. become sensitive to $\begin{cases} \text{finite size of projectile and target} \\ \text{nucl. force interacting between nucleons.} \end{cases}$

For the moment we are interested in the size of nucleus. So we avoid the complications due to finite size of the projectile and the nucl. int. between projectile and target.

Leptons are the ideal probs; $\begin{cases} \text{They are point particles} \\ \text{They do not participate in strong ints.} \end{cases}$

i) Neutrinos would have been the best choice, since they interact with the nucleus at extremely short-range weak ints.

But $\begin{cases} \text{1- Their detection is difficult} \\ \text{2- " cross section is small} \\ \text{3- Preparing a good beam is hard.} \end{cases}$

ii) Electrons (e) and to a lesser extent muons, become the common lepton probs used to measure nuclear sizes and charge distributions.

But one of the drawback of electromag. probs. is that they <u>are not</u> sensitive to the distribution of <u>neutral</u> neutrons in nucleus.

$\Rightarrow$ The study of nuclear matter distribution requires <u>hadronic</u> probs.

From electron scatt.:

$$R_{ch} = r_0 A^{1/3}$$ charge radius (on average)

$r_0 = 1.2$ fm

When the energy of the electrons is increased to that;

$\lambda_d << $ size of nucleon

$\Rightarrow$ the scatt. becomes sensitive to charge distribution within each individual nucleon, and the quarks begin to play a role in the measured results.

4-2 Electron Scatt. Form Factor

In derivation of Rutherford formula we assumed:

1) Relativistic effects are ignored.
2) Both projectile and target are considered to be spinless, ($J=0$)
3) " " " " " assumed to be point charges.

For α-particle:

For low energy α-particles cond. (1) is satisfied.
For α-particle scattering off even-even nuclei (ground state $J=0$) cond. (2) is also satisfied.

However cond. (3) is violated by the finite size of nuclear charge distribution.

Scattering of point particles:

Electron scattering off nuclei is different from low energy α-particle scattering.

i) Electron is a Dirac particle ($s=\frac{1}{2}$)
ii) m_{0e} = small

$\rightarrow$ Electrons with $\lambda_d \lesssim$ nuclear dimensions, $\left(\lambda=\frac{h}{p}\right)$ are already relativistic particles, with $E \gg m_{0e}c^2$

For scattering relativistic electrons off point-charged particles in CM;

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \left\{ \left[\frac{1}{4\pi\epsilon_0}\right] \frac{e^2 Z E}{2p^2c^2 S^2(\frac{\theta}{2})} \right\}^2 \left\{ 1 - \frac{p^2c^2}{E^2} S^2(\tfrac{\theta}{2}) \right\}$$

$$= \left\{ \frac{\alpha \hbar c Z E}{2p^2c^2 S^2(\frac{\theta}{2})} \right\}^2 \left\{ 1 - \beta^2 S^2(\tfrac{\theta}{2}) \right\} \quad \text{Mott formula}$$

where $E = \sqrt{(Pc)^2 + (m_{0e}c^2)^2}$, $\beta = \frac{v}{c}$

In nonrelativistic limit; $E \simeq m_e c^2$ $\quad ((Pc)^2 << (m_e c^2)^2)$

$$T = mc^2 - m_{0e}c^2 = \left(\frac{m_{0e}}{\sqrt{1-\beta^2}} - m_0\right)c^2 = m_{0e}c^2\left[(1-\beta^2)^{-\frac{1}{2}} - 1\right]$$

$$= m_{0e}c^2\left[1 + \tfrac{1}{2}\beta^2 + \dots - 1\right] \simeq m_{0e}c^2 \frac{\beta^2}{2} = \frac{1}{2} m_{0e} V^2 = \frac{p^2}{2m_{0e}}$$

$$\frac{E}{2p^2c^2} \approx \frac{m_{0e}c^2}{2p^2c^2} = \frac{1}{4T} \qquad \text{for } v << c \ (\beta \to 0)$$

$$\text{Mott formula} \xrightarrow{\beta \to 0} \text{Rutherford formula}$$

It is often useful to express the scatt. results as a func. of the momentum transfer $\hbar \bar{q}$ from the electron to the nucleus.

In the nonrelativistic limit, we can define a 3-momentum transfer:

$$\bar{q} = \bar{k}_i - \bar{k}_f$$

$\hbar\bar{k}_i$: incident electron momentum
$\hbar\bar{k}_f$: final " "

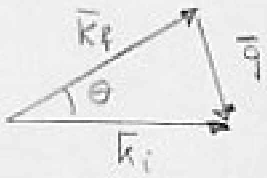
If the electron energy is high, it is more appropriate to use the 4-momentum transfer

$$t = \frac{(E_i - E_f)^2}{\hbar^2 c^2} - (\bar{k}_i - \bar{k}_f)^2 \qquad \text{Lorentz scalar}$$

For elastic scattering $E_f = E_i = E$ (in CM)

$$q^2 = -t = 4k^2 \sin^2(\tfrac{\theta}{2})$$

In the limit of $E \to pc = \hbar k c$ (relativistic)



$(\beta \to 0)$ $\begin{cases} q^2 = k^2 + k^2 - 2k^2\cos\theta \\ \quad = 2k^2(1-\cos\theta) \\ \quad = 4k^2\sin^2\frac{\theta}{2} \end{cases}$

$$q^2 \simeq \left(\frac{2E}{\hbar c}\right)^2 \sin^2\left(\frac{\theta}{2}\right)$$

where $k_i \simeq k_f = k$

$$q = f(\theta, E)$$

Also;

$$d\Omega = dq\, d(\cos\theta) \qquad d\Omega' = d(\cos\theta)\int dq = 2\pi\, d(\cos\theta)$$

$$q^2 = 4k^2 \sin^2\frac{\theta}{2} \quad \rightarrow \quad dq^2 = 4k^2\, d(\sin^2\frac{\theta}{2})$$

$$\rightarrow dq^2 = 2k^2\, d(1-\cos\theta) \qquad dq^2 = -2k^2\, d(\cos\theta)$$

$$d(\cos\theta) = -\frac{dq^2}{2k^2}$$

$$d\Omega' = -\frac{\pi}{k^2} dq^2 \qquad |d\Omega'| = \frac{\pi}{k^2} dq^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \left\{\frac{\alpha\hbar c Z E}{2p^2c^2 \sin^2(\frac{\theta}{2})}\right\}^2 \left(1-\beta^2 \sin^2(\frac{\theta}{2})\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \left\{\frac{\alpha\hbar c Z E}{2p^2c^2 \sin^2(\frac{\theta}{2})}\right\}^2$$ (neglecting the second term due to $\sin^2(\frac{\theta}{2})$) electron scatt.

(the same result can be obtained by $\frac{1}{4\pi} \rightarrow \frac{E}{2p^2c^2}$ prescription)

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dq^2}\frac{k^2}{\pi}$$

$$\frac{d\sigma}{dq^2} = \frac{\pi}{k^2}\left\{\frac{\alpha\hbar c Z E}{2p^2c^2 \sin^2(\frac{\theta}{2})}\right\}^2 = \frac{\pi}{k^2}\left\{\frac{\alpha\hbar c Z(\hbar k c)}{2p^2c^2 \sin^2(\frac{\theta}{2})}\right\}^2$$

in the limit of $E \rightarrow pc = \hbar kc$ (relativistic limit)

$$\frac{d\sigma}{dq^2} = \frac{\alpha^2 Z^2 (\hbar k c)^2 (\hbar c)^2}{4\,(\hbar k c)^4 \sin^4(\frac{\theta}{2})}\frac{\pi}{k^2}$$

$$\left(\frac{d\sigma}{dq^2}\right)_{Rutherford} \simeq \frac{4\pi Z^2\alpha^2}{q^2} \qquad \frac{d\sigma}{dq^2} = f(q) \text{ not } f(E) \text{ explicitly}$$

Since $M_{\text{nucleon}} \gg m_e$

$\xrightarrow[\text{to express}]{\text{it is more convenient}}$ the scatt. cross section in the <u>lab.</u> frame (target, initially at rest)

$\longrightarrow E_i - E_f = E_{\text{Target}}$ the energy taken away by the recoil of the target

On the other hand:

$$\bar{P}_i - \bar{P}_f = \bar{P}_T \longrightarrow P_i^2 + P_f^2 - 2 P_i P_f \cos\theta = P_T^2$$

$$\underbrace{(P_i - P_f)^2}_{\approx 0} + 2 P_i P_f (1 - \cos\theta) = P_T^2 \longrightarrow 2 P_i P_f (1 - \cos\theta) = P_T^2$$

$$\frac{2}{2M} P_i P_f (1 - \cos\theta) = \frac{P_T^2}{2M}$$

Since $E_T = \frac{P_T^2}{2M} \longrightarrow E_i - E_f = \frac{P_i P_f}{M} 2 \sin^2(\frac{\theta}{2})$

$$E_i - E_f = \frac{(P_i c)(P_f c)}{Mc^2} 2 \sin^2(\frac{\theta}{2})$$

In the limit $E \to PC$ (relativistic)

$$\frac{E_f}{E_i} = \frac{1}{1 + \frac{2E_i}{Mc^2} \sin^2(\frac{\theta}{2})} \qquad \text{(lab. frame)}$$

In the limit $m_e c^2 \simeq$ ignorable, for elastic scatting of unpolarized electrons off spinless ($J=0$) point-charged particles, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} = \left(\frac{Z\alpha\hbar c}{2E_i S^2(\frac{\theta}{2})}\right)^2 \frac{E_f}{E_i} C_o^2\left(\frac{\theta}{2}\right) \quad (\text{lab. frame})$$

For targets with a finite spin, additional contribution from the magnetic scattering is also present.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Dirac}} = \left(\frac{Z\alpha\hbar c}{2E_i S^2(\frac{\theta}{2})}\right)^2 \frac{E_f}{E_i} \left\{\underbrace{C_o^2\left(\frac{\theta}{2}\right)}_{\text{electric term}} + \underbrace{\frac{(\hbar q)^2}{2(Mc)^2} S^2\left(\frac{\theta}{2}\right)}_{\text{mag. term}}\right\}^?$$

(lab. frame)

Relative contribution of these two terms:

Using $q^2 \simeq \left(\frac{2E}{\hbar c}\right)^2 S^2\left(\frac{\theta}{2}\right)$

$$\frac{(\hbar q)^2}{2(Mc)^2} \simeq 2\left(\frac{E}{Mc^2}\right)^2 S^2\left(\frac{\theta}{2}\right)$$

If E (electron energy) $\ll Mc^2$ (target rest mass)

$\rightarrow$ Second term $\ll 1$

$\rightarrow$ The mag. scatt. term may be ignored in elastic scatt.

The exceptions are in $\left\{\begin{array}{l}\text{high energies}\\ \text{backword angles}\end{array}\right.$

At $\theta \to \pi \longrightarrow \begin{cases} \cos^2(\frac{\theta}{2}) \to 0 \\ \sin^2(\frac{\theta}{2}) \to 1 \end{cases}$

For inelastic scatt. : $\begin{cases} \text{the mag. term dominates} \\ \text{the electric term is forbidden by selection rules.} \end{cases}$

Charge form factor:

Scatt. described by <u>Mott</u> or <u>Dirac</u> formula are for point charges.

Such a picture is correct for nuclei where;

$\lambda_{incident} \gg$ nucl. dimensions (low energy)

As the $E_{bombarding}$ increases $\longrightarrow$ the <u>charge</u> and magne<u>tization</u> density distribution in a nucleus become <u>visible</u> to the incident electrons

$\longrightarrow$ The scatt. results are sensitive to <u>finite</u> distributions of charge and magnetism in the target.

Let $\psi(\bar{r})$: the ground state of a nucleus

$$\rho_{ch}(\bar{r}) = Z\,|\psi(\bar{r})|^2 \qquad \text{charge density dist.}$$

where $\int \rho_{ch}(\bar r)\, d\bar r^3 = Z$ number of protons

The Fourier tr. of $\rho_{ch}(\bar r)$ is given by:

$$F(\bar q) = \int \rho_{ch}(\bar r)\, e^{iq\cdot r}\, dV$$

charge or longitudinal form factor

For nuclear size studies; we are primarily interested in the radial dependence of the density.

For this reason $\longrightarrow$ we can take an average over the angular distribution and consider only $\rho_{ch}(r)$, the radial dist. of the charge density.

$$F(q) = \int \rho_{ch}(r)\, e^{iqr\cos\alpha}\, dV = \int_0^\infty dr\, r^2 \int_0^{2\pi} d\varphi \int d(\cos\alpha)\, e^{iqr\cos\alpha}\, \rho_{ch}(r)$$

$$= 2\pi \int r^2 dr \left[\frac{1}{iqr}\, e^{iqr\cos\alpha}\right]_{-1}^{1} \rho_{ch}(r)$$

$$= -\frac{i2\pi}{q} \int r\, dr\, \rho_{ch}(r) \left(e^{iqr} - e^{-iqr}\right)$$

$$= \frac{4\pi}{q} \int \rho_{ch}(r)\, \sin(qr)\, r\, dr$$

We will see F is a func of q^2 (since only q^2 is a proper Lorentz scalar).

$$\longrightarrow F(q^2) = \frac{4\pi}{q} \int \rho_{ch}(r)\, \sin(qr)\, r\, dr$$

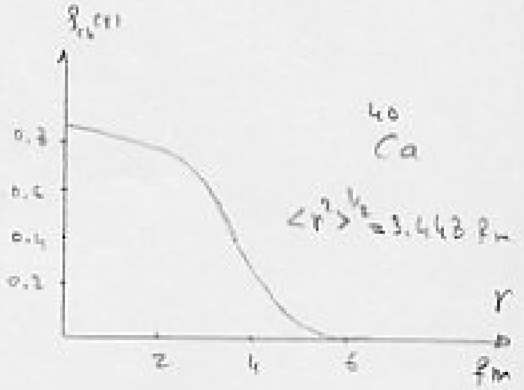
The cross section for a charge distribution (not point charge) for elastic scatt. of electrons of a finite nucleus;

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{point} |F(q^2)|^2$$

with the groundstate of target with $J=0$

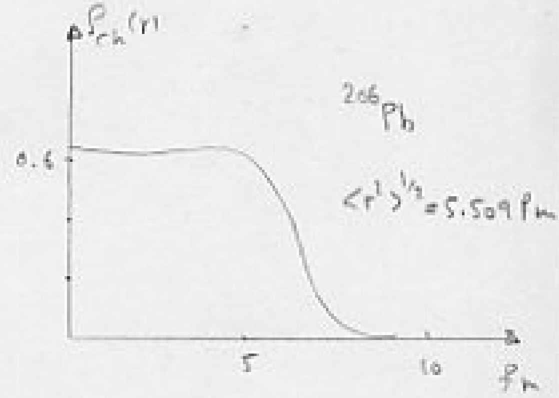
By invers tr.:

$$\rho_{ch}(r) = \frac{1}{2\pi^2 r}\int_0^\infty F(q^2)\,\Sigma(qr)\,q\,dq$$



Transverse form factor:

For elastic scatt. off $J=0$ states, only the longitudinal form factor can contribute to $\frac{d\sigma}{d\Omega}$.



The reason:

This is the only operator having $K=0$ (spherical tensor rank) and hence, allowed by angular momentum coupling requirement.

More generally:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{point} \left\{ |F(q^2)|^2 + \left(\frac{1}{2} + \tan^2\left(\frac{\theta}{2}\right)\right) |F_T(q^2)|^2 \right\}$$

$F_T(q^2) \xrightarrow[\text{decomposed into}]{\text{can be further}}$ (electric + magnetic) multipole components

Since tensorial ranks of these operators > 0 $\longrightarrow$

i) their contributions are left for elastic scatt. involving states with $J > 0$

ii) and for inelastic scatt. (other than those with both initial and final spin zero)

4-3 Charge Radius and Charge Density:

Charge Radius:

From electron scatt. $\longrightarrow$ $F(q^2)$ can be obtained

$\longrightarrow$ rms radius of a nucleus can be obtained in a model-indep. way.

$$F(q^2) = \frac{4\pi}{q} \int \rho_{ch}(r) \left(qr - \frac{1}{3!}(qr)^3 + \dots \right) r\, dr$$ for low momentum transfer (q).

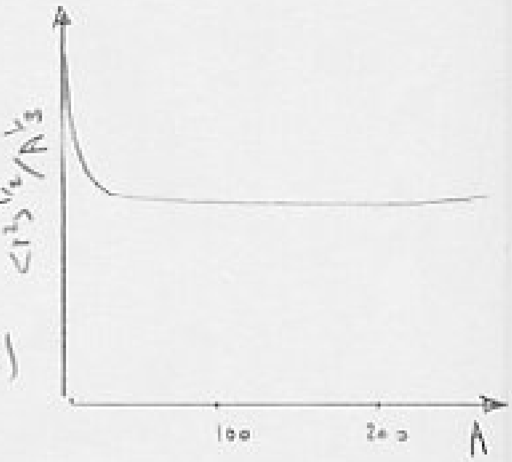
$$F(q^2) = \int \rho_{ch}(r)(4\pi r^2)\,dr - \frac{1}{6} q^2 \int r^2 \rho_{ch}(r)(4\pi r^2)\,dr + \cdots$$

$$= Z\left\{1 - \frac{1}{6} q^2 \langle r^2\rangle + \cdots\right\}$$

$$F(q^2) \simeq Z\left(1 - \frac{1}{6} q^2 \langle r^2\rangle\right) \quad \text{at low } q$$

From electron scatt.;

$$\frac{\langle r^2\rangle^{1/2}}{A^{1/3}} = 0.97 \pm 0.04\ \text{fm}$$ except for the light nuclei (A=small)



$\langle r^2\rangle^{1/2}$ data deduced from electron scatt.

This lends support to the idea that → nucleus is made of <u>incompressible</u> fluid.

Note that

$$\langle r^2\rangle^{1/2} \neq R \quad \text{nuclear radius}$$

Illustrative example:

$$\rho(r) = \begin{cases} \rho_0 & \text{for } r \leq R \\ 0 & r > R \end{cases}$$

R: radius of const. density sphere

$$V = 4\pi \int_0^\infty \rho(r) r^2\,dr = \frac{4\pi}{3} R^3 \rho_0 \quad \text{volume of the sphere}$$

$$\langle r^2\rangle = \frac{\int_0^\infty \rho(r) r^4\,dr}{\int_0^\infty \rho(r) r^2\,dr} \rightarrow \langle r^2\rangle = \frac{3}{R^3\rho_0}\int_0^\infty \rho(r)\, r^4\,dr = \frac{3}{5}R^2$$

$$\rightarrow R = \left[\frac{5}{3}\langle r^2\rangle\right]^{1/2} = 1.29\,\langle r^2\rangle^{1/2}$$

For more realistic radial density distribution,

$\frac{R^2}{\langle r^2\rangle}$ is slightly <u>smaller</u> than $\frac{5}{3}$.

$$\left\{\begin{array}{l} \frac{R^2}{\langle r^2\rangle} \simeq \frac{5}{3} \\ \langle r^2\rangle^{1/2} \simeq 0.97 A^{1/3} \end{array}\right. \longrightarrow R = r_0 A^{1/3} \quad (r_0 = 1.2\ fm)$$

Fourier-Bessel coeffs.:

Nuclear charge densities $\rho_{ch}(r)$, <u>deduced</u> from $F(q^2)$, <u>measured</u> in electron scatt. experiments are often given in terms of Fourier-Bessel coeffs.

$$\rho_{ch}(r) = \begin{cases} \sum_k a_k\, j_0\!\left(\frac{k\pi r}{R_c}\right) & \text{for } r \leq R_c \\ 0 & \text{" } r > R_c \end{cases}$$

R_c: cutoff radius

j_0: spherical Bessel func of order zero

Charge density OP is a spherical tensor of rank = 0
(carries no angular moment)

$\longrightarrow$ only j_0 ($k=0$) enters

For an inelastic transitions involving multipole excitation of order λ, spherical Bessel func. $j_\lambda(\xi)$ comes instead of $j_0(\xi)$

Since: $j_0(\xi) = \dfrac{\sin\xi}{\xi}$ and $\displaystyle\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \frac{1}{2}\delta_{mn}$

m, n: integer

$$\rho_{ch}(r) = \sum_k a_k\, j_0(k\pi \frac{r}{R_c}) \qquad r \leq R_c$$

$$\int \rho_{ch}(r)\, j_0(m\pi\frac{r}{R_c})\, dr\,(r^2) = \int \sum_k a_k\, j_0(k\pi\frac{r}{R_c})\, j_0(m\pi\frac{r}{R_c})\, dr\,(r^2)$$

$$\int \rho_{ch}(r)\, j_0(m\pi\frac{r}{R_c})\, r^2 dr = \sum_k a_k \frac{R_c^2}{mk\pi^2}\int \sin(k\pi\frac{r}{R_c})\sin(m\pi\frac{r}{R_c})\, dr$$

$$a_m = \frac{2m^2\pi^2}{R_c^3}\int_0^{R_c} \rho_{ch}(r)\, j_0(m\pi\frac{r}{R_c})\, r^2 dr$$

In practice;

$F(q^2)$ can be measured only up to some maximum $\underline{q}$.

$\longrightarrow$ $\rho_{ch}(r)$ may be determined only up to a certain accuracy.

$\xrightarrow{\text{implies}}$ there is only a finite number of Fourier-Bessel coeffs. that can be found from a given measurement.

The accuracy achieved in using a finite number of Fourier-Bessel coeffs. to represent a charge density depends somewhat on the choice of the cut off radius R_c.

Usually, R_c is taken to be slightly beyond where the density essentially drops off to zero.

$R_c \simeq 8$ fm for light nuclei

$R_c \simeq 12$ " " heavy "

Other forms of charge density:

For many practical applications, density distributions in terms of Fourier-Bessel coeffs remains too complicated.

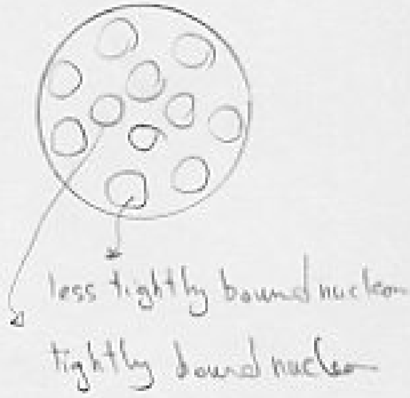
As can be seen from the Fig. (P 219)

$\rho(r)$ = essentially const except for $\begin{cases} r \simeq 0 \\ r \simeq R \end{cases}$ (particularly in heavy nuclei)

In general $r \simeq 0$ region is <u>unimportant</u>; because

$\begin{cases} \text{1- it occupies only a } \underline{\text{small}} \text{ fraction of nuclear volume} \\ \text{2- most studies are sensitive primarily to surface region.} \end{cases}$

Because of short nature of nuclear force, nucleons near the surface of the nucleus are <u>less tightly</u> bound



$\longrightarrow \rho(r) \xrightarrow{\text{drops off}} \sim e^{-r}$

$$\longrightarrow \rho_{2PF}(r) = \frac{\rho_0}{1 + e^{\frac{r-c}{z}}}$$ (const. central region and diffused edge)

Two parameter Fermi form

or, Woods - Saxon form

C and z can be determined by fitting them to densities derived from measured form factors.

ρ_0 is given by normalization.

Interpretation of c and z:

c : the radius at which $\rho(c) = \frac{\rho(\text{central})}{2}$

z : Diffuseness $\xrightarrow{\text{related}}$ surface thickness.

A better description:

Nucleus	$\langle r^2\rangle^{1/2}$	c	z	w
^{16}O	2.730±0.015	2.608	0.513	−0.051
^{28}Si	3.086±0.018	3.340	0.580	−0.233
⋮				
^{206}Pb	5.509±0.029	6.61	0.545	

$$\rho_{3PF}(r) = \rho_0 \frac{1 + w\left(\frac{r}{c}\right)^2}{1 + e^{\frac{r-c}{z}}}$$

modified Fermi form

or three-parameter Fermi form

Other forms:

$$\rho_{3PG} = \rho_0 \frac{1 + w\left(\frac{r}{c}\right)^2}{1 + e^{\frac{r^2-c^2}{z^2}}}$$ three parameter Gaussian

$$\rho_{HO} = \rho_0 \left\{1 + z\left(\frac{r}{c}\right)^2\right\} e^{-\left(\frac{r}{c}\right)^2}$$ Harmonic oscillator model

4-4 Nucleon Form Factor:

At high energies ~ Gev ;

$\lambda_{d, \text{electron}} << $ typical nucleus

Scatt. result $\xrightarrow{\text{is dominated}}$ by charge dist. of individual nucleons

$\longrightarrow$ the structure of nucleons become visible, rather than the structure of nucleus.

At these energies: we are concerned with nucleon form factors (instead of nuclear form factors).

Nucleon form factors:

Since $S_{\text{nucleon}} = \frac{1}{2} \longrightarrow$ both $\left\{\begin{array}{l}\text{electric} \\ \text{magnetic}\end{array}\right.$ scatt. contribute

to the electron scatt. cross section.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab.}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} \left\{ \frac{G_E^2(q^2) + \xi G_M^2(q^2)}{1+\xi} + 2\xi G_M^2(q^2) \tan^2\left(\frac{\theta}{2}\right) \right\}$$

Rosenbluth formula

G_E, G_M Sachs form factors

$$\xi = \left(\frac{\hbar q}{2Mc}\right)^2$$

The relation between $\begin{cases} F(q^2) \\ F_T(q^2) \end{cases}$ and $\begin{cases} G_E(q^2) \\ G_M(q^2) \end{cases}$ may be obtained by comparing the relevant formulas.

$$G_E(0) = \begin{cases} 1 & \text{for proton} \\ 0 & \text{" neutron} \end{cases} \qquad G_M(0) = \begin{cases} \mu_p & \text{for proton} \\ \mu_n & \text{" neutron} \end{cases}$$

(in terms of nuclear magnetons)

In the place of $G_E(q^2)$ and $G_M(q^2)$, the scatt. cross section may also be written in terms of Dirac and Pauli form factors $F_1(q^2)$ and $F_2(q^2)$,

$$G_E(q^2) = F_1(q^2) - \xi F_2(q^2)$$
$$G_M(q^2) = F_1(q^2) + F_2(q^2)$$

The Dirac form factor $F_1(q^2)$ represents the helicity preserving part of scatt.

The Pauli form factor $F_2(q^2)$ represents the helicity filipping part.

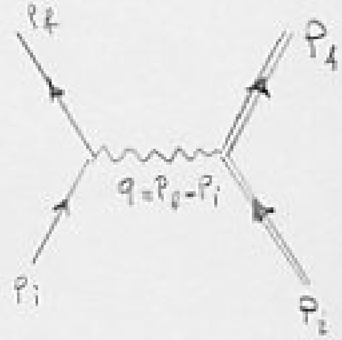
Def. - Helicity $\frac{\sigma \cdot P}{|P|}$ (the projection of σ of say electron along its direction of motion $\frac{P}{|P|}$)

Positive helicity (Right handed) $\longrightarrow$ σ and p in the same direction

Negative " (Left ") $\longrightarrow$ " " " anti parallel

Rosenbluth formula has been derived using first Born approx. involving only one photon exchange between electron and nucleon.

Higher order corrections due to two or more photon exchanges are needed. But still this formula works well, to fairly high energies.



Asymptotic form:

In the limit of q = large

$$G_E^p(q^2) = \frac{1}{\mu_p} G_M^p(q^2) = \frac{1}{|\mu_n|} G_M^n(q^2) = G(q^2) \qquad (1)$$

The func. $G(q^2)$ may be described by a dipole form:

$$G(q^2) = \frac{1}{\left\{1 + \left(\frac{q}{q_0}\right)^2\right\}^2}$$

$\hbar q_0 = 0.84$ GeV/c (empirically)

Using the dipole form:

$$\rho_{ch}(r) = \rho_0 \, e^{-q_0 r} \qquad \text{for proton}$$

using this:

$$\langle r^2 \rangle_E = \frac{\int r^2 \rho_{ch}(r) r^2 dr}{\int \rho_{ch}(r) r^2 dr} = \frac{12}{q_0^2} = (0.81 \text{ fm})^2$$

for Proton (square of charge radius)

Because of equ(1) (P 229)

$$\langle r^2 \rangle_E = \langle r^2 \rangle_M$$

square of magnetic radius for proton

Note that:

$$0.81 \text{ fm (rms)} \underset{\text{slightly}}{<} \frac{\langle r^2 \rangle^{1/2}}{A^{1/3}} = 0.97 \quad \text{for nuclei}$$

$G_E^n(q^2)$ is only known at small q ($q < 10 \text{ GeV}/c$), and:

$$G_E^n(q^2) << G_M^n(q^2)$$

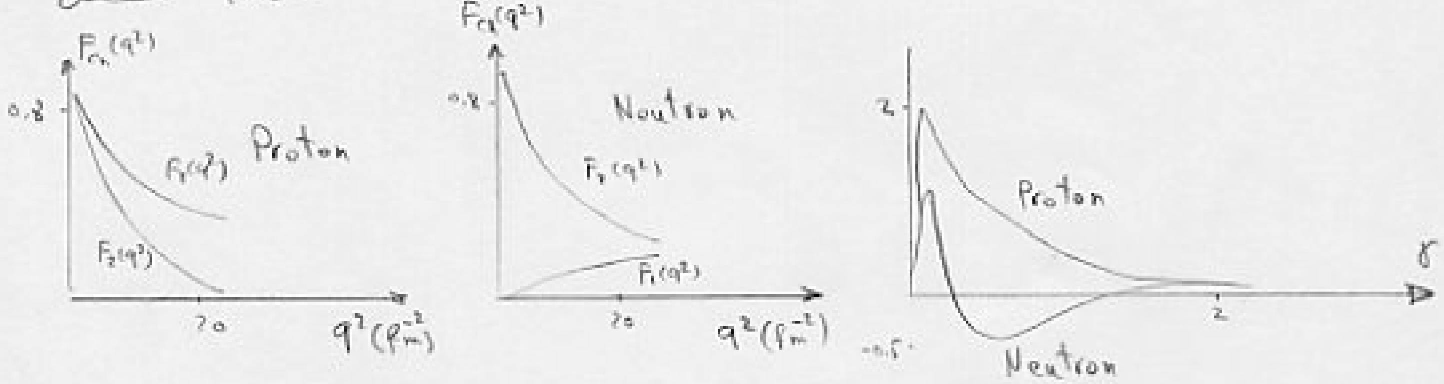
and this makes difficult its determination.

There are two other reasons which makes difficult $G_E^n(q^2)$ measurements at high q:

i) as q increases $\longrightarrow$ ξ increases

and $\frac{d\sigma}{d\Omega}$ at high q is dominated by the second term in the equ. (P227) ; i.e. magnetic term.

ii) The absence of a fixed neutron target, and our experimental knowledge of neutrons must be deduced indirectly from scatt. off such targets as deutron and ^3He.



High Energy Lepton Scattering:

As the energy of electron increases;

$\longrightarrow$ $\lambda_{\text{incident electrons}} < \sigma \sim$ nuclear size

$\longrightarrow$ the extended size of nuclear charge distribution comes into play.

$\longrightarrow$ $\frac{d\sigma}{d\Omega}$ is modified by $F_{ch}(q^2)$

In elastic scatt.;

$$\underbrace{\hbar\omega \equiv E_i - E_f}_{\text{electron}} = \underbrace{\frac{(\hbar q)^2}{2M}}_{\substack{\text{recoil of} \\ \text{nucleus}}} \quad (1)$$

M: nucleus mass

The relation between ω and q^2 given above may be taken as the def. of elastic scatt.

Quasi-elastic Scatt.:

As the energy of the incident electron is increased further

$\lambda_{\text{elect.}} \sim$ Nucleon size

At this point;

the coherence in the scatt. from several nucleons, at the same time, is lost, and

$\longrightarrow$ the scatt. takes place essentially from individual nucleons. (bound)

For elastic scatt. of electrons off free nucleons, the energy transfered is given by;

$$\hbar\omega = \frac{(\hbar q)^2}{2M_N} \quad (2)$$

M_N: nucleon mass

Equ. (1) and (2) are different: $M_N \neq M$

$\longrightarrow$ The scatt of an <u>electron</u> from a <u>bound</u> nucleon is <u>no longer</u> a <u>true</u> elastic scatt.

$\longrightarrow$ It is, instead, a <u>quasi</u>-elastic scatt.

Also there is another difference;

The nucleons in a nucleus are not <u>stationary</u> with respect to the nuclear center of mass.

$$P_F \sim \frac{\hbar}{R}$$ the average momentum of a nucleon (Fermi momentum)

R: the size of the potential well binding a nucleon to the nucleus

$R \sim$ order of the size of nucleus $\sim$ a <u>few</u> fm

$\longrightarrow P_F \sim 100-200\ \mathrm{MeV}/c$

$\longrightarrow$ there is a spread of the order of 100 MeV in the energy transfered in the <u>quasi-elastic</u> scatt. (around %10 of the total).

Structure Functions:

At θ = small (forward angles) $\longrightarrow q$ = small

$\frac{d\sigma}{d\Omega}$ is dominated by elastic scatt.

Since $F_1(q^2)$, $F_2(q^2) \xrightarrow{\text{quickly}}$ small as: $q \longrightarrow$ large

$\longrightarrow \frac{d\sigma}{d\Omega}$ (elastic) $\longrightarrow$ small

$\longrightarrow$ the importance of inelastic scatt becomes apparent.

Now,

The reaction cross section depends on $\begin{cases} \hbar\omega \\ q \end{cases}$

The scattering result is usually expressed as a double differential cross section in these two variable;

$$\frac{d^2\sigma}{dq^2\, d\omega} = \frac{4\pi Z^2\alpha^2}{q^4}\, \frac{\hbar E_f}{E_i Mc^2} \left\{ \frac{Mc^2}{\hbar\omega} F_2(q^2,\omega) \cos^2\left(\frac{\theta}{2}\right) + 2F_1(q^2,\omega) \sin^2\left(\frac{\theta}{2}\right) \right\}$$

where

$\frac{4\pi Z^2\alpha^2}{q^4}$ Rutherford scatt. cross section off a point charge

$F_1(q^2,\omega)$ and $F_2(q^2,\omega)$, (related to the form factors) defined earlier for (single) differential cross section $\frac{d\sigma}{d\Omega}$, are often referred to as the nucleon structure functions (since they express the difference of nucleon from a point particle.

still $\quad \bar{q} = \bar{k}_i - \bar{k}_f$

$$t = \frac{(E_i - E_f)^2}{\hbar^2 c^2} - (\bar{k}_i - \bar{k}_f)^2$$

$$t = \frac{E_i^2}{\hbar^2 c^2} + \frac{E_f^2}{\hbar^2 c^2} - \frac{2E_iE_f}{\hbar^2 c^2} - k_i^2 - k_f^2 + 2k_ik_f \cos\theta$$

$$E^2 = (pc)^2 + (mc^2)^2 \longrightarrow \frac{E^2}{\hbar^2 c^2} = k^2 + \frac{m^2c^2}{\hbar^2}$$

$$t = \frac{m_e^2 c^2}{\hbar^2} + k_i^2 + \frac{m_e^2 c^2}{\hbar^2} + k_f^2 - \frac{2E_iE_f}{\hbar^2 c^2} - k_i^2 - k_f^2 + 2k_ik_f \cos\theta$$

$$t = 2\frac{m_e^2 c^2}{\hbar^2} - 2\frac{E_iE_f}{\hbar^2 c^2} + 2k_ik_f \cos\theta$$

$$E_i E_f = \sqrt{(p_ic)^2 + (m_e c^2)^2}\sqrt{(p_fc)^2 + (m_e c^2)^2} = (p_ic)(p_fc)\sqrt{1 + \left(\frac{m_ec^2}{p_ic}\right)^2}\sqrt{1 + \left(\frac{m_ec^2}{p_fc}\right)^2}$$

$$E_iE_f = (p_ip_fc^2)\left[1 + \frac{1}{2}\left(\frac{m_ec^2}{p_ic}\right)^2\right]\left[1 + \frac{1}{2}\left(\frac{m_ec^2}{p_fc}\right)^2\right]$$

$$= (p_ip_fc^2)\left[1 + \frac{1}{2}\left(\frac{m_ec^2}{p_ic}\right)^2 + \frac{1}{2}\left(\frac{m_ec^2}{p_fc}\right)^2 + \phi\right]$$

$$= p_ip_fc^2 + \frac{1}{2}\frac{p_f m_e^2 c^4}{p_i} + \frac{1}{2}\frac{p_i m_e^2 c^4}{p_f} \approx p_ip_fc^2 + m_e^2c^4$$

$$t = 2\frac{m_e^2c^2}{\hbar^2} - 2\frac{P_iP_f}{\hbar^2} - 2\frac{m_e^2c^2}{\hbar^2} + 2k_ik_f\cos\theta$$

$$t \simeq -2k_ik_f + 2k_ik_f\cos\theta = -4k_ik_f\sin^2(\tfrac{\theta}{2})$$

Now;

$$t = \frac{(E_i-E_f)^2}{\hbar^2c^2} - (\bar{k}_i - \bar{k}_f)^2 = \cancel{\frac{(E_i-E_f)^2}{\hbar^2c^2}} - q^2$$

$$t \simeq -q^2$$

$$q^2 = 4k_ik_f\sin^2(\tfrac{\theta}{2}) \qquad K = \frac{P}{\hbar} = \frac{Pc}{\hbar c} \simeq \frac{E}{\hbar c}$$

In the limit $m_ec^2 \rightarrow$ ignorable

$$\rightarrow (\hbar cq)^2 = 4E_iE_f\sin^2(\tfrac{\theta}{2})$$

$$E_i - E_f \longrightarrow \begin{cases} \text{1- Target particle recoil} \\ \text{2- Exciting particles} \end{cases}$$

(in inelastic scatt.)

For high-energy electron scatt., it is customary to express the scattering in terms of two dimensionless quantities:

$$x = \frac{\hbar q^2}{2M\omega} \qquad y = \frac{\hbar\omega}{E_i} \qquad (1)$$

instead of q^2 and ω.

$$\begin{cases} (\hbar c q)^2 = 4E_iE_f \sin^2(\frac{\theta}{2}) \\ \cos^2\frac{\theta}{2} = 1 - \sin^2(\frac{\theta}{2}) \end{cases} \longrightarrow \cos^2(\tfrac{\theta}{2}) = 1 - \frac{(\hbar c q)^2}{4E_iE_f} \simeq 1 \qquad (2)$$

For high energy scatt. in the forward dir.

Also $\sin^2(\frac{\theta}{2}) = \frac{(\hbar c q)^2}{4E_iE_f} = \frac{Mc^2}{2E_f}\,xy \qquad (3)$

and $q^2 = \frac{2ME_i}{\hbar^2}\,xy$

$$(2),(3) \longrightarrow \frac{d^2\sigma}{dq^2\,d\omega} = \frac{4\pi Z^2\alpha^2}{q^4}\frac{1}{\omega}\left\{F_2(q^2,x)(1-y) + F_1(q^2,x)\,xy^2\right\}$$

double diff. cross-section for inelastic scatt.

where we have used $\frac{E_f}{E_i} = 1 - y$

Also,

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi Z^2\alpha^2}{q^4}\,\frac{2ME\nu}{\hbar^2}\left\{F_2(q^2,x)(1-y) + F_1(q^2,x)\,xy^2\right\}$$

The only assumption in the derivation was;

$m_e c^2 \longrightarrow$ ignorable

$\longrightarrow$ The formula can be applied to describe the scatt. of other charged leptons, such as muons at high energies.

For neutrino scatt.;

$$\frac{4\pi\alpha^2}{q^4}\ (\text{in Rutherford scatt.}) \xrightarrow[\text{to}]{\text{is replaced}} G_F^2/2\pi$$

G_F: Fermi coupling Const. for weak ints.

EMC effect in deep-inelastic scattering:

Many different types of final states can be reached in high-energy scatt.

If $\frac{d\sigma}{d\Omega}$ includes $\{$all the possible final states $\rightarrow$

the scatt. is called inclusive scatt. (deep inelastic scatt.)

In exclusive scatt. a particular final state is observed.

One of the interesting questions in high-energy deep inelastic lepton scatt. off nuclei and nucleons

concerns $\rightarrow$ the quark substructure of nucleons

Partons: Point like objects inside nucleons in lepton-nucleon scatt. Known as partons at the time of discovery (quarks).

The effect of quark substructure in lepton scatt. may be formulated in terms of the nucleon structure funcs.

$F_1(q^2, x)$ and $F_2(q^2, x)$ in the $\frac{d^2\sigma}{dx\,dy}$ (P 238)

This equ. (P 238) applies equally well to $\begin{cases} \text{1- scatt. leptons off nucleons} \\ \text{2- as well as: off nuclei} \end{cases}$

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi Z^2\alpha^2}{q^4}\,\frac{2ME_i}{A\hbar^2}\left\{F_2^A(q^2,x)(1-y) + F_1^A(q^2,x)\,x\,y^2\right\}$$

per nucleon

for nuclei

$$F^A(q^2,x) = \frac{F(q^2,x)}{A}$$

Now we compare the structure funcs. per nucleon obtained from high energy lepton scatt. $\begin{cases} \text{off free nucleons} \\ \text{and off bound nucleons} \end{cases}$

$F_{1,2}(q^2,x) \xrightarrow{\text{are different}}$ for $\begin{cases} \text{Neutron} \\ \text{Protons} \end{cases}$

$F_{1,2}(q^2,x) =$ average between contributions from $\begin{cases} n \\ p \end{cases}$.

Assume $N = Z$ for light nuclei

$\longrightarrow$ the contributions from each part have equal weights

Since deutron is a loosely bound system, we can treat it as essentially free nucleons.

and still $N = Z = 1$

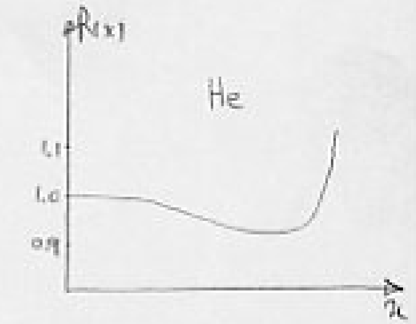
For simplicity, we take $F_1(q^2, x) \simeq 0$, since its contributions is small (in high energy).

If,

the quark structure of a free nucleon = the quark structure of a bound nucleon

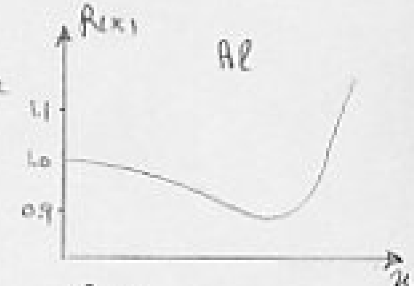
$$\longrightarrow \quad R(x) = \frac{F_2^A(q^2, x)}{F_2^d(q^2, x)} \xrightarrow{\text{must be}} 1$$

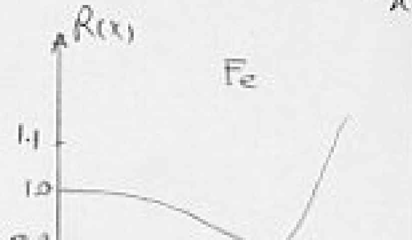
The experimental results, first reported by the European Muon Collaboration group (1983) showed:



$$R(x) \neq 1 \quad \text{and} \quad x\text{-dep.}$$

But because of large uncertainties, the available data at the moment are also compatible with calculated results (assuming that the only difference in bound and free nucleon structure funcs. is due to binding energy and not because of the difference in quark structure).





4-6 Matter Density and Charge Density:

Since electron $\underline{\text{ int. }}$ nucleus is electromagnetic

$\longrightarrow$ the observed density dist. is predominantly charge dist.

What is the influence of neutrons on the charge dist.?

One way to answer:

is to measure the isotopic difference (the difference in the charge dists. of nuclei with the same A but different N).

If charge dist is indep. of neutrons $\longrightarrow$ isotopic shifts in the charge dists. for different isotopes = negligible

The experimental results indicate that:

the shifts = small but not zero

Muonic Atom:

Muon properties $\xrightarrow{\text{very similar to}}$ Electron

But $m_\mu = 207\, m_e$

Consider the case of a simple:

Hydrogen-like atom with $\begin{cases} Z\text{-Proton in nucleus} \\ \text{a single electron outside} \end{cases}$

Using Bohr model:

$$r_n(\bar{e}) = [4\pi\epsilon_0] \frac{n^2\hbar^2}{Zm_e e^2} = \frac{1}{\alpha\hbar c} \frac{n^2\hbar^2}{Zm_e} \qquad \alpha: \text{structure const.}$$

For hydrogen atom $(Z=1)$, in the ground state $(n=1)$

$$a_0 = \frac{\hbar}{\alpha c m_e} = 5.29\times10^{-11}\,\text{m} \quad \text{Bohr radius}$$

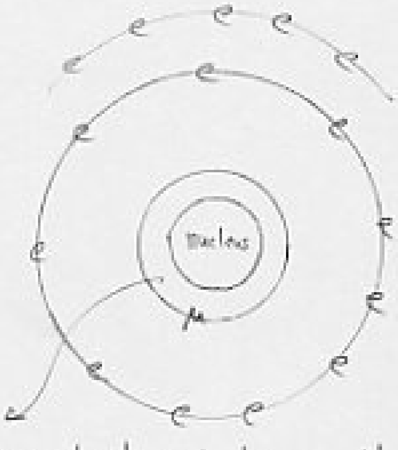
Analogously; $r_n(\mu^-) = \dfrac{1}{\alpha\hbar c} \dfrac{n^2\hbar^2}{Zm_\mu} = a_0 \dfrac{n^2 m_e}{Zm_\mu}$

Also; $$E_n = -\frac{m_\mu}{2} \frac{(Z\alpha c)^2}{n^2}$$

For $Z>1$, it is assumed that all electrons except one are stripped off, so as to remove the influences due to other electrons.

Such screening effects are in general quite difficult to calculate accurately.

We don't have such a problem in muonic atom with only one muon.



No electron between the nucleus and the orbit of μ
→ No screening effect
(because the radius of μ is much smaller than electrons)

For a heavy nucleus like ^{208}Pb $(Z=82)$

$$r_1(\mu^-) = a_0 \frac{0.511}{82 \times 106} = 3.1 \times 10^{-15} \text{ m}$$

3.1 fm

$$m_\mu = 106 \text{ MeV}/c^2$$

$$R_{Pb} = r_0 A^{1/3} = 1.2 \times (208)^{1/3} = 7.1 \text{ fm}$$

$$r_1(\mu^-) < R_{Pb}$$

A more elaborate calculation $\xrightarrow{\text{show}}$ muon spends %50 of the time inside a heavy nucleus

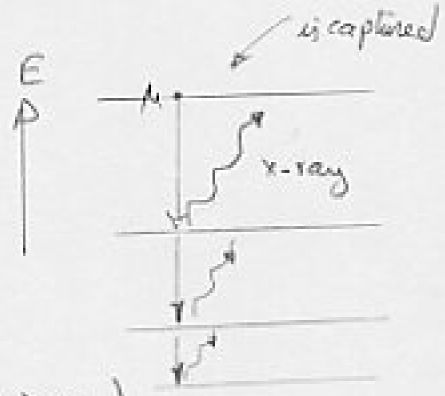
→ Actual muonic orbits $\neq r_1(\mu^-)$

Since low-lying muonic orbits are very close to nuclear surface → the detailed charge-dist. in nucleus play an important role

→ The modifications to the muonic orbits may be observed from → {the changes in the energy level positions

→ which in turn affects the energy of x-ray emitted when the muonic atom decays from one level to another

When a μ is captured by an atom $\longrightarrow$ it is likely that initially the μ will be in one of the higher levels.



The muonic atom then $\xrightarrow{\text{decays}}$ to lower states by emitting x-ray

Since $\Delta E_{n,n-1\,(\mu^-)} > \Delta E_{n,n-1\,(e^-)}$ (because $m_\mu > m_e$)

$\longrightarrow$ X-ray is emitted

X-ray energies $\xrightarrow{\text{we can deduce}}$ muonic energy levels

The differences between these and those given by E_n (P243) provide a measure of charge dist. in nucleus.

Isotopic shifts in Calcium Isotops:

Influence of neutrons on the measured $\rho_{ch}(r)$ using the even calcium isotopes (as example).

Charge dist. of Ca isotops (isotopic shift data)

Nucleus	$\langle r^2\rangle^{1/2}$ (fm)	t (fm)	c (fm)	z (fm)	w
^{40}Ca	3.4869	2.681	3.6758	0.5851	-0.1017
^{42}Ca	3.5166	2.724	3.7278	0.5911	-0.1158
^{44}Ca	3.5149	2.630	3.7481	0.5715	-0.0948
^{48}Ca	3.4762	2.351	3.7444	0.5255	-0.03
^{48}Ti	3.5844	2.580	3.8551	0.5626	-0.0761

t: Surface tickness (distance between %90 and %10 of the peak density)

$$^{40}Ca \longrightarrow {}^{48}Ca \quad \Longrightarrow \quad \langle r^2\rangle^{1/2} \xrightarrow{\text{decreases by}} 0.01$$

Neutron no.: $20 \longrightarrow 28$

$\rightarrow$ The addition of neutrons to the Calcium isotops reduces the size of the charge dist. of the same 20 protons.

But acc. to $R = r_0 A^{1/3}$; in going from $^{40}Ca \xrightarrow{\text{to}} {}^{48}Ca$

R must increase by 6%

In contrast; when we compare ^{40}Ca with ^{48}Ti:

$p = 20 + 2 \qquad n = 20 + 6$

$\longrightarrow$ the size of charge dist. is increased by 0.1 fm,

Consistent with $A^{\frac{1}{3}}$ - dependence

Two possible explanations;

1- the addition of neutrons makes the protons more tightly bound $\xrightarrow{\text{hence}}$ the smaller charge radius.

This is, however, not true for nuclei in general.

2- The central part of charge dist. in neutron is positive and near the surface is negative.

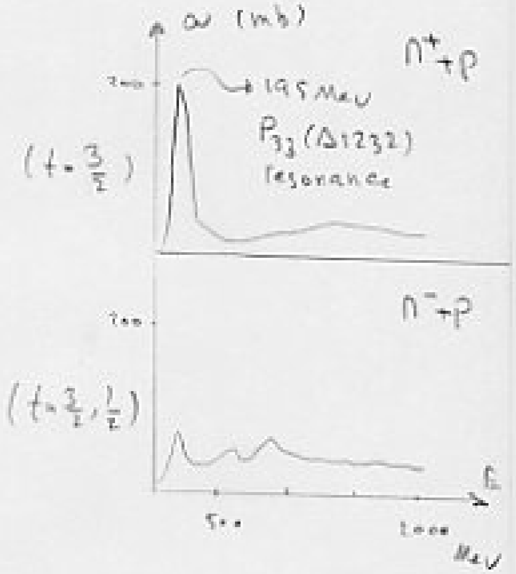
(The detailed charge dist. inside a neutron is not well known because of the difficulty in measuring the small charge form factors)

However, a small excess of negative charge in the surface region can produce a decrease in the charge radius in going from $^{40}Ca \xrightarrow{\text{to}} {}^{48}Ca$ (about $\frac{1}{3}$ of the decrease).

The other $\frac{2}{3}$ may be attributed to the spin-dependence in the ints. of protons with other nucleons.

Pion - Nucleus Scatt.:

We have seen earlier: that there is a strong pion-nucleon resonance in the $\begin{cases} t = \frac{3}{2} \\ S = \frac{3}{2} \end{cases}$ channel at $E_{\pi(lab)} \simeq 195$ MeV



Six possible pion-nucleon scatt.:

a) $\pi^+ + p \longrightarrow \pi^+ + p$

b) $\pi^- + n \longrightarrow \pi^- + n$

c) $\pi^+ + n \longrightarrow \pi^+ + n$

d) $\pi^- + p \longrightarrow \pi^- + p$

e) $\pi^+ + n \longrightarrow \pi^0 + p$

f) $\pi^- + p \longrightarrow \pi^0 + n$

Detection of π^0 are much harder than $\pi^\pm$ (we ignore them)

In (a) and (b), $|t_z| = \frac{3}{2} \longrightarrow$ purely $t = \frac{3}{2}$

For (c) and (d), $|t_z| = \frac{1}{2} \longrightarrow$ there is isospin mixing of $t = \frac{1}{2}$ and $t = \frac{3}{2}$.

$|\pi^+, p\rangle = |t=\frac{3}{2}, t_z=\frac{3}{2}\rangle$

$|\pi^-, n\rangle = |t=\frac{3}{2}, t_z=-\frac{3}{2}\rangle$

$$|\pi^+, n\rangle = \sum_t (1 1, \tfrac{1}{2} -\tfrac{1}{2} / t \tfrac{1}{2}) |t, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, |t=\tfrac{3}{2}, t_z=\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\, |t=\tfrac{1}{2}, t_z=\tfrac{1}{2}\rangle$$

$$|\pi^-, p\rangle = \sum_t (1 -1, \tfrac{1}{2} \tfrac{1}{2} / t -\tfrac{1}{2}) |t -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, |t=\tfrac{3}{2}, t_z=-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\, |t=\tfrac{1}{2}, t_z=-\tfrac{1}{2}\rangle$$

i.e. $(\sqrt{\frac{1}{3}})^2$ of the scatt. amplitude in the last two cases in the $t=\frac{3}{2}$ channel

$(\sqrt{\frac{2}{3}})$ " " " " " , $t=\frac{1}{2}$ "

Since isospin is conserved:

a) $f_{\pi^+ p}(\theta) = f_{t=\frac{3}{2}}(\theta)$ b) $f_{\pi^- n}(\theta) = f_{t=\frac{3}{2}}(\theta)$

c) $f_{\pi^+ n}(\theta) = \frac{1}{3} f_{t=\frac{3}{2}}(\theta) + \frac{2}{3} f_{t=\frac{1}{2}}(\theta)$ d) $f_{\pi^- p}(\theta) = \frac{1}{3} f_{t=\frac{3}{2}}(\theta) + \frac{2}{3} f_{t=\frac{1}{2}}(\theta)$

At energies near P_{33}-resonance $\sigma_{t=\frac{1}{2}} << \sigma_{t=\frac{3}{2}}$

We ignore $\sigma_{t=\frac{1}{2}}$

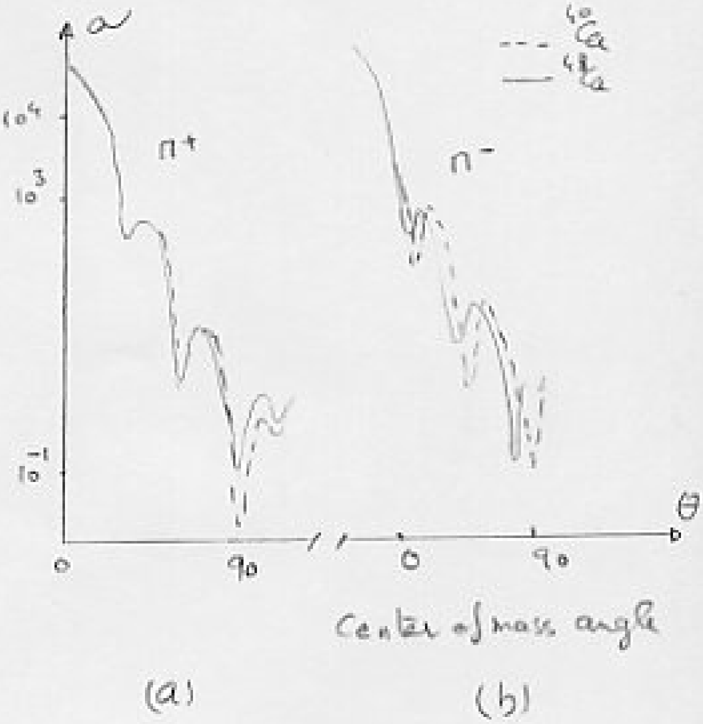
Since $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$

$$\rightarrow \quad \frac{\sigma(\pi^+ p)}{\sigma(\pi^+ n)} \simeq \frac{\sigma(\pi^- n)}{\sigma(\pi^- p)} \simeq 9$$

i.e. π^+ is sensetive to proton
π^- " " " neutron

The method is particulary useful, since pion scatt. is <u>sensetive</u> mainly to the nucl. surface region (where the most differences in the neutron and proton densities are expected).

The Fig. Shows the $\pi^\pm$ scatt. off ${}^{40}Ca$ and ${}^{48}Ca$ at $E_\pi = 180$ MeV (near P_{33}-resonance)



In (a) since π^+ is sensetive to proton, and since there is a small difference between the scatt. results of ${}^{40}Ca$ and ${}^{48}Ca$ ⟶ adding neutrons does not change essentially the proton dsit.

On the other hand in (b), since π^- is sensetive to neutrons and since there is a definite difference between the results of ${}^{40}Ca$ and ${}^{48}Ca$ ⟶ adding neutrons changes the neutron dist.

Nucleon-nucleus Scatt.:

A proton of an intermediate energy (100 - 1000 MeV) spends sufficiently short time in the nucleus;

$\longrightarrow$ it is unlikely to suffer mulitiple scatt.

$\longrightarrow$ The incident proton interacts only with one of the nucleons in the target.

$\longrightarrow$ the scatt. is sensetive to the density of nucleon dist. (not nucleus)

The incident proton interacts with both protons and neutrons in the nucleons.

There is a small difference between these two interactions (comming from the isospin channel). (comparison can be made with the scatt. results deduced from electron scatt. which is sensetive to protons.)

Since isospin-dependence of nucl. forces is not strong enough, one can not differentiate between isospin effects and charge in neutron and proton densities.

(In $\pi^{\pm}$ scatt. off nucleus, the isospin dependence is strong

i.e. $\frac{\sigma(\pi^+ p)}{\sigma(\pi^+ n)} \approx \frac{\sigma(\pi^- n)}{\sigma(\pi^- p)} \approx 9$)

4.7 Nuclear Shape and Electromagnetic Moments:

Multipole Expansion of Charge Density:

$$\varphi(\bar{r}) = \left[\frac{1}{4\pi\epsilon_0}\right] \frac{q}{|\bar{r}-\bar{r}'|} \qquad \text{(point charge)} \qquad \bar{r} = (r,\theta,\varphi)$$

For $r > r'$

$$\varphi(\bar{r}) = \left[\frac{1}{4\pi\epsilon_0}\right] \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi q}{2\ell+1} \frac{r'^{\ell}}{r^{\ell+1}} Y^*_{\ell m}(\theta',\varphi') \, Y_{\ell m}(\theta,\varphi)$$

For a charge dist. $\rho_{ch}(\bar{r}')$

$$\varphi(\bar{r}) = \left[\frac{1}{4\pi\epsilon_0}\right] \sum_{\ell} \sum_{m} \frac{4\pi}{2\ell+1} \frac{1}{r^{\ell+1}} (Z Q_{\ell m}) \, Y_{\ell m}(\theta,\varphi)$$

$$Q_{\ell m} = \frac{1}{Z} \int e \, r'^{\ell} \, Y^*_{\ell m}(\theta',\varphi') \, \rho_{ch}(\bar{r}') \, dv'$$

Along z-axis, $\cos\theta = 1$, $\quad Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \, P^m_{\ell}(\cos\theta) \, e^{im\varphi}$

$$\varphi(\bar{r}) \rightarrow \varphi(r) = \left[\frac{1}{4\pi\epsilon_0}\right] \frac{1}{r} \sum_{\ell} \sum_{m} \sqrt{\frac{4\pi}{2\ell+1}} \, \frac{Z Q_{\ell m}}{r^{\ell}}$$

For nearly spherical charge dist. $\varphi(r)$ converges very fast.

Using the normalization given by

$$\rho_{ch}(\bar{r}) = Z \, |\Psi(\bar{r})|^2$$

$$Q_{\ell m} = \int e r^\ell Y^*_{\ell m}(\theta,\varphi) \, |\psi(\bar r)|^2 \, dv$$

$$Q_{\ell m} = \langle \psi(\bar r) | e r^\ell Y^*_{\ell m}(\theta,\varphi) | \psi(\bar r) \rangle$$

$\rightarrow$ $O_{\ell m}(E) = e r^\ell Y^*_{\ell m}(\theta,\varphi)$ Electric multipole op.

$$O_{\ell m}(E) = e \sum_{\text{Protons}} r_i^\ell Y^*_{\ell m}(\theta_i,\varphi_i) = \sum_{i=1}^{A} e(i) r_i^\ell Y^*_{\ell m}(\theta,\varphi)$$

$$e(i) = \begin{cases} 1e & \text{for proton} \\ 0 & \text{" neutron} \end{cases}$$

(mainly we are concerned with $m=0$ component)

If the charge dist. is spherical in shape;

$$Q_{00} = e \quad , \quad Q_{\ell m} = 0 \quad \begin{cases} \ell \neq 0 \\ m \neq 0 \end{cases}$$

$\rightarrow$ $Q_{\ell m} \neq 0$ ($\ell \neq 0, m \neq 0$) is a measure of the departure from a spherical shape.

However, some multipole moments may also vanish because of symmetry reasons.

$$Y_{\ell m}(\theta,\varphi) \xrightarrow{\Pi} Y_{\ell m}(\pi-\theta, \varphi+\pi) = (-1)^\ell Y_{\ell m}(\theta,\varphi)$$

requires $\rightarrow$ all odd electric multipole coeffs. be zero.

i.e. $Q_{\ell=\text{odd}, m} = 0$ for all nuclei.

Any observation of nonvanishing value (for odd ℓ) implies a violation of this symmetry.

Electric dipole moment is of interest.

For neutron, $Q_{00} = 0$ (no net charge)

$Q_{1m} = ?$

If both <u>time</u> - and <u>parity</u> invariance symmetries are <u>violated</u>;

$\xrightarrow{\text{can happen}}$ $Q_{1m} \neq 0$

$Q_{10} = (-1.1 \pm 0.8) \times 10^{-25}$ e-cm (measured value for neutron)

$\simeq 0$ $\xrightarrow{\text{but}}$ it shows the existence of a small symmetry violation contribution.

For our purposes, we shall assume both <u>parity</u> and <u>time</u>-invariance are <u>exact</u> symmetries $\xrightarrow{\text{and}}$ only <u>even</u> order electric multipole moments may be different from zero.

Second restriction:

The is another restriction on the multipole coeff. a state can have.

$O_{\ell m}(\vec{r}) = r^{\ell} Y_{\ell m}(\theta, \varphi)$ is a tensor of rank (ℓ, m)

and carries an angular momentum $\bar{\ell}$.

$\langle J | O_{\lambda\mu}(\vec{r}) | J \rangle \neq 0$ if J, ℓ, J form a closed triangle

$\longrightarrow \langle | \quad | \rangle = 0$ if $2J < \lambda$

for this reason $\longrightarrow$ For $J=0$ the only $\lambda=0$ has $\langle | \quad | \rangle \neq 0$

regardless of the intrinsic shape of the nucleus (i.e. regardless of ℓ).

Using $\langle JM | T_{kq} | J'M' \rangle = (-)^{J-M} \begin{pmatrix} J & k & J' \\ -M & q & M' \end{pmatrix} \langle J \| T_k \| J' \rangle$

$$\langle JM | O_{\ell m} | JM \rangle = (-)^{J-M} \begin{pmatrix} J & \ell & J \\ -M & m & M \end{pmatrix} \langle J \| Q_\ell \| J \rangle$$

Convension:

We define the multipole moment as the expectation value of the multipol op. in the state of maximum M.

Electric quadrapole moment:

Since the dipole moment vanishes for symmetry reasons,

$\longrightarrow$ the lowest order electric moment that can give some idea of the shape of a nucleus is the quadrapole moment ($\ell = 2$).

Nuclei near a closed shell are $\approx$ spherical

$\longrightarrow |Q_{20}| =$ small

Nuclei in the middle of a major shell are $\approx$ deformed

$\longrightarrow |Q_{20}| =$ large

$$Q = \sqrt{\frac{16\pi}{5}}\, e \langle J, M=J | r^2 Y_{20}(\theta,\varphi) | J, M=J \rangle$$

$$= e \langle J, M=J | (3z^2 - r^2) | J, M=J \rangle$$

For a spherical nucleus $\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = \frac{1}{3}\langle r^2 \rangle$

$\longrightarrow Q = 0$

$Q=0$ spherical shape

$Q<0$ oblate shape

$Q>0$ prolate shape

The next higher order, nonvanishing, static electric multipole moment is the hexadecapole moment ($\ell=4$)

For $\ell=4$, $\langle \; | \; | \; \rangle = 0$ except for $J \geq 2$

For such a state $Q \neq 0$ (usually)

$\xrightarrow{\text{as a result}}$ It is hard to measure the hexadecapole moment as it is not, in general, to differentiate its effect from those due to quadrapole moment.

On the other hand since most of the nuclei are very nearly spherical in shape,

$\longrightarrow Q_{lm} \xrightarrow{\text{very quickly}}$ as l increases

$\longrightarrow Q_{20}$ is the dominant term.

The lack of data is also due in part to the fact that:

Measurements are easy to be carried out in the ground state and there are very few stable nuclei with ground state spin $J \geq 2$.

Magnetic moments:

In addition to charge dist., a deformed nucleus may also possess a nonspherical magnetic charge dist.

Nuclear magnetism originates from the;

$$\begin{cases} \text{intrinsic magnetic dipole moment of nucleons} \\ \text{orbital motion of protons} \end{cases}$$

$\rho_M(\bar{r}) = -\nabla\cdot M(\bar{r})$ mag. charge density

$\bar{M}(\bar{r})$ magnetization density (mag. moment density)

$J_M(\bar{r}) = \frac{c}{[c]} \nabla\times M(r)$ magnetization current (effective current density)

As far as $J_M(\bar{r})$ is concerned, we can again adopt a model that nuclei are made of point nucleons having an intrinsic spin but no internal structure.

Remark:

$$\begin{cases} \Delta A(x) = \frac{J(x')\Delta V}{c|x-x'|} + \frac{M(x')\times(x-x')}{|x-x'|^3}\Delta V \\ A(x) = \frac{1}{c}\int \frac{J(x') + c\nabla'\times M(x')}{|x-x'|} d^3x' \end{cases}$$

$$\begin{cases} \mu_p = \frac{1}{2} g_p \\ q_p = e \end{cases} \qquad \begin{cases} \mu_n = \frac{1}{2} g_n \\ q_n = 0 \end{cases}$$

Since $\mu_\ell = \frac{e\hbar[c]}{2Mc}\ell$ $\quad(\ell\hbar = r\times p)$ $\quad \mu_s = \frac{e\hbar[c]}{2Mc} s$

$\longrightarrow \mu_\ell(i) = g_\ell \mu_N \ell(i) \qquad \mu_s(i) = g_s \mu_N s(i)$

and $\nabla\times(r\times P) = r(\nabla\cdot P) - P(\nabla\cdot r) + (P\cdot\nabla)r - (r\cdot\nabla)P \quad , \nabla\cdot r = 3$

In such point particle picture,

$$J(r) = \sum_{i=1}^{A} \Big\{ e g_e(i) \underbrace{\frac{p(i)}{M_N}}_{\text{convective part}} + \frac{e\hbar}{2M_N} g_s(i) \underbrace{\nabla \times S(i)}_{\text{magnetization part}} \Big\} \delta(r - r(i)) \qquad (1)$$

where in units of nucl. magneton μ_N,

$$g_e = \begin{cases} 1 & \text{for } p \\ 0 & \text{" } n \end{cases} \qquad g_s = \begin{cases} 5.586 & \text{for } p \\ -3.826 & \text{" } n \end{cases}$$

Similar to a charge dist., we may decompose a magnetization density dist. in terms of multipole coeffs. given by

$$\mathcal{M}_{lm} = \int r^l Y_{lm}^*(\theta,\varphi) \rho_M(r)\, dv = -\int r^l Y_{lm}^*(\theta,\varphi) \nabla \cdot M(r)\, dv$$

Because of divergence op., the parity of the integrand for mag. multipole moment of order l is $(-)^{l+1}$!

$\longrightarrow$ even order mag. multipoles vanish.

The lowest order nonvanishing mag. multipole $\rightarrow$ dipole

Dipole op $\sim r Y_{1m}$

$r Y_{1m} \sim r_m$

r_m : a component of vector $\bar{r}$ in spherical coords.

$$\mathcal{M}_{1m} = -\int r Y_{1m}\, \nabla \cdot M(r)\, dv$$

Integration by parts;

$$\mathcal{M}_{1m} = -\int r_m\, \nabla \cdot M(r)\, dv = -\int r_m \left(\sum_{i=-1}^{1} \frac{\partial}{\partial r_i} M_i(r) \right) dv$$

$$\mathcal{M}_{1m} = -r_m \left(\sum_{i=-1}^{1} M_i(r) \right) \Big|_0^\infty + \int \left(\sum_{i=-1}^{1} \frac{\partial}{\partial r_i} r_m\, \hat{e}_i \right) \cdot M(r)\, dv$$

$$\mathcal{M}_{1m} = \int M_m(r)\, dv$$

Contribution from nucleon intrinsic magnetic moment to dipole op.;

$$O_{1m}(M1, s) = \frac{e\hbar\,[c]}{2M_N c} \sum_{i=1}^{A} g_s(i)\, s_m(i) \qquad (2)$$

Also note that; the contributions from orbital motion;

$$\mathcal{M}_{1m}(\ell) = \int M_m(\ell)\, dv = \frac{1}{2} \int \left(r \times (\nabla \times M(\ell) \right)_m dv = \frac{[c]}{2c} \int (r \times \underset{\sim}{J}(\ell))_m\, dv$$

(3)

(integration by parts)

Contribution to the dipole op. due to proton orbital motion;

$$(1)(3) \rightarrow O_{1m}(M1,\ell) = \frac{[c]}{2c}\left(r \times \sum_{i=1}^{A} e g_e(i)\,\frac{p_i}{M_N}\right)_m$$

$$= \frac{e\hbar[c]}{2M_N c}\sum_{i=1}^{A} g_e(i)\,\ell_m(i) \qquad (4)$$

$$(2)(4) \rightarrow O_{1m}(M1) = O_{1m}(M1,\ell) + O_{1m}(M1,s)$$

$$= \frac{e\hbar[c]}{2M_N c}\sum_{i=1}^{A}\left\{g_e(i)\,\ell_m(i) + g_s(i)\,S_m(i)\right\}$$

$$= \mu_N \sum_{i=1}^{A}\left\{g_e(i)\,\ell_m(i) + g_s(i)\,S_m(i)\right\}$$

In general;

$$O_{\lambda\mu}(M\lambda) = \mu_N \sum_{i=1}^{A}\left\{\frac{2}{\lambda+1} g_e(i)\,\ell(i) + g_s(i)\,S(i)\right\}\cdot\nabla_i\left(r_i^{\lambda}\, Y_{\lambda\mu}^{*}(\theta_i\varphi_i)\right)$$

(Shalit & Talmi)

4.8 Magnetic Dipole Moment of Odd Nuclei;

The mag. dipole moment op. $O_{1m}(M1)$ carries <u>one</u> unit of angular momentum,

$$\rightarrow \quad \langle \, | O_{1m}(M1) | \, \rangle \neq 0 \qquad \text{if } J \geq \tfrac{1}{2}$$

The magnetic dipole moment μ of a state is defined as the expectation value of the op. in a state with $M=J$:

$$\mu = \langle J, M=J \,|\, \mathcal{M}_{10} \,|\, J, M=J \rangle$$

$$= \sum_{i=1}^{A} \langle J, M=J \,|\, (g_\ell(i)\, \ell_0(i) + g_s(i)\, S_0(i)) \,|\, J, M=J \rangle$$

(in units of μ_N)

In general we are interested to the ground states;

In the ground state;

$$\left\{ \begin{array}{l} (\text{Pair of Protons}) \\ (\;\;\text{"}\;\;\;\;\text{"}\;\;\text{Neutrons}) \end{array} \right. \xrightarrow{\text{tend}} \text{to couple Zero angular momentum}$$

For such a <u>Zero-coupled pair</u> contributions to $\mu \rightarrow$ <u>vanish</u>.

This can be seen from;

1- Angular momentum selection rules;

2 — For a <u>pair</u> of <u>identical</u> nucleons;

acc. to the Pauli exclusion principle;

$(L, S) = (0,0)$ or $(L,S) = (1,1)$

The second one has a <u>higher</u> energy $\longrightarrow$ does not belong to the ground state

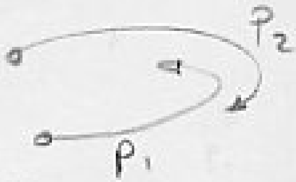
$(L,S) = (0,0) \longrightarrow J = 0$

$S = 0 \longrightarrow s_1, s_2 = \uparrow\downarrow$ or $\downarrow\uparrow$

$\longrightarrow$ Their contributions <u>cancel</u> each other.

Similarly for a pair of protons coupled to $L = 0$,

$\longrightarrow$ the <u>net</u> contribution to the mag. dipole moment due to their <u>orbital</u> motion is <u>also zero</u>.



$\longrightarrow$ All the <u>neutron pairs</u> and <u>proton pairs</u> <u>don't</u> make any contributions to μ.

i) For <u>even-even</u> nuclei: $\xrightarrow{\text{we expect}} \mu = 0 \quad (J = 0)$

This is observed to be true for all <u>stable</u> even-even nuclei.

ii) For odd-mass nuclei, there is a single unpaired nucleon;

The expression for μ reduces to the expectation value of the unpaired nucleon alone

$$\mu_{S.P} = \mu_N \langle j, m=j \,|\, g_\ell l_z + g_s S_z \,|\, j, m=j \rangle$$

$|j, m\rangle$: single particle wave func of the unpaired nucleon

Using the Landé formula (as we have done earlier),

$$\mu_{s.p.} = \frac{1}{j(j+1)} \langle j, m=j \,|\, (\mu \cdot j)\, J_z \,|\, j, m=j \rangle$$

where $\bar{\mu} = g_\ell \bar{\ell} + g_s \bar{S}$

$$\ell \cdot j = \ell \cdot (\ell + s) = \ell^2 + \tfrac{1}{2}(j^2 - \ell^2 - s^2)$$
$$s \cdot j = s \cdot (\ell + s) = s^2 + \tfrac{1}{2}(j^2 - \ell^2 - s^2)$$

$$\longrightarrow \mu_{s.p} = j \left\{ g_\ell \pm \frac{g_s - g_\ell}{2\ell + 1} \right\} \quad \text{for } j = \ell \pm \tfrac{1}{2}$$

If all the nucleons except one are members of zero-coupled pairs $\longrightarrow$ $J = j$

nucleus (J), single particle (unpaired) (j)

For a given j-value $\longrightarrow$ $\ell = j \pm \frac{1}{2}$

The choice between them is determined by the ground state Parity.

For a pair $\quad \Pi = (-1)^{\ell}(-1)^{\ell}$

$\longrightarrow \Pi_{\text{nucleus}} = \Pi_{\substack{\text{unpaired} \\ \text{nucleon}}} = (-1)^{\ell}$

From experimental data for $\left\{ \begin{matrix} J \\ \Pi \end{matrix} \right.$ of the nucleus $\longrightarrow$ ℓ of the unpaired nucleon can be determined

$\longrightarrow$ $\mu_{s.p}$ can be calculated

The results fall into groups characterized by the values j and ℓ and are known as the Schmidt values.

The observed values for μ are within the limits given by the Schmidt values with only a small number of exceptions.

In more realistic ground state wave func. will introduce other components which may be characterized by the number of broken zero-coupled pairs of nucleons coupled to $J > 0$

These non-zero coupled pairs cannot give a coherent contribution to the unpaired nucleon.

Thus these effects tend to decrease the absolute value of μ given by single particle model.

$\longrightarrow$ Schmidt values give the limits of possible ground-state mag. dipole moment for odd nuclei:

The model can be checked in the simple cases near the closed shell: ^{3}He, ^{15}N, ^{17}O, ^{17}F

$$\text{Single-particle description} + \underset{\text{model}}{\text{Corrections to single particle}}$$

$$= \mu_{\text{observed}}$$

However

$$\text{Corrections is} > |\text{the required values to explain the discrepancy}|$$

This means that there are some other sources of adjustments.

In our calculations we have assumed that each nucleon inside the nucleon has essentially the same properties as a free nucleon (impulse approx.).

For example we have taken $\begin{cases} \mu_p = \frac{1}{2} g_p \\ \mu_n = \frac{1}{2} g_n \end{cases}$ for both free and bound nucleons.

There are <u>two</u> reasons that this approx. may fial for μ caclulation of odd-mass nuclei

1– Mesonic currents of charged mesons, which produces magnetic moment.

2– We have assumed that nucleons behave like point particles carrying the same charge and mag. dipole moment as free nucleons. Instead <u>effective</u> values should be used.

4.9 Ground State Spin and Isospin;

The ground state properties commonly observed are;

i) binding energy, ii) spin, iii) isospin, iv) static electromag. moments

Ground state spin:

Since for each nucleon $S = \frac{1}{2}$, l = integer

$\rightarrow$ j = half-int.

$$J = \sum_{i=1}^{A} j_i = \begin{cases} \text{half-int. for } A = \text{odd} \\ \text{integer} \quad : \quad A = \text{even} \end{cases}$$

Similarly; $T = \sum_{i=1}^{A} t_i$

Even-mass nuclei $\begin{cases} N = \text{even}, Z = \text{even} \\ N = \text{odd}, Z = \text{odd} \end{cases}$

For even-even nuclei $J = 0$ (total spin) without exception

$\longrightarrow$ which is a reflection of fundamental property of nucl. int, Known as Pairing.

Since for odd-odd nuclei $J \neq 0$

$\longrightarrow$ Pairing effect is important between two identical nucleons.

For example for deutron $J = 1$ $(T = 0)$

There is no pairing effect between p and n in deutron

If there were a pairing effect $J \xrightarrow{\text{would be}} 0$

In terms of isospin decomposition;

Pairing force reveals itself in the $T = 1$ state of two nucleon-system but not in the $T = 0$ state.

Because of antisymmetrization requirements:

a pair of $\begin{cases} n \\ p \end{cases}$ in $l=0$ can be either

$\begin{cases} S=0 \\ T=1 \end{cases}$ singlet-state (isovector) or $\begin{cases} S=1 \\ T=0 \end{cases}$ triplet-state (isoscalar)

$\omega = \frac{1}{4}$ of Time $\qquad \omega' = \frac{3}{4}$ of time

$\xrightarrow{\text{This shows}}$ T=1 pairing force cannot be very strong (since $\omega' > \omega$).

Because of pairing force $\longrightarrow$ J of odd-mass nucleus is given by the j-value of the unpaired nucleon.

i.e. odd-mass nucleus = (even-even nucleus) + single nucleon

j-value of the unpaired nucleon may be found from the single-particle energy level spectrum.

$\longrightarrow J_{\text{odd-mass}}$ can usually be deduced from the number of neutrons and protons.

For odd-odd nuclei the prediction of J is somehow difficult -

$$(\text{odd-odd nucleus}) = (\text{even-even nucleus}) + p + n$$
$$J \qquad\qquad J' = 0 \qquad j_p \quad j_n$$

$$|j_p - j_n| \leq J \leq j_p + j_n$$

j_p can be estimated by the grandstate spin of the neighboring odd-mass nuclei with one less neutron.

Similarly, the value of j_n may be obtained from neighboring odd mass nucleus with one less proton.

i.e.

$$j_p = J \text{ of } (\text{odd-odd, one less neutron})$$
$$j_n = J \text{ of } (\text{odd-odd, one less proton})$$

There is also empirical Nordheim rules

Strong rule $\quad J = |j_p - j_n| \quad$ for $\eta = 0$

Weak rule $\quad J =$ either $|j_p - j_n|$ or $j_p + j_n$ for $\eta = \pm 1$

where $\eta = j_p - l_p + j_n - l_n$.

But there are many exceptions to these rules.

Ground State Isospin;

The possible T of a nucleus $\xrightarrow{\text{may be deduced}}$ from the $\begin{cases} \text{Proton numbers} \\ \text{Neutron} \ \text{"} \end{cases}$

For a system of $\begin{cases} Z\text{ - Proton} \\ N\text{ - Neutron} \end{cases}$

$$T_z = \tfrac{1}{2}(Z-N)$$

$$T_{min} = |T_z|$$

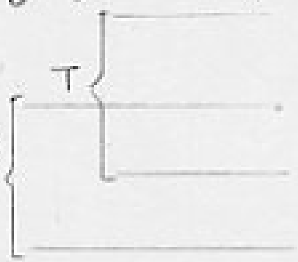
Since $|t_z| = \frac{1}{2} \longrightarrow |T_z|_{max} = \frac{1}{2}A$

$$\longrightarrow T_{max} = \tfrac{1}{2}A$$

$$\longrightarrow \tfrac{1}{2}|Z-N| \leq T \leq \tfrac{1}{2}(Z+N)$$

The isospin dependence of nuclear forces is not large enough to separate states belonging to different T into isolated groups in energy!

i.e.



However, the lowest member of each allowed T-value is well separated in energy from each other (except for odd-odd nuclei).

Since there is a bound two-nucleon state;

$\begin{cases} \text{for } T=0 & \text{(deutron)} \\ \text{but not for } T=1 & \end{cases}$

We can infer: Nuclear force favors the min. value

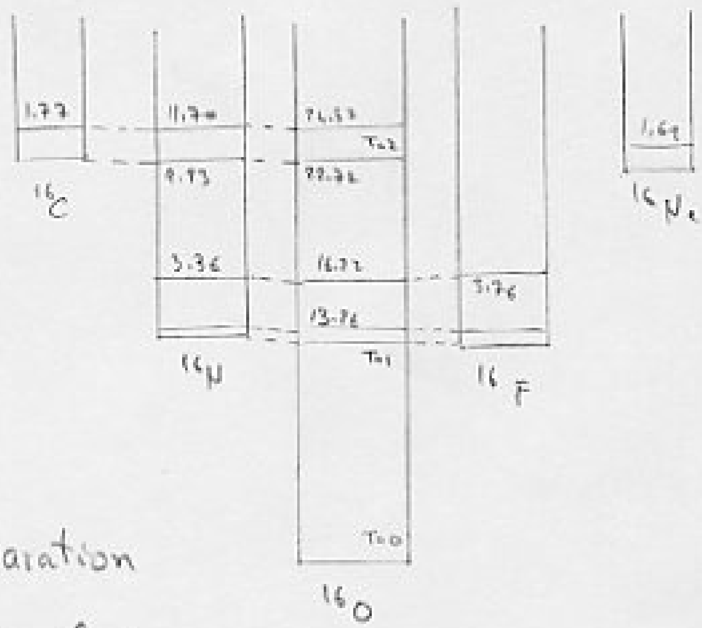
$T = \frac{1}{2}|Z-N|$ for the ground state.

For higher T, the lowest member of each group usually appears at successively higher energies.

For example, in ^{16}O;

the ground state has $T=0$

The lowest $T=1$ state occurs $\sim$ 13 Mev excitation, and the lowest $T=2$ state at 23 Mev.



For odd-odd nuclei, the separation in energy between the lowest member of the two lowest possible isospin is often quite small.

This is the result of competition between the T- and J- dependence of the nucl. force.

Because of this; the ground state of isospin is often a choice between

$$\tfrac{1}{2}|Z-N| \text{ and } \tfrac{1}{2}|Z-N|+1$$

For example;

${}^{26}_{13}Al$ has a groundstate $(J^{\pi}, T) = (5^{+}, 0)$

and the first excited state $(J^{\pi}, T) = (0^{+}, 1)$ 0.229 MeV

${}^{42}_{21}Sc$ has a ground state $(J^{\pi}, T) = (0^{+}, 1)$

and the lowest state with $T=0$, $(J^{\pi}, T) = (7^{+}, 0)$ 0.6 MeV

We have assumed that the nuclear force depends;

$$\begin{array}{rl} \text{on} & \left\{\begin{array}{ll} \text{isospin} & (t) \\ & \\ \text{charge states} & (t_z) \end{array}\right. \\ \text{but not on} & \end{array} \quad \text{of interacting nucleons.}$$

But still there exist charge dependent effects.

Although Coulomb int. is much weaker than nuclear forces and may be ignored for most purposes on the nucleon-nucleon level, it is not true for a nucleus as a whole.

Nucl. force is a short-range force $\sim A$

Coulomb force is long in range $\sim Z^2$

$\longrightarrow$ Coulomb force may become quite significant in heavy nuclei.

$\longrightarrow$ Nuclear many-body system is no longer invariant under exchange of n and p.

$\longrightarrow$ T is not a good quantum number.

Isospin mixing:

Consider a nuclear Hamiltonian H, with two eigenstates of the isospin-conserving part of the Hamiltonian,

$$H = H_0 + H_1(T)$$

$$H_0 |J T, x\rangle = E |J, T, x\rangle$$

$$H_0 |J', T', y\rangle = E' |J', T', y\rangle$$

x, y all other quantum lables other than sipn and isospin.

Since $H_1(T)$ is an isospin-breaking term in H (for example Coulomb int.), these two states are not the eigenstates of the total H.

We can find the eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ of the complete H using $|JTx\rangle$ and $|J'T'y\rangle$ as the basis states and solve the eigenvalue prob. in the two dimensional model.

$$\{H\} = \begin{pmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{pmatrix}$$

$$H_{xx} \equiv \langle JTx|H|JTx\rangle \qquad H_{yy} \equiv \langle J'T'y|H|J'T'y\rangle$$

$$H_{xy} \equiv \langle JTx|H|J'T'y\rangle \qquad H_{yx} \equiv \langle J'T'y|H|JTx\rangle$$

H_{xx} and H_{yy} can be large, because there are contributions from nuclear int. H_0.

In contrast H_{xy} and H_{yx} are small (isospin-breaking terms).

Furthermore Coulomb force is;

1- invariant under rotation (Preserves rotational symmetry)

2- " " time-reversal

$\rightarrow \{H\}$ is <u>real</u> and <u>symmetric</u>.

$$H_{xy} = H_{yx} = S\,\delta_{JJ'}$$

S: the size of the off-diagonal matrix element.

$$|\psi_1\rangle = C\theta\,|J\,T\,x\rangle + S\theta\,|J'\,T'\,y\rangle$$

$$|\psi_2\rangle = -S\theta\,|J\,T\,x\rangle + C\theta\,|J'\,T'\,y\rangle$$

$$\tan\theta = \frac{2S}{H_{xx} - H_{yy}}$$

Note: $\langle\psi_1|\psi_1\rangle = S^2\theta + C^2\theta = 1 \qquad \langle\psi_2|\psi_2\rangle = S^2\theta + C^2\theta = 1$

$\langle\psi_1|\psi_2\rangle = S\theta C\theta - S\theta C\theta = 0$

$|\psi_1\rangle' = |J\,T\,x\rangle + \tan\theta\,|J'\,T'\,y\rangle$
if we choose units

Remember: $|n\rangle = |n^\circ\rangle + g \sum\limits_{m \neq n} \frac{\langle m^\circ|V|n^\circ\rangle}{E_n^\circ - E_m^\circ}\,|m^\circ\rangle$

The mixing depends on $\left\{ \begin{array}{l} S \\ H_{xx}-H_{yy} \end{array}\right.$

In realistic nucleus the number of states that can be admixed by isospin-breaking forces may be much larger (than two states taken here).

There are two points:

1- The admixture is important only between states whose unperturbed states are close in energy.

2- Since isospin-breaking term $H_1(T)$ is rotational invariant $\langle J \cdots | H | J' \rangle = k\, \delta_{JJ'}$

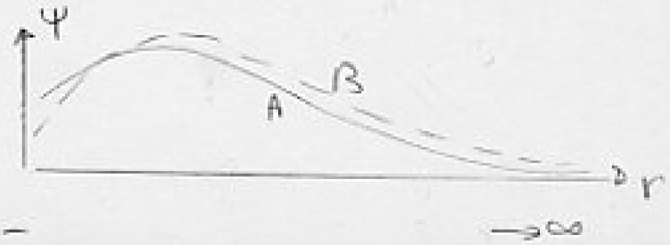
$\longrightarrow$ Isospin admixture can take place only between states of the same J.

Since Coulomb int. is long in range:

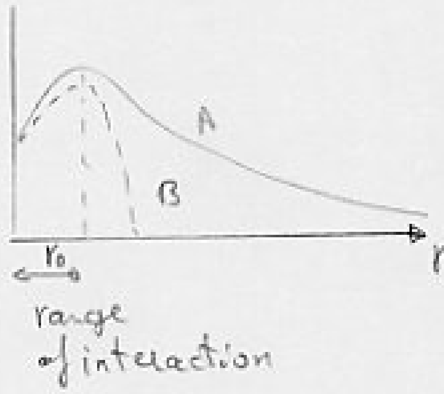
$\langle A | H_1 | B \rangle = $ small

except $|A\rangle \underset{\text{very similar}}{\approx} |B\rangle$ (except for isospin term)

$\longrightarrow$ large S



If the interaction were short-range the similarity of $|A\rangle$ and $|B\rangle$ would not be a necessary cond., because the integration is carried out within a short interval



Exterene example:

Consider an infinite range interaction, i.e. radial dependence of int = const.

$$\langle JT, x | H_1 | JT', y \rangle = K \delta_{xy} \qquad (H_1 = \underset{\text{const.}}{\alpha} f(T))$$

$\longrightarrow$ The two states must be identical except for isospin.

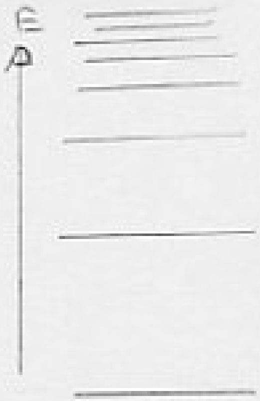
In reality, we find that isospin mixing is important only between <u>nearby</u> states
1- having the same spin (J)
2- a large overlap between their spatial wave funcs.

Isospin Purity in the ground states:

i) In light nuclei there are two reasons for this purity:

1- Since Z = small $\longrightarrow$ Coulomb force is relatively weak

$\longrightarrow$ δ (the size of the off diagonal matrix element) = small (in general)



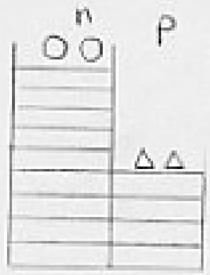
2- Density of states = relatively low in the low-lying regions of interest to us.

$\longrightarrow$ It is difficult to find two (unperturbed) states of <u>different isospin</u> near each other in energy as well as having a <u>large</u> overlap in their wave funcs.

i.e. $H_{xx} - H_{yy} =$ large

ii) In heavy nuclei, the isospin purity has <u>quite</u> different reasons.

1- Since $N > Z$ $\longrightarrow$ Fermi energy for neutrons is much higher than that for protons.

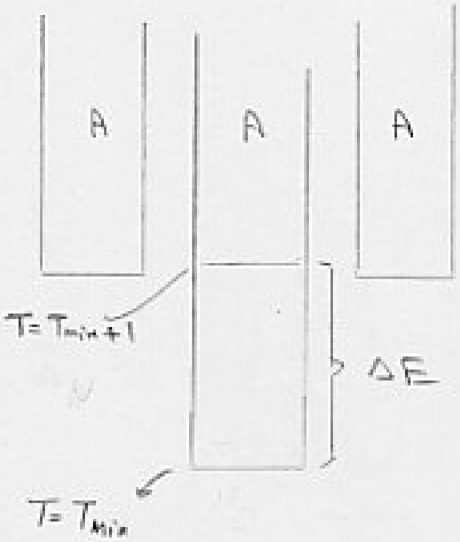


$T_{min} = \frac{1}{2} |Z - N|$

and the <u>dominant</u> component in the ground-state wave func. is given by the configuration with all nucleons occupying the lowest available single-particle states.

Admixtures of isospin will have to come from states with $T = T_{min} + 1$ -

The location (in energy) of such a state can be found by a method discussed in the next argument using the analogue states (neighboring isobars) -



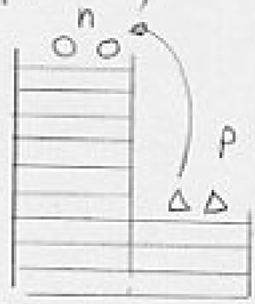
ΔE is large in heavy nuclei

$\longrightarrow$ It is difficult to have any significant isospin mixing from the states like ($T = T_{min}+1$) in the ground-states of heavy nuclei.

As we will see in the following argument;

$$E(T_{min}+1) - E(T_{min}) \simeq \Delta E_B$$ (the nucleus and neighboring isobar with one more neutron and one less proton)

Looking the Fig, it is clear that in heavy nuclei this difference is large.



Isobaric analogue states:

In <u>light</u> nuclei the different members of an isobar (Isobaric analogue states) show similitaries in their properties.

Examples are $A = 11$, $A = 21$ and $A = 16$.

In $A = 16$

$^{16}N \begin{cases} Z = 7 \\ N = 9 \end{cases}$, The ground state isospin is $T = 1$, $T_z = -1$

$$\underset{(T=1)}{T_+ |^{16}N, T_z = -1 >_G} = \underset{(T=1)}{|^{16}O, T_z = 0 >_{Ex}}$$

$\begin{cases} G: \text{ground state} \\ Ex: \text{excited state} \end{cases}$

$^{16}O \begin{cases} Z = 8 \\ N = 8 \end{cases}$ $\underset{G}{(T = 0, T_z = 0)}$ $\underset{Ex}{(T = 1, T_z = 0)}$

T_+ change one $n \xrightarrow{\text{to}}$ one p

The structure of the state remains <u>unchanged</u>.

Also since $H_{nucl.}$ is charge-independent (assumption)

$[H, T_z] = 0 \longrightarrow T_+ |T, T_z>$ is also an <u>eigenstate</u> of $H \xrightarrow{\text{hence}}$ Corresponds to an observable state ^{16}O.

This state (i.e. $|{}^{16}O, T=1, T_z=0\rangle$) must have very similar properties as the ground states of ^{16}N (since their wave funcs. are the same except for T_z)

Two states $|{}^{16}N, T=1, T_z=-1\rangle_G$ and $|{}^{16}O, T=1, T_z=0\rangle_{Ex}$ are called isobaric analogue states (IAS) of each other.

We can estimate the location of such an excited state ($^{16}O_{Ex}$).

If the nuclear forces are completely charge-indep;

$\longrightarrow$ Excitation energy of ^{16}O $(T=0 \to T=1) = E_B(^{16}O) - E_B(^{16}N)$

where $E_B(^{16}O) = 127.62$ MeV $E_B(^{16}N) = 117.98$ MeV

$\Delta E = 9.64$ MeV

Two corrections:

1- Contributions of the Coulomb int. to $E_B \sim$ number of P

Using a simple model, of a uniformally charged sphere with $R = 1.2A^{1/3} = 3.52$ fm $\rightarrow \Delta E_c = 4.00$ MeV i.e. 9.64 Mev $\longrightarrow$ 13.64 Mev

2- Correction due to $\Delta m = m_n - m_p$

$\longrightarrow$ 0.78 Mev

13.64 $\longrightarrow$ 13.64 − 0.78 = 12.86 Mev

The ground state of ^{16}N $\longrightarrow$ $J^{\pi} = 2^{-}$ (T=1)

There is a $J^{\pi} = 2^{-}$ state at 12.97 mev in ^{16}O;

The two states $\left\{\begin{array}{ll} J^{\pi}=2^{-} & {}^{16}N \\ J^{\pi}=2^{-} & {}^{16}O \end{array}\right.$ have also very

similar properties (by experiment)

$\left\{\begin{array}{l} \text{Our estimate} = 12.86 \text{ Mev} \\ \text{Observed value} = 12.97 \end{array}\right. \rightarrow$ difference = 0.11 Mev

This difference is attributed to:

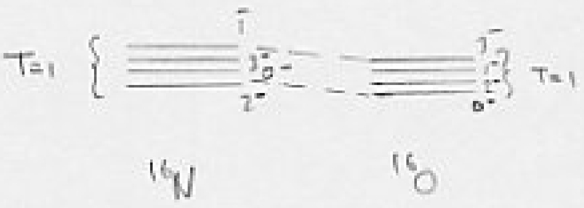
1- Crudness of our Coulomb energy Calculation

2- Small difference in the wave fncs.;

$|{}^{16}O, 12.97\,\text{mev}\rangle$ and $T_{+}|{}^{16}N\rangle$

A careful examination;

A quartet states with $J^\pi = \begin{cases} 2^- \\ 0^- \\ 3^- \\ 1^- \end{cases}$



in the ground state of ^{16}N (T=1).

The same quartet of T=1 states is also found in ^{16}O.

However the sequence of 4-levels is different from that in ^{16}N $\xrightarrow{\text{showing}}$ a small violation of the IAS idea in the actual nuclei.

4-10 Nuclear Binding Energy:

$$E_B(Z,N) = \left(Z M_H + N M_n - M(Z,N) \right) c^2$$

$M(Z,N)$: the mass of neutral atom (including the mass of electrons and their binding energies)

(given in the tables)

$m_e c^2 = 0.5110034$ MeV

Atomic mass unit (amu):

$$u = \frac{\text{mass of } ^{12}C \text{ atom}}{12} = \frac{1\,\text{kg}}{N_A}$$

$$= 1.6605402(10) \times 10^{-27}\,\text{kg} = 931.49432(28)\,\text{MeV}/c^2$$

when $N_A = 6.0221367(36) \times 10^{26}$

$\longrightarrow M_p = 1.007276470(12)\,u$

$M_n = 1.008664904(14)\,u$

$M_{^{12}C} = 12\,u$

Since $E_B << M_0(Z,N) \longrightarrow$

$M_0(Z,N)$ in amu $\approx A = Z+N$ (numerically)

It is therefore more convenient to express nuclear mass difference in terms of mass excess $\Delta(Z,N)$ (or mass defect)

$$\Delta(Z,N) \equiv (M(Z,N) \text{ in } u - A) \times 931.49432$$

For example $\Delta(H) = (1.007276470 - 1) \times 931.49432 + 0.5110$
$= 7.2890\,\text{MeV}$

and for <u>free</u> n;

$$\Delta(n) = (1.008664904 - 1) \times 931.49432 = 8.0713\,\text{MeV}$$

$$\longrightarrow E_B(Z,N) = Z\,\Delta(H) + N\,\Delta(n) - \Delta(Z,N)$$

Saturation of nuclear force:

$$\frac{\overline{E_B}}{A} \simeq \text{const.} \quad 8-8.5 \text{ MeV} \qquad \text{except for } A<30$$

shows → Short range nature of nucl. force

i) For two-body, infinite-range int.

$E_B \sim$ number of pairs of interacting particles, $\sum_{ij} \sim A^2$

$$\longrightarrow \frac{\overline{E_B}}{A} \sim A$$

ii) The fact that $\frac{\overline{E_B}}{A} = \text{const.}$ → each nucleon interacts mainly with a few of its nearby neighbors.

$\left(\frac{\overline{E_B}}{A}\right)_{Max}$ occurs at around $A \simeq 56$ (most tightly nuclei)

As A increases ⟶ $\frac{\overline{E_B}}{A}$ decreases (slightly)

This is due to rising of the Coulomb repulsion.

For this reason there is no stable nuclei beyond $Z=82$,

→ Nuclei $(Z>82)$ $\xrightarrow{\text{Fission}}$ $(\text{Nuclei})_1 + (\text{Nuclei})_2 + \cdots$

light nuclei

Among light nuclei;

$\frac{\bar{E}_B}{A}$ shows many sharp peaks occuring at $A = 2n$ for $n = 1, 2 \ldots$. All these nuclei are even-even with $Z = N$, ^4He, ^8Be, , ^{12}C, and ^{16}O, etc.

For example;

$\frac{\bar{E}_B}{A} =$ 2.83 for ^3_1H (triton), 2.57 for ^3_2H, 5.48 for ^5He

5.27 for ^5_3Li

while 7.07 MeV for ^4He

Also > 7 MeV for ^8Be, ^{12}C, ^{16}O

This phenomenon is known as the saturation of nuclear force.

i.e. Nucl. force is strong in α-particle-like cluster consisting of two protons and two neutrons.

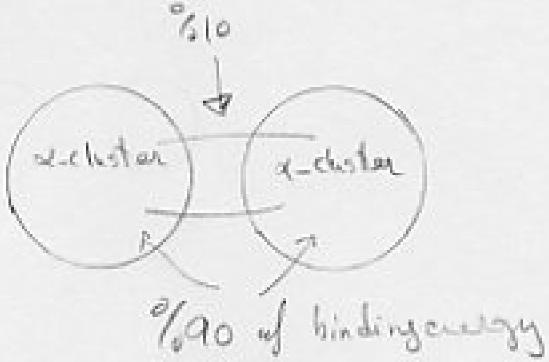
On the other hand the int. between α-clusters is relatively weak.

For example $^8Be \xrightarrow{\text{unstable}} {}^4He + {}^4He$

$\frac{\overline{E}_B}{A}$ for ^{12}C and ^{16}O are 7.68 MeV and 7.98 MeV respectively

these are not much larger than the value 7.07 MeV for ^4He.

A simple interpretation:

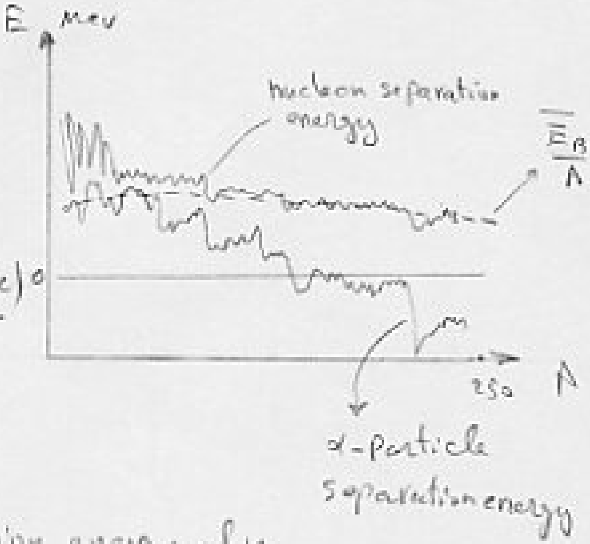


The rise in $\frac{\overline{E}_B}{A}$ of 4n-nuclei is no longer visible beyond ^{16}O.

This is because the increase in $\frac{\overline{E}_B}{A}$ due to addition of a new α-cluster is averaged over a large number of nucleons.

However, the saturation effect persists to heavy nuclei. ⟶

(separation energy of α-particle in 4n nuclei is large)



$$S_n(Z,N) = E_B(Z,N) - E_B(Z,N-1)$$

separation energy of n

$$S_p(Z,N) = E_B(Z,N) - E_B(Z-1,N)$$

" " " p

1- In addition to an increase in the separation energies for Un-nuclei because of saturation property of nuclear force

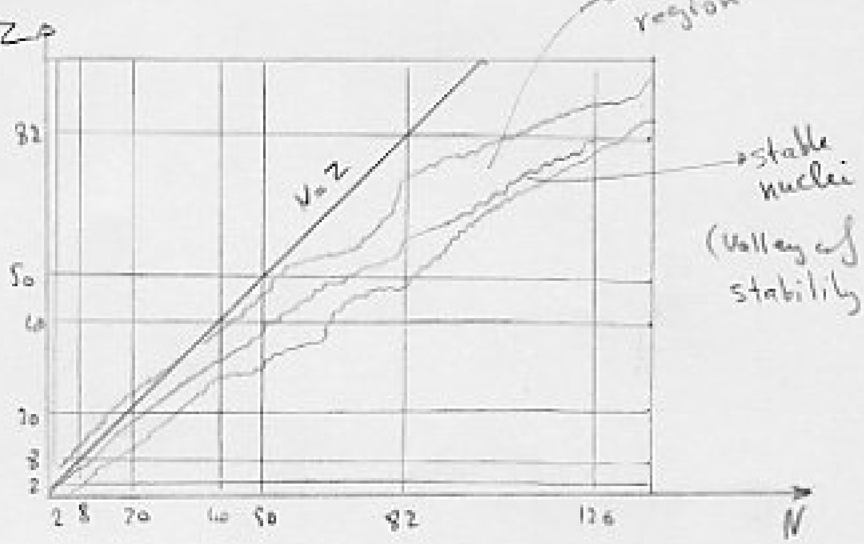
2- $S_n(Z,N)$ are larger for even-N than the corresponding values for nearly odd-N nuclei.

3- Similar effects are seen in $S_p(Z,N)$ between neighboring nuclei with odd-Z and even-Z.

These are the consequence of pairing effects.

Additional variation of separation energy are also found whenever N and Z is near one of the magic numbers,

2, 8, 20, 50 (40), 82 and 126



Valley of Stablity:

As A ⟶ large

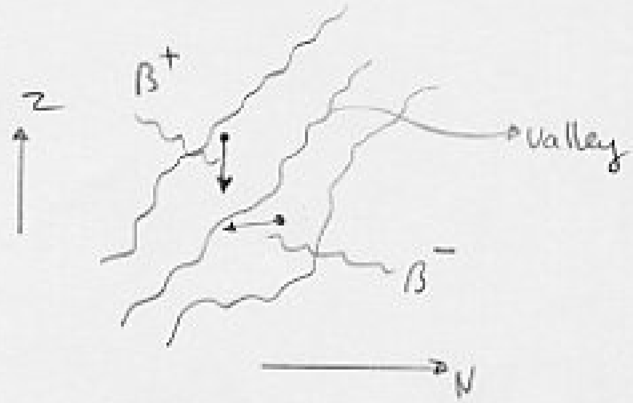
⟶ N > Z

to overcome the Coulomb repulsion between a large number of protons.

In most regions the number of stable nuclei for a given N, Z or A is small.

Lifetime of unstable nuclei on both sides of valley decreases as we move away from the central region.

If the p or n excess is large $\longrightarrow$ particle emission becomes becomes a possible channel of decay, and since int. process is strong $\longrightarrow$ lifetimes decrease dramatically.



Why the stable nuclei are clustered along a narrow region with $Z \simeq N$?

From deutron case we remember that

$$\text{Nucl. force is} \begin{cases} \text{attractive between non-identical nucleons} \\ \text{repulsive} \quad " \quad \text{identical} \quad " \end{cases}$$

In the first case E_B increases with NZ

In the second " E_B decreases with $\sim N^2$ and Z^2 (Proton pairs and neutron pairs).

The net result $\longrightarrow$ Nuclei with $N = Z$ are the more stable ones.

However as $Z \longrightarrow$ increases, Coulomb repulsion requires additional attraction, which is compensated by the excess of neutrons.

4-11 Semi-Empirical Mass Formulae

If we ignore small local departures, it is possible to develop simple mass formulae.

Weizaker mass formula;

This formula is based on the analogy of a nucleus to a drop of incompressible fluid.

The evidence in support of this model comes from, the fact that

$$V_{\substack{vol. \\ nucl.}} \sim A$$

We have seen;

$$E_B(Z,N) = \alpha_1 A$$

Since the binding energy on the surface is less than interior region;

$$E_B(Z,N) = \alpha_1 A - \alpha_2 A^{2/3}$$

There is also a correction due to Coulomb repulsion;

$$E_c = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{3}{5}\frac{Z(Z-1)e^2}{R} = \frac{3\alpha\hbar c}{5r_0}\frac{Z(Z-1)}{A^{1/3}}$$

$$= 0.72\frac{Z(Z-1)}{A^{1/3}} \text{ MeV}$$ for uniform spherical charge of $R = 1.2 A^{1/3}$ fm.

A real nucleus has a diffused surface (not spherical) $\longrightarrow$ We leave the strength of E_c as an adjustable parameter.

$$E_B(Z,N) = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z(Z-1)}{A^{1/3}}$$

So far classical ideas have been used in obtaining this formula.

Quantum mechanical effects (no classical analogues);

1- Isospin dependence of nucl. forces;

Except for Coulomb repulsion, stable nuclei prefer to have $N \simeq Z$

The effect is proportional to $(N-Z)^2$.

$$E_B(Z,N) = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z(Z-1)}{A^{1/3}} - \alpha_4 \underbrace{\frac{(N-Z)^2}{A}}_{\text{symmetry energy}}$$

The new term is divided by A, since N and Z and hence (N-Z) increase with A, whereas isospin effect is essentially const. indep. of A.

2- Pairing effect;

even-even nuclei $\underset{\text{more tight}}{>}$ odd-odd nuclei (with the same A)

and odd-mass nuclei nearly have intermediate values between them.

$$\begin{cases} N = \frac{A}{2}+\nu \\ Z = \frac{A}{2}-\nu \\ \Delta E = \nu^2 \Delta \end{cases} \rightarrow \nu = \frac{N-Z}{2}$$

$\Delta E = 3 \times 3 \Delta$

$$E_B(Z,N) = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z(Z-1)}{A^{1/3}} - \alpha_4 \frac{(N-Z)^2}{A} + \Delta$$

$$\Delta = \begin{cases} \delta & \text{for even-even nuclei} \\ 0 & \text{" odd-mass "} \\ -\delta & \text{" odd-odd "} \end{cases}$$

Weizacker mass formula

The consts. $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and δ may be obtained by fitting the known binding energies.

Most common set:

$\alpha_1 = 16$ MeV $\qquad \alpha_2 = 17$ MeV $\qquad \alpha_3 = 0.6$ MeV

$\alpha_4 = 25$ MeV $\qquad \delta = \frac{25}{A}$ MeV

This formula is useful for example, in obtaining fission energies.

However it does not include the effects such as local increase in binding energy for nuclei near closed shells

The δ parameter is perhaps, the least well determined quantity in the set and is found to be much larger in heavy nuclei.

Kelson - Garvey mass formula;

The aim of Weizacker mass formula is to obtain global agreement, and it does not give good results for the binding energy differences of nearby nuclei.

Instead of liquid drop model, a microscopic model is used.

$$E_B = \sum_i V_i + \sum_{ij} V_{ij}$$ (in this model)

For different mass regions ($A - A'$ = large)

V_i, and V_{ij} may vary, but for ($A - A'$ = small) they must essentially be consts.

V_i and V_{ij} may be obtained from the known binding energies of nuclei and then to be used to predict the unknown binding energies in the same region.

Illustrating the model by a simple example:

Consider one-body int.;

$$E_B(Z, N) = \alpha N + \beta Z$$

α: average values of the int. of a neutron with the rest of the nucleons.

β: " " " " " " proton " " "

In the neighborhood (α = const, β = const.);

$$E_B(Z+1, N) = \alpha N + (Z+1)\beta$$

$$E_B(Z, N+1) = \alpha(N+1) + \beta Z$$

$\rightarrow \alpha = E_B(Z, N+1) - E_B(Z, N)$, $\beta = E_B(N+1, Z) - E_B(N, Z)$

Also;

$$E_B(Z+1,N+1)-E_B(Z,N)=\alpha+\beta=$$

$$=E_B(Z,N+1)-E_B(Z,N)+E_B(Z+1,N)-E_B(Z,N)$$

$$\rightarrow E_B(Z+1,N+1)+E_B(Z,N)-E_B(Z,N+1)-E_B(Z+1,N)=0$$

Using this relation, the binding energy of <u>any</u> one of the <u>four</u> nuclei may be deduced from the known values of the other three.

We can test the validity of this relation by using the actual binding energies of 4-nuclei.

$$\Delta E_B=E_B(Z+1,N+1)+E_B(Z,N)-E_B(Z,N+1)-E_B(Z+1,N)$$

We find $\Delta E_B \neq 0 \rightarrow$ large

$\longrightarrow$ This is due to the ignored two-body int.

Generalization;
We include two-body int.

Two-body ints. are between the nn, np, pp pairs.

$$\text{Number of pairs} \begin{cases} \frac{1}{2} Z(Z-1) & \text{for } pp \\ \frac{1}{2} N(N-1) & " \ nn \\ NZ & " \ pn \end{cases}$$

$$\rightarrow E_B = aN + bZ + cN(N-1) + dZ(Z-1) + eNZ$$

$$\begin{cases} E_B(Z+1, N-1) = b(Z+1) + dZ(Z+1) + e(N-1)(Z+1) + f(a,c,N) \\ -E_B(Z, N-1) = -bZ - dZ(Z-1) - e(N-1)Z - f(a,c,N) \end{cases}$$

$$E_B(Z+1, N-1) - E_B(Z, N-1) = b + 2dZ + e(N-1)$$

$$\begin{cases} E_B(Z, N+1) = bZ + dZ(Z-1) + e(N+1)Z + g(a,c,N) \\ -E_B(Z-1, N+1) = -b(Z-1) - d(Z-1)(Z-2) - e(N+1)(Z-1) - g(a,c,N) \end{cases}$$

$$E_B(Z, N+1) - E_B(Z-1, N+1) = b + 2d(Z+1) + e(N+1)$$

$$\begin{cases} E_B(Z+1, N) = b(Z+1) + dZ(Z+1) + eN(Z+1) + h(a,c,N) \\ -E_B(Z-1, N) = -b(Z-1) - d(Z-1)(Z-2) - eN(Z-1) - h(a,c,N) \end{cases}$$

$$E_B(Z+1, N) - E_B(Z-1, N) = 2b + 2d(2Z+1) + 2eN$$

By eliminating b, d and e from these three equs.;

$$E_B(Z+1,N-1)+E_B(Z-1,N)+E_B(Z,N+1)-E_B(Z,N-1)$$
$$-E_B(Z+1,N)-E_B(Z-1,N+1)=0$$

$$\Delta E = E_B(Z+1,N-1)+E_B(Z-1,N)+E_B(Z,N+1)-E_B(Z,N-1)$$
$$-E_B(Z+1,N)-E_B(Z-1,N+1)$$

Although $\Delta E \neq 0$ but it is small (for arbitrary group) and it is distributed randomly around Zero.

$$\frac{1}{N}\left\{\sum_{i=1}^{N}(\Delta E_i)^2\right\}^2 \approx 0.1\ \text{MeV} \quad \text{(standard deviation)}$$

Each E_B from the six can be obtained from the other five.

It is more useful to obtain E_B of unstable nuclei (which are unknown and far way from the valley) from the known E_B of the stable nuclei in the valley by by extrapolation.