Chapter 3

Nuclear Force and Two-Nucleon Systems

3-1 The Deuteron

Binding Energy: — The deuteron is a very <u>unique</u> nucleus in many respects.

1- Only loosely bound nucleus
2- having binding energy much less than the average value between a pair of nucleons in all the other stable nuclei.

$$E_B = \{ [\sum M_{p_i} + \sum M_{n_i}] - M_{nucleus} \} c^2 \qquad (M_p \simeq M_H)$$

$$E_B = \{ [938.783 + 939.566] - 1876.124 \} = 2.225 \text{ Mev}$$

$E_B = 2.22457312(22)$ Mev more precise

This value is obtained, using the radiative capture of a neutron by hydrogen.

$$H + n \longrightarrow \gamma + d \qquad H(n,\gamma)d$$

If $E_n \simeq 0 \longrightarrow E_\gamma \simeq E_B$

Partly because of small E_B, the deutron has no excited state.

Ground state Property		Value
Binding energy	E_B	2.22457312(22) MeV
Spin & Parity	J^π	1^+
Isospin	T	0
Magnetic dipole moment	μ_d	0.857406(1) μ_N
Electric quadrupole "	Q_d	0.28590(30) e fm^2
Radius	r_d	1.963(4) fm

Parity, Spin and Isospin;

From experimental information $\longrightarrow \Pi_d = +$

$$\Psi_d = \Psi_n \Psi_p \Psi_{rel.}$$

intrinsic (r-dep.)   (l-dep)

$\Psi_n \Psi_p = +$ Since P and n are two different states of a single particle nucleon, their intrinsic wave funcs. have the same parity.

$\longrightarrow \Psi_{rel.}$ determines the parity of deuteron

$$Y_{LM}(\theta,\varphi) \xrightarrow{\Pi} Y_{LM}(\pi-\theta, \pi+\varphi) = (-1)^L Y_{LM}(\theta,\varphi)$$

Since $\Pi_d = + \longrightarrow L = \text{even}$ $\left\{ \begin{array}{l} \bar{r} \equiv \bar{r}_1 - \bar{r}_2 \xrightarrow{\Pi_{12}} -\bar{r} \\ (r,\theta,\varphi) \xrightarrow{\Pi_{12}} (r, \pi-\theta, \varphi+\pi) \end{array} \right\}$

The ground state of deuteron has J=1

$$\bar{J} = \bar{L} + \bar{S} \qquad S = 0, 1$$

$$\left\{ \begin{array}{l} J=1 \\ L=\text{even} \end{array} \right. \longrightarrow \cancel{S=0} \quad \cancel{L>2}$$

$\rightarrow (L,S) = (0,1)$ and $(2,1)$ for ground state

(0,1): dominant part   (2,1): small but significant amount admixture

$$\Psi_d = \Psi_r \chi_s \xi_t$$

Ψ_r: sym χ_s: sym ξ_t: $\hookrightarrow$ should be antisym.

$\longrightarrow T = 0 \quad \cancel{T=1}$

$$\chi^S_{m_S} = \begin{cases} \chi^1_1 = \alpha_1\alpha_2 \\ \chi^1_0 = \frac{1}{\sqrt{2}}\{\alpha_1\beta_2 + \alpha_2\beta_1\} \\ \chi^1_{-1} = \beta_1\beta_2 \end{cases} \qquad \begin{matrix} \alpha = |\tfrac{1}{2}, \tfrac{1}{2}\rangle_s \\ \beta = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_s \end{matrix}$$

$$\xi^t_{t_z} = \xi^0_0 = \frac{1}{\sqrt{2}}\{\alpha'_1\beta'_2 - \alpha'_2\beta'_1\} \qquad \begin{matrix} \alpha' = |\tfrac{1}{2}, \tfrac{1}{2}\rangle_t \\ \beta' = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_t \end{matrix}$$

Alternatively, <u>if</u>

$$\xi^1_{t_z} = \begin{cases} \xi^1_1 & \text{(p-p system)} \\ \xi^1_0 & \text{(n-p system)} \\ \xi^1_{-1} & \text{(n-n system)} \end{cases} \qquad (\text{if deuteron} \in \xi^1_0)$$

→ we would expect the other two states. However, <u>no</u> such states have been observed.

i) The existance of p-p system may be rejected since the Coulomb repulsion is of order 1 Mev (large fraction of E_B)

ii) This limitation, however, <u>does not</u> apply for n-n system.

Conclusion $T=1$ bound state: impossible

Therefore; possible states are:

$T=0$, $S=1$, $L=0$: ${}^3S_1$ (triplet S-state)

$T=0$, $S=1$, $L=2$: ${}^3D_1$ (" D- ")

If L and S are good quantum number;

$$[H, L^2] = [H, S^2] = 0$$

$\longrightarrow$ deuteron would have to be in either one of these two states.

However, this is not the case.

3-2) Deuteron Magnetic Dipole Moment;

Mag. dipole operator;

Mag. dipole moment has two sources;

$\begin{cases} 1- \mu_n, \mu_p & \text{coming from the intrinsic spin and orbital motion of quarks.} \\ 2- \mu_p^{orb} & \text{Coming from orbital motion of Proton.} \end{cases}$

From electromagnetic theory;

$$\mu_i^{orb.} = \frac{e\hbar [c]}{2M_p c} l_i \qquad \text{in cgs } [c] \text{ is ignored.}$$

$$\mu_i^{orb.} = g_e(i) l_i \qquad g_\ell(i) = \begin{cases} 1\,\mu_N & \text{for } P \\ 0 & \text{" } n \end{cases}$$

$$\mu_i^{spin} = g_s(i) S_i$$

Since $S = \frac{1}{2}$ $\longrightarrow g_s(i) = \begin{cases} g_p = 2\mu_p = 5.585695\ \mu_N & \text{for } P \\ g_n = 2\mu_n = -3.826085\ \mu_N & \text{" } n \end{cases}$

for <u>free</u> nucleons

If we assume $\text{nucleon structure}_{\text{bound}} = \text{nucleon structure}_{\text{free}}$

We may use then values in nuclei as well.

$$\mu_d = g_p S_p + g_n S_n + l_p$$

Since $m_p \simeq m_n \longrightarrow l_p \simeq \frac{1}{2} L$ (rel. angular mom.) or angular mom. of deuteron

$$\mu_d = g_p S_p + g_n S_n + \frac{1}{2} L$$

Contribution from $^3S_1$-state:

For this state $L = 0$, we will see soon that;

$$\mu_d(^3S_1) = \mu_p + \mu_n = 0.879805\ \mu_N$$

There is a small difference with the observed value;

$$\mu_d(\text{obs.}) - \mu_d(^3S_1) = 0.857406 - 0.879805 = -0.022399\ \mu_N$$

This difference may be attributed to three possibilities:

1) $\text{nucleon structure}_{\text{bound}} \neq \text{nucleon structure}_{\text{free}}$

$$\longrightarrow g_{p_{\text{bound}}} \neq g_{p\,\text{free}} \qquad g_{n_{\text{bound}}} \neq g_{n\,\text{free}}$$

2) There are contributions from charged mesons <u>exchanged</u> between the proton and the neutron, and these have not been included in our calculations.

3) There are small admixtures of $^3D_1$-state in the ground state of deuteron.

i) Item 1 is <u>extremely</u> unlikely, since the deuteron is a loosely bound system;

Because $E_B = 2.22\ \text{Mev} << E_{B(\text{quarks})} \sim$ hundreds of Mev

$\longrightarrow$ $E_B = 2.22\ \text{Mev}$ is not expected to affect the motion of quarks.

ii) The effect of mesonic currents is possible.

iii) Item 3 is more likely to be <u>major</u> case for this discrepancy

Expectation value of the magnetic dipole operator;

By <u>convention</u>, the mag. moment, similar to other static electromagnetic moments is defined as the expectation value of the <u>z-component</u>, or $q=0$ component in spherical tensor notation of the operator in the substate of Max. M, $M=J$.

$$\mu_d = \langle J, M=J | \mu_z | J, M=J \rangle$$

Now first we show;

$$\langle \alpha', jm' | V_q | \alpha, jm \rangle = \frac{\langle \alpha', jm | J \cdot V | \alpha, jm \rangle}{j(j+1)} \langle jm' | J_q | jm \rangle$$

$$J \cdot V = \sum_q (-1)^q J_{1q} V_{1,-q}$$

$$\langle \alpha', jm | J \cdot V | \alpha, jm \rangle = \langle \alpha', jm | (J_0 V_0 - J_{+1} V_{-1} - J_{-1} V_{+1}) | \alpha, jm \rangle$$

$$= m \langle \alpha', jm | V_0 | \alpha, jm \rangle + \frac{1}{\sqrt{2}} \sqrt{(j+m)(j-m+1)} \langle \alpha', jm-1 | V_{-1} | \alpha\, jm \rangle$$

$$- \frac{1}{\sqrt{2}} \sqrt{(j-m)(j+m+1)} \langle \alpha', jm+1 | V_{+1} | \alpha, jm \rangle = C_{jm} \langle \alpha' j || V || \alpha j \rangle \quad (1)$$

Remember $J_{\pm} = \mp \frac{1}{\sqrt{2}} (J_x \pm i J_y) = \mp \frac{1}{\sqrt{2}} J_{\pm}'$, $J_0 = J_z$

C_{jm} does not depend on α', α and V.

Also since $J \cdot V$ is a scalar op $\longrightarrow$ C_{jm} is independent of m

$\longrightarrow \quad C_{jm} \rightarrow C_j$

If $V \rightarrow J$

$\longrightarrow \langle \alpha, jm | J^2 | \alpha, jm \rangle = C_j \langle \alpha j || J || \alpha j \rangle \quad (2)$

By Wigner-Eckart theorem;

$$\langle \alpha', j'm' | T_k^q | \alpha, jm \rangle = (jk, mq | jk j'm') \langle \alpha' j' || T_k || \alpha j \rangle$$

$$\frac{\langle \alpha', jm' | V_q | \alpha, jm \rangle}{\langle \alpha, jm' | J_q | \alpha, jm \rangle} = \frac{\langle \alpha' j \| V \| \alpha j \rangle}{\langle \alpha j \| J \| \alpha j \rangle}$$

(1) and (2) $\longrightarrow$ $$\frac{\langle \alpha' j \| V \| \alpha j \rangle}{\langle \alpha j \| J \| \alpha j \rangle} = \frac{\langle \alpha', jm | J \cdot V | \alpha, jm \rangle}{\langle \alpha, jm | J^2 | \alpha, jm \rangle}$$

Since $\langle \alpha, jm | J^2 | \alpha, jm \rangle = j(j+1)$

$$\rightarrow \langle \alpha', jm' | V_q | \alpha, jm \rangle = \frac{\langle \alpha', jm | J \cdot V | \alpha, jm \rangle}{j(j+1)} \langle jm' | J_q | jm \rangle$$

Now
$$\langle jm | \mu_z | jm \rangle = \frac{\langle jm | \mu \cdot J | jm \rangle}{j(j+1)} \langle jm | J_z | jm \rangle$$

$$= \frac{m}{j(j+1)} \langle jm | \mu \cdot J | jm \rangle$$

Projection of μ on J

$$\mu_d = g_p s_p + g_n s_n + \frac{1}{2} L = \frac{1}{2} \left\{ (g_p + g_n) S + (g_p - g_s)(s_p - s_n) + L \right\}$$

But
$$\langle S\, m_s | (s_p - s_n) | S' m_s' \rangle = \begin{cases} 0 & S = S' \\ \neq 0 & S = 1,\ S' = 0 \\ \neq 0 & S = 0\ \ S' = 1 \end{cases}$$

But we are interested in $S = S' = 1$ case. Therefore its contribution in our case is <u>Zero</u>.

This reduces μ_d to a func. of L and S only (in our case).

$$\langle Jm|\mu_z|Jm\rangle = \frac{m}{J(J+1)}\langle Jm|\tfrac{1}{2}\{(g_p+g_n)(S\cdot J)+(L\cdot J)\}|Jm\rangle$$

$$S\cdot J = S\cdot(L+S) = S^2+\tfrac{1}{2}(J^2-L^2-S^2) = \tfrac{1}{2}(J^2-L^2+S^2)$$
$$L\cdot J = L\cdot(L+S) = L^2+\tfrac{1}{2}(J^2-L^2-S^2) = \tfrac{1}{2}(J^2+L^2-S^2)$$

$$\mu_d = \langle J\ m{=}j|\mu_z|J, m{=}j\rangle = \frac{1}{4(J+1)}\Big\{(g_p+g_n)\big(J(J+1)-L(L+1)+S(S+1)\big) + \big(J(J+1)+L(L+1)-S(S+1)\big)\Big\}$$

For $^3S_1$-state, $L=0$, $J=1$, $S=1$

$$\mu_d(^3S_1) = \mu_p+\mu_n$$

For $^3D_1$-state $L=2$, $J=1$, $S=1$

$$\mu_d(^3D_1) = \frac{1}{8}\{(g_p+g_n)(-2)+6\} = 0.310\ \mu_N$$

Since $\mu_d(^3D_1) < \mu_d(^3S_1)$

$\longrightarrow$ any admixture of $^3D_1$-state will reduce the value of μ_d from that given by $\mu_d(^3S_1)$.

$$\mathcal{Y}_{\ell SJ}^{M} = \sum_{m_s m_\ell} (\ell s\, m_\ell m_s / J M)\, \chi_s^{m_s}\, Y_\ell^{m_\ell}$$

With the notation: $\quad {}^{b}X_{a} \quad \begin{cases} X: \ell \\ a: J \\ b: 2S+1 \end{cases}$

${}^3S_1$: $\quad J=1 \quad M=1 \quad \ell=0 \quad S=1$

$$\mathcal{Y}_{011}^{1} = (0101/11)\, Y_0^0\, \alpha(1)\alpha(2) = Y_0^0\, \alpha(1)\alpha(2)$$

${}^3D_1$: $\quad J=1 \quad M=1 \quad \ell=2 \quad S=1$

$$\mathcal{Y}_{211}^{1} = \sum_{m_s} (21\,(1-m_s)\, m_s / 11)\, Y_2^{1-m_s}\, \chi_1^{m_s}$$

$$= (212\,{-1}/11)\, Y_2^2 \chi_1^{-1} + (2110/11)\, Y_2^1 \chi_1^0 + (2101/11)\, Y_2^0 \chi_1^1$$

$$= \sqrt{\tfrac{3}{5}}\, Y_2^2\, \beta(1)\beta(2) - \sqrt{\tfrac{3}{10}}\, Y_2^1\, \frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) + \beta(1)\alpha(2)\right] + \frac{1}{\sqrt{10}}\, Y_2^0\, \alpha(1)\alpha(2)$$

where $\begin{cases} \chi_1^1 = \alpha(1)\alpha(2) \\ \chi_1^0 = \frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) + \beta(1)\alpha(2)\right] \\ \chi_1^{-1} = \beta(1)\beta(2) \end{cases}$

${}^3S_1$: $\quad \Phi_{011}^{1}(r) = \dfrac{u_0(r)}{r}\, \mathcal{Y}_{011}^{1} = \dfrac{u_0(r)}{r}\, Y_0^0\, \chi_1^1$

${}^3D_1$: $\quad \Phi_{211}^{1}(r) = \dfrac{u_2(r)}{r}\, \mathcal{Y}_{211}^{1}$

Admixture of ${}^3D_1$-state:

$$|\Psi_d\rangle = a\,|{}^3S_1\rangle + b\,|{}^3D_1\rangle$$

experimental

$$\begin{cases} \langle \Psi_d | \mu_z | \Psi_d \rangle = a^2 \mu_d({}^3S_1) + b^2 \mu_d({}^3D_1) = 0.857\ \mu_N \\ \quad \text{(no off-diagonal matrix element for } \mu_z\text{)} \\ \langle \Psi_d | \Psi_d \rangle = a^2 + b^2 = 1 \end{cases}$$

$\rightarrow b^2 = 0.04 \longrightarrow$ 4% admixture of the ${}^3D_1$

3-3 Deuteron Electric Quadrupole Moment:

Multipole Expansion of potential:

$$\Phi(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

$$\Phi(x) = \int \frac{\rho(x')}{|x-x'|} d^3x'$$

For the potential outside the charge distribution; $r_< = r'$, $r_> = r$.

$$\Phi(x) = 4\pi \sum_{lm} \frac{1}{2l+1} \left[\int Y^*_{lm}(\theta',\varphi') r'^l \rho(x') d^3x' \right] \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

$$\rightarrow q_{lm} = \int Y^*_{lm}(\theta',\varphi') r'^l \rho(x') d^3x'$$

$$q_{l,-m} = (-1)^m q^*_{lm}$$

$$q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(x')\, d^3x' = \frac{1}{\sqrt{4\pi}}\, q$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int (x'-iy')\, \rho(x')\, d^3x' = -\sqrt{\frac{3}{8\pi}}\, (P_x - iP_y)$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int z'\, \rho(x')\, d^3x' = \sqrt{\frac{3}{4\pi}}\, P_z$$

$$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int (x'-iy')^2\, \rho(x')\, d^3x' = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\, (Q_{11} - 2i\, Q_{12} - Q_{22})$$

$$q_{21} = -\sqrt{\frac{15}{8\pi}} \int z'(x'-iy')\, \rho(x')\, d^3x' = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\, (Q_{13} - iQ_{23})$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int (3z'^2 - r'^2)\, \rho(x')\, d^3x' = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{33}$$

$$P = \int x' \rho(x')\, d^3x'$$

$$Q_{ij} = \int (3x_i' x_j' - r'^2 \delta_{ij})\, \rho(x')\, d^3x'$$

$$\Phi(x) = \frac{q}{r} + \frac{P \cdot x}{r^3} + \frac{1}{2} \sum_{ij} Q_{ij} \frac{x_i x_j}{r^5} + \cdots$$

In a nucleus it is not possible to observe the charge distribution density directly, and the measured multipole moments may be used to infer somthing about the source.

Since $Y_{l,m}(\theta,\varphi) \xrightarrow{\Pi} (-1)^l\, Y_{lm}(\theta,\varphi)$

All odd multipoles electric operators give <u>vanishing</u> expectation values (for pure ground states).

$\rightarrow$ The lowest <u>non-vanishing</u> multipole moment $\rightarrow$ quadrupole moment

$Q_d = 0.28590\, e\, fm^2$ (measured)

Quadrupole operator:

For a spherical nucleus;

$$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle$$

As a result;

$$\langle r^2 \rangle = \langle x^2 + y^2 + z^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle$$

$$= 3 \langle z^2 \rangle$$

Note:

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

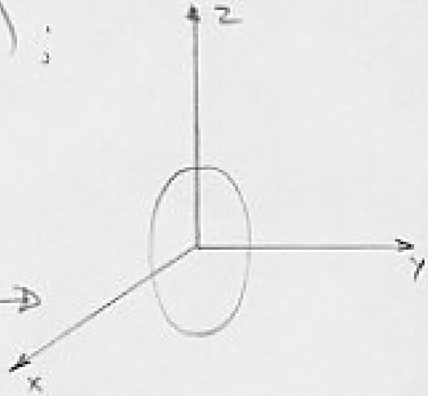
The charge quadrupole op. which measures the lowest order departure from a spherical symmetry is defined by;

$$Q_z = e(3z^2 - r^2)$$

For spherical nucleus (charge distribution);

$$\langle Q_z \rangle = 0$$

For deuteron: $\langle Q_z \rangle = +0.29$ $\longrightarrow$



$$Q_z = e(3z^2 - r^2) = er^2(3\cos^2\theta - 1)$$

$$= \sqrt{\frac{16\pi}{5}}\, er^2\, Y_{20}(\theta, \varphi)$$ spherical tensor of rank 2

$$Q_A = \langle j\ m=j \,|\, Q_z \,|\, j\ m=j \rangle$$ Electric quadrupole moment of a nucl. state

Remark:

$$\langle \alpha', j'm' | T_q^k | \alpha, jm \rangle = 0 \quad \text{unless } m' = q + m$$

also $\quad |j-k| \leq j' \leq j+k \quad$ (triangle cond.)

Here $k=2$

$\longrightarrow$ For $J<1 \longrightarrow Q_A = 0$

At the same time since Q_2 operates on only in the coordinate space; it is independent of total spin S.

$\xrightarrow{\text{i.e.}}$ $L \geq 1$ (must be)

$\longrightarrow \langle {}^3S_1 | Q_2 | {}^3S_1 \rangle = 0$

The nonvanishing $Q_d = 0.28590 \longrightarrow$ Presence of ${}^3D_1$-component in the ground state

Expectation value of the Quadrupole operator:

Assumption: Deuteron ground state is a <u>definite</u> L.

$$|LS;JM\rangle = \sum_{M_L M_S} (LM_L, SM_S / JM)\, |LM_L\rangle |SM_S\rangle$$

$$Q_d(L) = \langle LS; JM | Q_z | LS; JM\rangle$$

$$= \sum_{M_L M_S} \sum_{M_L' M_S'} (LM_L, SM_S / JM)(LM_L', SM_S' / JM) \langle LM_L, SM_S | Q_z | LM_L', SM_S'\rangle$$

where $|LM_L, SM_S\rangle \equiv |LM_L\rangle |SM_S\rangle$

Since $\langle SM_S | SM_S'\rangle = \delta_{M_S M_S'}$

$$Q_d(L) = \sum_{M_L} (LM_L S\,(M-M_L) / JM)^2 \langle LM_L | Q_z | LM_L\rangle$$

($M_L = M_L'$, because $Q_z = 0$ for $M_L \neq M_L'$)

$$|LM_L\rangle = R_L(r)\, Y_{LM_L}(\theta,\varphi)$$

$$\int_0^\infty R_L^*(r) R_L(r)\, r^2\, dr = 1$$

$$\langle LM_L | e(3z^2 - r^2) | LM_L\rangle = \langle LM_L | \sqrt{\frac{16\pi}{5}}\, e r^2\, Y_{20}(\theta,\varphi) | LM_L\rangle$$

$$= e\sqrt{\frac{16\pi}{5}} \int_0^\infty R_L^*(r)\, \underline{r}^2 R_L(r)\, r^2 dr \int_0^{2\pi}\int_{-1}^{1} Y_{LM_L}^*(\theta,\varphi)\, Y_{20}(\theta,\varphi)\, Y_{LM}(\theta,\varphi)\, d(\cos\theta)\, d\varphi$$

Since

$$\langle \alpha JM | T_q^k | \alpha' J' M' \rangle = (-)^{J-M} \begin{pmatrix} J & k & J' \\ -M & q & M' \end{pmatrix} \langle \alpha J \| T^k \| \alpha' J' \rangle$$

where

$$\langle \alpha J \| T^k \| \alpha' J' \rangle = (2k+1) \sum_{MM'} (-)^{J-M} \begin{pmatrix} J & k & J' \\ -M & q & M' \end{pmatrix} \langle \alpha JM | T_q^k | \alpha' J' M' \rangle$$

Here $T_q^k = Y_0^2$

$$\langle L M_L | Y_0^2 | L M_L \rangle = (-)^{L-M} \begin{pmatrix} L & 2 & L \\ -M_L & 0 & M_L \end{pmatrix} \langle L \| Y^2 \| L \rangle$$

But $\langle L \| Y^k \| L' \rangle = (-)^L \sqrt{\frac{(2\ell+1)(2k+1)(2\ell'+1)}{4\pi}} \begin{pmatrix} L & k & L' \\ 0 & 0 & 0 \end{pmatrix}$

$$\langle L \| Y^2 \| L \rangle = (-)^L (2L+1) \sqrt{\frac{5}{4\pi}} \begin{pmatrix} L & 2 & L \\ 0 & 0 & 0 \end{pmatrix}$$

$$\langle L M_L | Y_0^2 | L M_L \rangle = (-)^{M_L} \begin{pmatrix} L & 2 & L \\ -M_L & 0 & M_L \end{pmatrix} (2L+1) \sqrt{\frac{5}{4\pi}} \begin{pmatrix} L & 2 & L \\ 0 & 0 & 0 \end{pmatrix}$$

and where

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} (j_1 m_1, j_2 m_2 / j_1 j_2 j_3 - m_3)$$

$$\langle 2\,2 | Y_0^2 | 2\,2 \rangle = -\frac{1}{7}\sqrt{\frac{5}{\pi}}$$

$$\langle 2\,1 | Y_0^2 | 2\,1 \rangle = +\frac{1}{14}\sqrt{\frac{5}{\pi}}$$

$$\langle 2\,0 | Y_0^2 | 2\,0 \rangle = +\frac{1}{7}\sqrt{\frac{5}{\pi}}$$

The Clebsch-Gordon coeffs. for; $S=1$, $L=2$, $M=J=1$

$$(L M_L, S(M-M_L) / J M)^2 = (2\,2\,1\,-1 / 1\,1)^2 = \frac{6}{10}$$

$$(2\,1\,1\,0 / 1\,1)^2 = \frac{3}{10}$$

$$(2\,0\,1\,1 / 1\,1)^2 = \frac{1}{10}$$

$$Q_d(L) \equiv Q_d(2) \equiv Q_d(^3D_1) = \frac{6}{10} \langle L=2, M_L=2 | Q_z | L=2, M_L=2 \rangle$$

$$+ \frac{3}{10} \langle L=2, M_L=1 | Q_z | L=2, M_L=1 \rangle + \frac{1}{10} \langle L=2, M_L=0 | Q_z | L_2, M_L=0 \rangle$$

$$= e\sqrt{\frac{16\pi}{5}} \left\{ \frac{6}{10} \left(-\frac{\sqrt{5/\pi}}{7} \right) + \frac{3}{10} \left(\frac{\sqrt{5/\pi}}{14} \right) + \frac{1}{10} \left(\frac{\sqrt{\frac{5}{\pi}}}{7} \right) \right\} \langle r^2 \rangle_D$$

$$= \frac{-e}{5} \langle r^2 \rangle_D$$

where $\langle r^2 \rangle_D = \int_0^\infty R_D^*(r)\, r^2 R_D(r)\, r^2 dr$

An estimate:

If we take $\langle r^2 \rangle_D =$ square of deuteron radius

$\longrightarrow Q_d(^3D_1) = -0.77\ e\,fm^2$

but $Q_d = 0.28590\ e\,fm^2$ $\qquad Q_d(^3D_1) \neq Q_d$ (experimental)

$\longrightarrow$ The ground-state of deuteron is not entirely $^3D_1$-state.

More realistic model:

$$|\Psi_d\rangle = a|^3S_1\rangle + b|^3D_1\rangle$$

$$Q_d = a^2 \langle ^3S_1\, M=1 | Q_z | ^3S_1\, M=1 \rangle + b^2 \langle ^3D_1 M=1 | Q_z | ^3D_1\, M=1 \rangle$$

$$+ 2ab \langle ^3S_1 M=1 | Q_z | ^3D_1\, M=1 \rangle$$

with the notation:

$$|\underset{0}{L}\, \underset{1}{S},\, \underset{1}{J}\, \underset{1}{M} \rangle \equiv |^3S_1\, M=1\rangle$$

$\langle {}^3S_1, M=1 | Q_z | {}^3S_1, M=1 \rangle = 0$ since $L=0$

The main contribution comes from the last term (<u>off</u>-diagonal matrix)

$$\langle {}^3S_1, M=1 | Q_z | {}^3D_1, M=1 \rangle \sim \int_0^\infty R_S^*(r)\, r^2 R_D(r)\, r^2 dr$$

$|b|^2 = 4\%$ to 7% (There are some parameters in the wave funcs. R_S and R_D to be fixed)

3-4 Tensor Force and the Deuteron D-state

The analysis of μ_d and Q_d $\longrightarrow$ $|\Psi_d\rangle = a|{}^3S_1\rangle + b|{}^3D_1\rangle$

$\longrightarrow$ <u>No</u> definite L-value $\longrightarrow$ L is not a good quantum number

$\longrightarrow [V \;\; L^2] \neq 0$

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 + V$$

V : nucleon-nucleon int.
μ : reduced mass

$$H|\Psi_d\rangle = E_d|\Psi_d\rangle$$

$|\Psi_d\rangle$: Ground-state wave func. eigenfunc. of H

From symmetry arguments;
the only contributions come from $L=0$, $L=2$

$$|\Psi_d\rangle = a|{}^3S_1\rangle + b|{}^3D_1\rangle$$

It may be convenient for us to think in terms of a matrix approach to the eigenvalue problem.

Since we are only interested in finding the amount of mixing between $^3S_1$ and $^3D_1$ states, we may use these two states as the basis to construct the Hamiltonian matrix;

$$\{H\} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

$H_{11} = \langle {}^3S_1 | H | {}^3S_1 \rangle$ $\quad H_{22} = \langle {}^3D_1 | H | {}^3D_1 \rangle$ $\quad H_{12} = H_{21} = \langle {}^3D_1 | H | {}^3S_1 \rangle$

In diagonalization, we find $\begin{cases} E_d \\ a, b \end{cases}$

But this is not of interest here;
We are interested to the type of H which can produce a mixing between $^3S_1$ and $^3D_1$ states.

If $H_{12} = H_{21} = 0 \longrightarrow$ H: diagonal in the basis states

The two eigenstates $|{}^3S_1\rangle$ and $|{}^3D_1\rangle$ without any admixtures between them.

The fact that ground state of deuteron is a mixture of these two states $\longrightarrow \langle {}^3D_1 | H | {}^3S_1 \rangle \neq 0$

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V$$

The kinetic energy term of H contributes only to the diagonal elements $\longrightarrow$ $\langle {}^3D_1 | V | {}^3S_1 \rangle \neq 0$

$\longrightarrow$ nuclear potential V <u>is not</u> diagonal in the basis span by states with <u>definite</u> orbital angular momentum and can therefore <u>mix</u> ${}^3S_1$ and ${}^3D_1$-states.

$$\langle {}^3D_1 | V | {}^3S_1 \rangle \neq 0$$

$l=2$ $\downarrow$ $l=0$ $\uparrow$

must have $l=2$

V must have a spatial part that is <u>spherical tensor</u> of rank 2, so as to be able to connect an S-state to a D-state.

$$\Psi_d = \Psi_r \, \Psi_{spin} \, \Psi_{Isospin} \qquad V = (\ r\ )(\ spin\ )(\ Isospin\ )$$

We ignore isospin part in the following discussion.

Since $[H, J^2] = 0$ (H conserves total angular momentum)

V must be a <u>scalar</u> in J

But we found that the spherical rank of spatial part of V is 2.

$\longrightarrow$ Intrinsic spin part must have rank 2, so that a scalar product of these two rank-2 operators can be constructed.

For spin $\frac{1}{2}$ system an arbitrary op may be expressed as a linear combination of:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In terms of spherical components:

$$\bar{\sigma} = \begin{cases} \sigma_{+1} = -\frac{1}{\sqrt{2}}(\sigma_x + i\sigma_y) = \sqrt{2}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ \sigma_{-1} = +\frac{1}{\sqrt{2}}(\sigma_x - i\sigma_y) = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \sigma_0 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases}$$

Its rank is $k=1$ (vector) and carries one unit of angular momentum.

In general the product of two spherical tensors <u>is not</u> a spherical tensor.

$$[V^{k_1} \times U^{k_2}]^{k}_{q} \equiv T^{k}_{q} = \sum_{q_1, q_2} (k_1 q_1 ; k_2 q_2 / k q)\, V^{k_1}_{q_1} U^{k_2}_{q_2}$$

The product is a mixture of tensors with ranks:

$$|k_1 - k_2| \leq k \leq k_1 + k_2$$

If $V^{k_1}_{q_1}$ and $U^{k_2}_{q_2}$ are <u>irreducible</u> spherical tensors of rank k_1 and k_2 respectively, then T^{k}_{q} a spherical (irreducible) tensor of rank k.

For $k = 2$:

$$T^{0}_{0} = \frac{-V \cdot U}{3} = \frac{1}{3}\left(V_{+1} U_{-1} + V_{-1} U_{+1} - V_0 U_0\right)$$ Irreducible

$\sqrt{3}$?

$$T^{1}_{q} = \frac{(V \times U)_q}{i\sqrt{2}}$$ "

$$T^{2}_{\pm 2} = V_{\pm 1} U_{\pm 1}$$

$$T^{2}_{\pm 1} = \frac{V_{\pm 1} U_0 + V_0 U_{\pm 1}}{\sqrt{2}}$$ "

$$T^{2}_{0} = \frac{V_{+1} U_{-1} + 2 V_0 U_0 + V_{-1} U_{+1}}{\sqrt{6}}$$

where $V_{\pm 1} = \mp \frac{1}{\sqrt{2}} (V_x \pm i V_y)$ . . .

$V_0 = V_z$ . .

Remark:

Cartesian tensor of rank 2 (dyadic):

$$T_{ij} \equiv A_i B_j$$

$$A_i B_j = \underbrace{\frac{1}{3}(A \cdot B)\delta_{ij}}_{P_{ij}} + \underbrace{\frac{1}{2}(A_i B_j - A_j B_i)}_{Q_{ij}} + \underbrace{\left[\frac{1}{2}(A_i B_j + A_j B_i) - \frac{1}{3}(A \cdot B)\delta_{ij}\right]}_{R_{ij}}$$

$$AB = \underbrace{\begin{pmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{pmatrix}}_{\text{Diagonal}} + \underbrace{\begin{pmatrix} 0 & Q_{12} & Q_{13} \\ -Q_{12} & 0 & Q_{23} \\ -Q_{13} & -Q_{23} & 0 \end{pmatrix}}_{\text{Anti-sym.}} + \underbrace{\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{12} & R_{22} & R_{23} \\ R_{13} & R_{22} & R_{33} \end{pmatrix}}_{\text{Sym.}}$$

$$T'_{ij} = \sum_{kl} R_{ik} R_{jl} T_{kl}$$

Under transformation:

Diagonal $\xrightarrow{tr.}$ Diagonal

Anti-sym $\xrightarrow{tr.}$ Anti-sym

Sym. $\xrightarrow{tr.}$ Sym.

Diagonal matrix:

$Tr(P) = const.$

Any tr. leave it invariant. (scalar)

Anti-sym. matrix:

It has 3-independent elements. Q_{12}, Q_{13}, Q_{23} (Vector)

Any tr. leave it antisymmetric.

Sym. matrix:

It has 5-independent elements:

$R_{11}, R_{22}, R_{33}, R_{12}, R_{13}, R_{23}$

with the cond. $Tr(R) = const.$

Any tr. leave it symmetric.

A two-body op in the nucleon intrinsic space may be constructed from the product of $\sigma(1)$ and $\sigma(2)$

$$[\sigma(1) \times \sigma(2)]^k_q$$

In Cartesian coord.:

$$\sigma(1) \cdot \sigma(2) = \sigma_x(1)\sigma_x(2) + \sigma_y(1)\sigma_y(2) + \sigma_z(1)\sigma_z(2)$$

In terms of $\sigma_\pm$, and σ_0:

$$\sigma(1) \cdot \sigma(2) = \sigma_0(1)\sigma_0(2) - \sigma_{+1}(1)\sigma_{-1}(2) - \sigma_{-1}(1)\sigma_{+1}(2)$$

which is in agreement with

$$J \cdot V = \sum_q (-1)^q J_{1q} V_{1,-q}$$

$$T_0^0 = [\sigma(1) \times \sigma(2)]^0_0 = -\frac{1}{\sqrt{3}}(\sigma(1) \cdot \sigma(2))$$

$$T_0^0 = \sum_{p,q} (1p, 1q/00)\, \sigma^1_p(1)\, \sigma^1_q(2) \qquad (p+q=0)$$

$p, q = -1, 0, 1$

Similarly

$$(\sigma(1) \times \sigma(2))^1_m = \sum_{pq} (1p, 1q/1m)\, \sigma^1_p(1)\, \sigma^1_q(2)$$

$$(p+q=m)$$

Also;

$$[\sigma(1)\times\sigma(2)]^2_m = \sum_{pq} (1p\,1q/2m)\ \sigma'_p(1)\,\sigma'_p(2)$$

$$[\sigma(1)\times\sigma(2)]^2_0 = \frac{1}{\sqrt{6}}\left\{\sigma_1(1)\,\sigma_{-1}(2) + \sigma_{-1}(1)\,\sigma_1(2) + 2\,\sigma_0(1)\,\sigma_0(2)\right\}$$

$$[\sigma(1)\times\sigma(2)]^2_{\pm1} = \frac{1}{\sqrt{2}}\left\{\sigma_{\pm}(1)\,\sigma_0(2) + \sigma_0(1)\,\sigma_{\pm1}(2)\right\}$$

$$[\sigma(1)\times\sigma(2)]^2_{\pm2} = \sigma_{\pm}(1)\,\sigma_{\pm}(2)$$

Max. rank of $[\sigma(1)\times\sigma(2)] = 2$

To form a scalar product in J:

rank of spin part $\overset{\text{must}}{=\!=}$ rank of orbital part

$\rightarrow$ Max rank of orbital part $= 2$

This is adequate for our purposes:

We obtained before the orbital angular part of V must have rank 2 (P131)

Tensor operator:

An operator, formed by the scalar product of a second rank op. in intrinsic spin space and a similar one in coord. space, is called tensor operator.

The quantities we can use to form V being a scalar one:

$\bar{r} = r\hat{n}$, $S = \frac{1}{2}\sigma$ $\bar{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ ...

$\bar{\nabla}$ is rejected, since it gives velocity-dep. forces (non-conservative)

i) $V = V(r) = V(\sqrt{r \cdot r})$ central

ii) $V = V(r)\,(\sigma_1 \cdot \sigma_2)$ spin-dep. but central

iii) $V = V(r)\,(\sigma_1 \cdot \hat{n})$
or $V = V(r)\,(\sigma_2 \cdot \hat{n})$
or $V = V(r)\,(\sigma_1 \times \sigma_2) \cdot \hat{n}$

are not scalars, they are pseudoscalars.

Because $\hat{n} \xrightarrow{\Pi} -\hat{n}$

$\sigma \xrightarrow{\Pi} \sigma$

There are two points:

1) Since $\hat{n}$ is the <u>only</u> polar vector it must occur an <u>even</u> number of times in any acceptable product

2) Any polynomial in σ can be reduced to an expression linear in σ by means of spin identities.

For example $(\sigma \cdot \hat{n})^2 = I$, $(\sigma \cdot n)^3 = (\sigma \cdot n)$

$\longrightarrow$ An expression <u>linear</u> in σ_1 and σ_2 is the most general one we can use.

iv) $V = V(r)\ (\sigma_1 \cdot \hat{n})(\sigma_2 \cdot \hat{n})$

or $V = V(r)\ (\sigma_1 \times \hat{n}) \cdot (\sigma_2 \times \hat{n})$

The second one is a linear combination of the first and $(\sigma_1 \cdot \sigma_2)$ by vector identities.

Hence, the combination $(\sigma_1 \cdot \hat{n})(\sigma_2 \cdot \hat{n})$ is essentially the only scalar leading to a non-central force.

It is useful to define the potential of the non-central force in such a way that its <u>average</u> over directions $\hat{n}$ <u>vanishes</u>.

Since:

$$\overline{(A\cdot\hat{n})(B\cdot\hat{n})}_{\text{all dir.}} = \frac{1}{3}(A\cdot B)$$

$\hat{n} = n(\theta,\varphi)$
$\downarrow$
L - part

$$S_{12} = 3(\sigma_1\cdot\hat{n})(\sigma_2\cdot\hat{n}) - \sigma_1\cdot\sigma_2$$
$$= \frac{3}{r^2}(\sigma_1\cdot\bar{r})(\sigma_2\cdot\bar{r}) - \sigma_1\cdot\sigma_2$$

$$S_1 = \frac{\hbar}{2}\sigma_1 \quad S_2 = \frac{\hbar}{2}\sigma_2 \qquad S = S_1 + S_2 = \frac{\hbar}{2}(\sigma_1 + \sigma_2)$$

$$s_1 = \frac{1}{2} \quad s_2 = \frac{1}{2} \qquad |s_1 - s_2| \leq s \leq s_1 + s_2$$

$$s = 0, 1$$

$$S^2 = \frac{\hbar^2}{4}(\sigma_1^2 + \sigma_2^2 + 2\sigma_1\cdot\sigma_2) = \frac{\hbar^2}{4}(3I + 3I + 2\sigma_1\cdot\sigma_2)$$

$$S^2 = \frac{3}{2}\hbar^2 I + \frac{\hbar^2}{2}\sigma_1\cdot\sigma_2 \longrightarrow \sigma_1\cdot\sigma_2 = \frac{2}{\hbar^2}S^2 - 3I$$

$$S\cdot\hat{n} = \frac{\hbar}{2}(\sigma_1 + \sigma_2)\cdot\hat{n} = \frac{\hbar}{2}(\sigma_1\cdot\hat{n} + \sigma_2\cdot\hat{n})$$

$$(S\cdot n)^2 = \frac{\hbar^2}{4}\left\{(\sigma_1\cdot\hat{n})^2 + (\sigma_2\cdot\hat{n})^2 + 2(\sigma_1\cdot\hat{n})(\sigma_2\cdot\hat{n})\right\}$$

But $(\sigma\cdot\hat{n})^2 = \sigma_i n_i \sigma_j n_j = \sigma_i \sigma_j r_i r_j = [\delta_{ij} I + i\epsilon_{ijk}\sigma_k] r_i r_j$

$$= 0 + \hat{n}^2 I = I$$

$$(S\cdot\hat{n}) = \frac{\hbar^2}{4}(I + I + 2(\sigma_1\cdot\hat{n})(\sigma_1\cdot\hat{n}))$$

$$\rightarrow (\sigma_1 \cdot \hat{n})(\sigma_2 \cdot \hat{n}) = \frac{2}{\hbar^2}(S \cdot \hat{n})^2 - I$$

$$S_{12} = \frac{6}{\hbar^2}(S \cdot \hat{n})^2 - 3I - \frac{2}{\hbar^2}S^2 + 3I$$

$$S_{12} = \frac{2}{\hbar^2}\left[3(S \cdot \hat{n})^2 - S^2\right]$$

It can be shown (Moshinsky 1958):

$$S'_{12} = V(r)\left(\frac{32}{5}\pi\right)^{\frac{1}{2}} Y_2(\theta,\varphi) \cdot X_2$$

$Y_2(\theta,\varphi)$: Racah tensor with the components $Y^2_m(\theta,\varphi)$

X_2 : 〃 〃 〃 $m=0$ component $X^2_0 = \frac{1}{\sqrt{2}}(3S_z^2 - S^2)$

3-5 Symmetry and Nuclear Force

One-body interaction: It acts only on a single body from an external source (central force).

Ex The force acting on each electron in the atom from the nucleus (electrostatic force).

Two-body interaction: The interaction between a pair of particles at a time.

Ex.

The force between any pair of the electrons in the atom (electrostatic force).

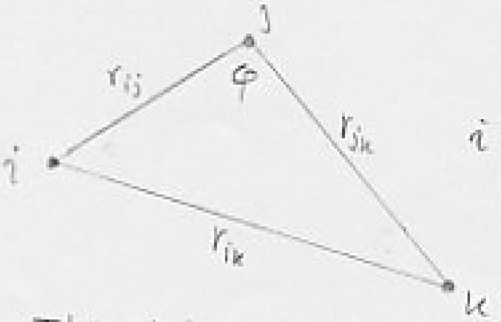
Three-body interaction: This interaction is present when at least there exist three particles.

Ex. The force between three nucleons in triton $\begin{cases} {}^3H & \text{tritium} \\ {}^3He \end{cases}$ system.

In a nucleus there is no fundamental one-body interaction

The only one-body op. is Kinetic energy op.

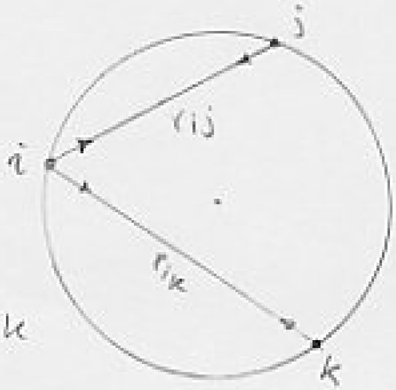
For convenience of carrying out calculations, one may, on occasion, construct an effective one-body term in the potential by taking an average of the two-body int. a particle experiences in the presence of all the other particles.



Three-body int.

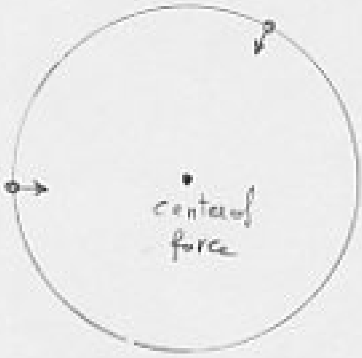
$V = V(r_{ij}, r_{jk}, r_{ik})$

or $V = V(r_{ij}, r_{jk}, \varphi)$



Two-body int

$V = V(r_{ij})$



one-body int.

$V = V(r_i)$

$$\rightarrow V = \sum_i V_i + \sum_{ij} V_{ij} + \sum_{ijk} V_{ijk}$$

i) With our present experimental equipment and theoretical Knowledge

ii) And since $\sum_{ijk} V_{ijk} << \sum_{ij} V_{ij}$

→ it is unlikely to detect any three-body effect.

For this reason → we ignore any possible 3-body effect.

For two-body int. investigation we make use of deuteron, but it is very limited system having only one bound state.

→ So, we will make use of scattering of one nucleon off another one.

Charge Independence:

We assume the nuclear force is charge-independent.
i.e. the interaction of n-n, p-p and n-p are the same if we remove the Coulomb interaction.

But still there are some symmetry-breaking effects, which are small, but significant.

Ex.
Mass-difference of charged and neutral pions exchanged between p-n and p-p, n-n causes a small difference in their interactions.

The observed strengths of charge-symmetry breaking terms are except for a few highly specialized cases, smaller than or comparable to the accuracy we can achieve.

Charge independence implies that;

$$[H, T_z] = 0$$

$$\rightarrow T_z |\psi\rangle = \frac{1}{2}(Z-N)|\psi\rangle$$

where $\quad H|\psi\rangle = E|\psi\rangle$

Charge symmetry breaking effects:

1) Coulomb
2) " perturbation
3) Center of mass motion (The center of mass of nucleus (in particular P) does not coincides with the center of single particle potential well)
4) Finite size of Proton
5) " " " neutron
6) Electromagnetic Spin-Orbit $(g_p \neq g_n)$
7) Vacuum polarization (virtual emision of $e^- e^+$ between P-P increases the repulsion)
8) P-n mass difference
9) Short range correlation
10) Isospin mixing in Core

Isospin Invariance;

In addition to $[H, T_z] = 0$, for a nuclear Hamiltonian we have

$$[H, T^2] = 0$$

$$\rightarrow \quad H|\psi\rangle = E|\psi\rangle$$

$$T^2|\psi\rangle = T(T+1)|\psi\rangle$$

Meaning:

$|\psi\rangle$ remains unchanged if we replace some p with n or vice versa.

Ex.

A = 14 system $\quad T = 1 \quad T_z = \begin{cases} 1 \\ 0 \\ -1 \end{cases} \begin{cases} {}^{14}O \\ {}^{14}N \\ {}^{14}C \end{cases} \quad \begin{matrix} 8p - 6n \\ 7p - 7n \\ 6p - 8n \end{matrix}$

${}^{14}O \xrightarrow[\text{into two } n]{\text{changing two } p} {}^{14}C$ members of isobar

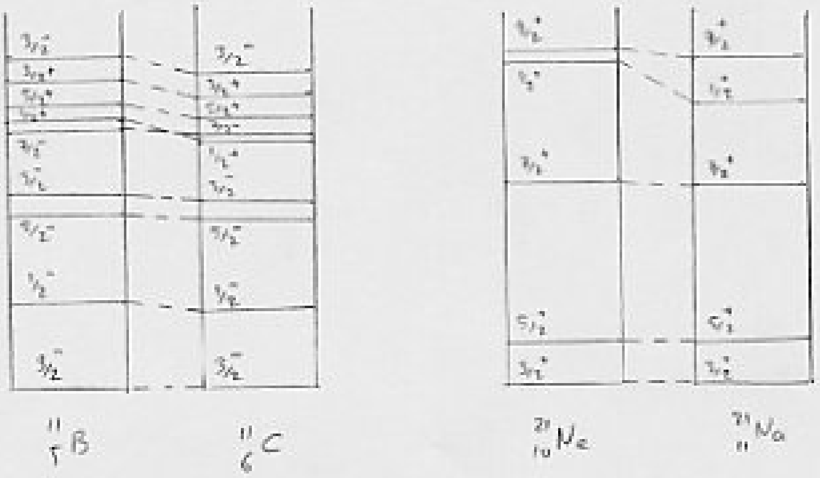
Mathematically, the only difference in $|\psi\rangle$ between the members is in their M_{tz}-quantum number.

$$|{}^{14}O\rangle = T_+|{}^{14}N\rangle = T_+T_+|{}^{14}C\rangle$$

$$|{}^{14}C\rangle = T_-|{}^{14}N\rangle \quad \cdots$$

Isobaric Analogue states (IAS): A group of states related by a rotation in the isospin space.

Among the light nuclei there are many groups with the property $[H, T^2] = 0$



In heavy nuclei: $F_{Coulomb}$ = Strong (because of large number of P)

Since; $[H^{em}, T^2] \neq 0$

$\longrightarrow T^2 |\Psi\rangle \neq T(T+1)|\Psi\rangle$

where $H|\Psi\rangle = E|\Psi\rangle$

In contrast with the previous situation;

$$T^2 |\Psi_i\rangle = \sum_j a_j |\Psi_j\rangle$$ ($|\Psi_i\rangle$: eigenstate of nuclear Hamiltonian)

In spite of the difficulty caused by electromagnetic interactions, the evidence for isospin invariance of the nuclear force itself is quite strong.

We will see;

Isospin symmetry limits the possible forms of nuclear potential.

Isospin Operators:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

All other single-particle isospin operators can be expressed as a linear combination of I and τ.

For a system consisting of A particles;

$$T = \frac{1}{2} \sum_{i=1}^{A} \tau_i$$

For two-particle system

$$T = \frac{1}{2}(\tau_1 + \tau_2)$$

Nucl. int. is $\begin{cases} \text{two-body} \\ \text{scalar} \end{cases}$

T is one-body op., since it operates on a particle at a time. Furthermore it is a vector.

$$T^2 = \frac{1}{4}(\tau_1^2 + \tau_2^2 + 2\,\tau_1 \cdot \tau_2) = \frac{1}{4}\underbrace{(\underbrace{3I + 3I}_{\text{one-body}} + \underbrace{2\,\tau_1 \cdot \tau_2}_{\text{two-body}})}_{\text{mixed particle rank}}$$

$$\tau_1 \cdot \tau_2 = 2T^2 - 3I$$

$$\langle T | \tau_1 \cdot \tau_2 | T \rangle = \begin{cases} -3 & \text{for } T = 0 \\ 1 & \text{"} \quad T = 1 \end{cases}$$

→ $\tau_1 \cdot \tau_2$ may be used to express the isospin dependence of the nucl. force. (There are only two linearly indep. scalar two-body ops., I and $\tau_1 \cdot \tau_2$)

Ex.

we have already seen, a bound state is found for $T=0$, the deuteron, but not for $T=1$.

Remark: The property of isospin invariance is quite different from isospin dependence of the nucl. force (above example).

Other symmetries and General Form of Nucl. Potential:

1) The nucl force between two nucleons must be invariant under a translation of two nucleons.

i.e. $V = V(|\bar{r}| = |\bar{r}_1 - \bar{r}_2|)$

2) Galilean invariance:

The potential may have $\bar{P}_1$ and $\bar{P}_2$ dependence. But since $\bar{P} = \bar{P}_1 + \bar{P}_2$ corresponds to the center of mass momentum, it can not appear in V.

The only possibility is

$$\bar{p} = \frac{1}{2}(\bar{p}_1 - \bar{p}_2) \qquad \text{(Galilean invariance)}$$

3) V may depend on isospin.

4) V invariant under space rotation

5) V " time-reversal

6) " " space reflection (parity)

7) " " permutation between two nucleons.

$$V = V(r, \sigma_1, \sigma_2, \tau_1, \tau_2, p)$$

Taking into account the symmetry requirements of nuclear potential, only a very limited number of linearly dependent set of single nucleon operators can be constructed.

Ex.

l is not indep. variable, since $\bar{l} = \bar{r} \times \bar{p}$

It can be shown (Okubo and Marshak Ann. Phys. 4 (1958) 166) the most general two-body potential takes the form:

$$V(r;\sigma_1,\sigma_2,\tau_1,\tau_2) =$$

$$V_0(r) + V_\sigma(r)\,\sigma_1\cdot\sigma_2 + V_\tau(r)\,\tau_1\cdot\tau_2 + V_{\sigma\tau}(r)\,(\sigma_1\cdot\sigma_2)(\tau_1\cdot\tau_2)$$
$$+ V_{LS}(r)\,L\cdot S + V_{LS\tau}(r)\,(L\cdot S)(\tau_1\cdot\tau_2)$$
$$+ V_T(r)\,S_{12} + V_{T\tau}(r)\,S_{12}\,\tau_1\cdot\tau_2$$
$$+ V_Q(r)\,Q_{12} + V_{Q\tau}(r)\,Q_{12}\,\tau_1\cdot\tau_2$$
$$+ V_{PP}(r)\,(\sigma_1\cdot P)(\sigma_2\cdot P) + V_{PP\tau}(r)\,(\sigma_1\cdot P)(\sigma_2\cdot P)(\tau_1\cdot\tau_2)$$

where $$L\cdot S = \tfrac{1}{2}(l_1+l_2)(\sigma_1+\sigma_2)$$

$$Q_{12} = \tfrac{1}{2}\left\{(\sigma_1\cdot L)(\sigma_2\cdot L) + (\sigma_2\cdot L)(\sigma_1\cdot L)\right\}$$

To determine $V_0(r)$, $V_\sigma(r)$, ... we need information in addition to that generated from symmetry arguments above.

Procedure:

i) For example, we may make use of our knowledge of the basic nature of the nucl. force such as the meson-exchange picture of Yukawa.

ii) Fitting some assumed form of the radial dependence to experimental data.

The 12-terms in the potential may be divided into 5-groups:

i) $V_{central} = V_0(r) + V_\sigma(r)\, \sigma_1 . \sigma_2 + V_\tau(r)\, \tau_1 . \tau_2 + V_{\sigma\tau}(r) (\sigma_1 . \sigma_2)(\tau_1 . \tau_2)$

Since tensorial ranks of the spatial parts of all 4-operators are zero, $(k=0)$.

1) $V_0(r)$ depends only on r and is invariant under a rotation of coord. system.

2) The spatial dependence of the second term is also on r only, but it depends on intrinsic spin through the op. $\sigma_1 . \sigma_2$.

$$\langle S | \sigma_1 . \sigma_2 | S \rangle = \begin{cases} -3 & \text{for } S=0 \\ 1 & \text{"} \quad S=1 \end{cases}$$

$S=0$ singlet state (scalar nucleon pair)

$S=1$ triplet " (Vector " ")

There are two linearly indep. scalar two-body op;

I and $\sigma_1 . \sigma_2$

3) Similar to case (2) but $\sigma_1 . \sigma_2$ is replaced by $\tau_1 . \tau_2$.

4) It has both spin and isospin dependence.

Since all 4-terms are scalars in intrinsic spin and orbital angular momenta

$$\longrightarrow [V_{Central}, S^2] = [V_{Central}, L^2] = [V_{Central}, J^2] = 0$$

ii) $V_{spin\text{-}orbit} = V_{LS}(r) L \cdot S + V_{LS_\tau}(r)(L \cdot S)(\tau_1 \cdot \tau_2)$ $\left\{ \begin{array}{l} \text{non-central} \\ L = L(\theta, \varphi) \end{array} \right.$

$$[V_{s\text{-}o}, L] \neq 0 \quad [V_{s\text{-}o}, S] \neq 0 \quad [V_{s\text{-}o}, J] = 0$$

1) The first term involves only $L.S$-term.

2) The second term has additional op. $(\tau_1 \cdot \tau_2)$, isospin-dep. -

For example different mesons exchanged between nucleons, may or may not have an isospin dependent int.

The spatial part in both cases involves L.

$L \xrightarrow{\Pi} L$ axial vector

To maintain parity invariance as well as rotational invariance of V; we need another axial vector to obtain a scalar product.

$L.L$ is not useful, since $[L^2, L] = [L^2, S] = 0$

$\longrightarrow$ which is a part of central force.

The only possibility: $L \cdot S$

However, $\langle LS | L\cdot S | L'S' \rangle = 0$ for $L \neq L'$

i) Because rank of $L = 1$ $\longrightarrow$ $\langle \quad \rangle = 0$ if $|L'-L| > 1$

$\longrightarrow$ For non-zero $\langle \ \rangle$ $L' = L$ or $L' = L \pm 1$

ii) On the other hand:

The parity of $\langle LS| = (-)^{L}$

" " $|L'S'\rangle = (-)^{L'}$

$S \xrightarrow{\Pi} S$

$L \xrightarrow{\Pi} L$

$\Longrightarrow \langle LS | L\cdot S | L'S' \rangle \xrightarrow{\Pi} -\langle LS | L\cdot S | L'S' \rangle$ if $L' = L \pm 1$

$\longrightarrow$ and must vanish.

iii) $V_{\text{Tensor}} = V_T(r)\, S_{12} + V_{T\tau}(r)\, S_{12}\, \tau_1 \cdot \tau_2$

iv) $V_{\text{Quadratic}} = V_Q(r)\, Q_{12} + V_{Q\tau}(r)\, Q_{12}\, \tau_1 \cdot \tau_2$

The quadratic spin-orbit terms enters only when there is momentum dependence in potential.

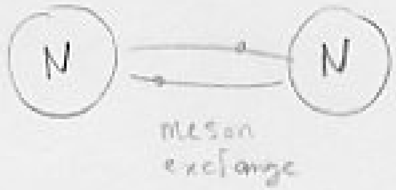
v) $V_{\text{inelastic}} = V_{pp}(r)\,(\sigma_1\cdot p)\,(\sigma_2\cdot p) + V_{pp\tau}(r)\,(\sigma_1\cdot p)\,(\sigma_2\cdot p)\,(\tau_1\cdot\tau_2)$

gives no contributions in elastic interaction.

3-6 Yukawa Theory of Nuclear Interaction:

Beyond the symmetry arguments in the previous section, meson-exchange idea by Yukawa (1934) is useful stating point for the examination of nucleon-nucleon int.

A proper derivation of a boson exchange potential requires a relativistic quantum field theory treatment.



However, the essence may be obtained by drawing an analogy to classical electrodynamics.

$\varphi(r)$: electrostatic pot.

is a sol of $\nabla^2\varphi(r) = 0$ (1) (in source-free region) Laplace's equ.

I the presence of a point source at origin:

$$\nabla^2\varphi(r) = -\left[\frac{1}{4\pi\epsilon_0}\right] 4\pi q\,\delta(r) \qquad (2)$$

Poisson's equ.

with the familiar sol. of Coulomb pot. $\varphi(r) = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{q}{r}$

When the electromagnetic field is quantized:

photons; imerge as the field quanta

Charge; the source of the field

Our aim is to find an equ similar to (2) and its analogue in quantum field theory for a short-range nucl. pot.

The equ. must be invariant under a Lorentz tr. so that to be correct in the relativistic limit as well.

$\longrightarrow$ The Schrödinger equ is ruled out

The field quanta exchanged between nucleons must be a boson. (because only bosons can be created and annihilated singly.

A fermion, on the other hand must be created and annihilated together with its antiparticle.).

The Dirac equ. is unsuitable (an equ for spin $\frac{1}{2}$ particles)

$\longrightarrow$ The prime candidate; Klein-Gordon equ.

$$E^2 = p^2c^2 + m^2c^4$$

We quantize this equ. by:

$$E \rightarrow i\hbar \frac{\partial}{\partial t}$$

$$\bar{p} \rightarrow -i\hbar \bar{\nabla}$$

(in the same way as in nonrelativistic quantum mechanics).

$$\rightarrow -\hbar^2 \frac{\partial^2}{\partial t^2} \varphi(r) = (-\hbar^2 c^2 \nabla^2 + m^2 c^4)\, \varphi(r)$$

m: mass of the field quantum.

$$\longrightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi(r) = \frac{m^2c^2}{\hbar^2}\varphi(r)$$

$$\rightarrow \left(\Box + \frac{m^2c^2}{\hbar^2}\right)\varphi(r) = 0$$

where $\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \bar{\nabla}^2$

This is only analogue to (1), since it does not yet contain a source term for field quanta.

This point may be further demonstrated by:

$$\begin{cases} m \rightarrow 0 \\ \text{and ignoring the time-dependence} \end{cases}$$

The result is quantized version of (1). (through the commutating relations)

Including a source:

For simplicity, we consider only the static limit, and ignore terms involving time derivatives.

For point source with strength g located at origin:

$$\nabla^2 \varphi(\bar{r}) = \frac{m^2c^2}{\hbar^2}\varphi(\bar{r}) - g\,\delta(\bar{r})$$

with sol. $\varphi(r) = \frac{g}{4\pi}\,\frac{e^{-\frac{mc}{\hbar}r}}{r}$ Yukawa pot.

In the limit $m \to 0$ and letting $g = [\frac{1}{4\pi\epsilon_0}]4\pi q$ it reduces to sol. of Coulomb potential.

If $m \neq 0$ the strength of the pot. drops by $\frac{1}{e}$ at

$r_0 = \frac{\hbar}{mc}$: range of the force mediated by boson

For pions ($m \sim 140\,\mathrm{MeV}$) $\longrightarrow r_0 \sim 1.4\,\mathrm{fm}$

We shall see that the exchange of a single pion gives a good representation of the long-range part of the nuclear pot.

3-7 Nucleon-Nucleon Scattering Phase Shifts:

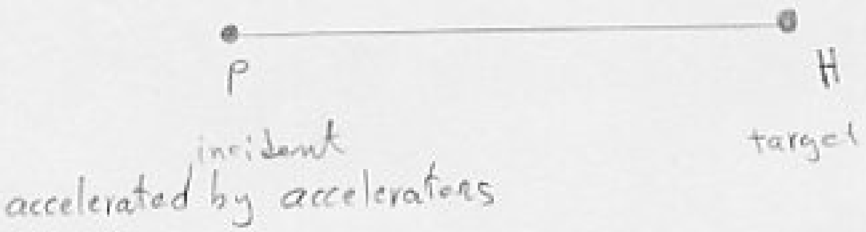
The general form of V in section 3-5 was obtained using the properties of the <u>deuteron</u> and <u>symmetries</u> of a two-nucleon system.

To make further progress, we need additional experimental information, and that is provided by scattering.

Nucleon-Nucleon Scattering:

There are 4-types of scattering:

i) p-p scattering:



ii) n-n scattering:

For incident; a) At low energies neutrons are used from nuclear reactions.

b) At higher energies, neutrons are used from proton bombardment

For example, through (P,n) reaction on a ^7Li target.

However, both $\left\{\begin{array}{l}\text{the intensity}\\ \text{energy resolution}\end{array}\right.$ of neutron beams obtained in this way are much more limited than those of Proton beams.

iii) The scattering of neutrons off proton target (n-p scattering); is an important source of information.

p-p system or n-n system can only be coupled to $T=1$, whereas a n-p system can be either $T=0$ or $T=1$.

iv) p-n scattering.

Since neutron is <u>unstable</u> (half-life ~ 10 min.), it is <u>not</u> possible to construct a <u>fixed</u> neutron target as it is with protons.

Getting round of this limitation:

1) To carry out a colliding experiment.

Such an experiment requires <u>high</u> intensities in both beams (which are not easily available).

2) Using deuterium.

Since the deuteron is a loosely bound system of n and p:

pn or nn scattering results may be obtained by

pd or nd scattering.

pn ~ pd − (pp)

nn ~ nd − (np)

The procedure is valid provided:

1 − The accuracy in pp or np scattering data is comparable or better than pd or nd data. The effect of deuteron binding energy can be corrected in a satisfactory manner.

2 − Three-body effects are negligible.

The information obtained from pn and nn-scattering <u>may not</u> be any different from that obtained from np and pp-scattering.

For example the difference between np and pn scattering is that the targets and incidents are replaced.

Under time-reversal invariance $\longrightarrow$ then two arguments should give identical results.

Isospin $\begin{Bmatrix} \text{up} \\ \text{down} \end{Bmatrix} \xrightarrow{\text{time reversal}} \begin{Bmatrix} \text{down} \\ \text{up} \end{Bmatrix}$

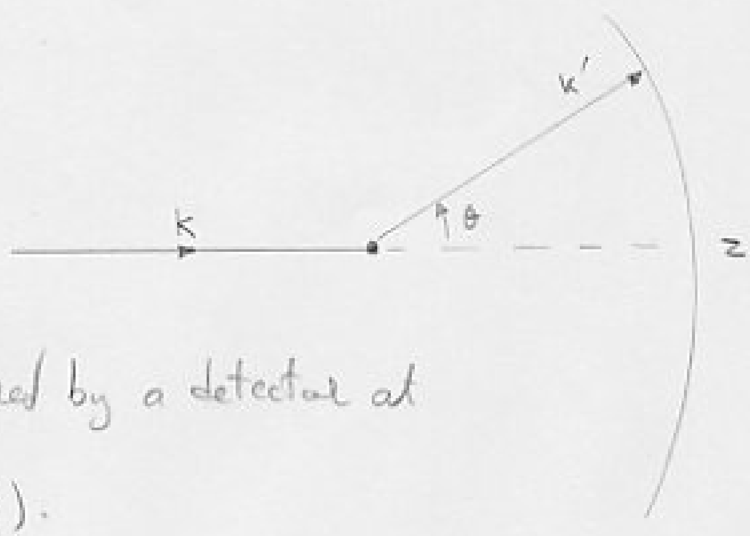
Both nn and pp are T=1 systems.

If nucl. force is charge-indep.:

$\longrightarrow$ nn-scattering result $=$ (pp-scattering result) $-$ Coulomb int.

But the accuracy in nn-scattering is still inadequate.

Scattering Cross Section:



The quantity measured in scattering experiment is:

The number of counts registered by a detector at a certain fixed angle (θ, φ).

$$\text{The counting rate} \approx \Delta\Omega,\ I,\ N,\ \frac{d\sigma}{d\Omega}$$

I: intensity of the incident beam

N: number of target nuclei

$\Delta\Omega$: the solid angle subtended by the detector.

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}(k, \theta, \varphi)$$

In nonrelativistic limit; the scattering wave-func. is the sol. of schrödinger equ.:

$$-\frac{\hbar^2}{2\mu}\nabla^2\Psi + (V-E)\Psi = 0 \qquad \mu\text{: reduced mass}$$

For short-range nucl. forces $\rightarrow V=0$ except for a small region scattering region

$$\Psi(r,\theta,\varphi) \xrightarrow[r\to\infty]{} e^{ikz} + f(\theta,\varphi)\frac{e^{ikr}}{r}$$

We assume elastic scattering; $|k| = |k'|$

$$\frac{d\sigma}{d\Omega}(\theta,\varphi) = |f(\theta,\varphi)|^2$$

For unpolarized incident beam and target (non aligned spins), the scattering is invariant with respect a rotation around z-axis, (φ-indep.)

$$f(\theta,\varphi) \longrightarrow f(\theta)$$

Partial wave analysis:

For a central pot. $\quad l$: conserved quantity in the reaction (relative ang. momentum)

$\underrightarrow{\text{under such conds.}}$ It is useful to expand the wave func. in the following way:

$$\Psi(r,\theta) = \sum_{l=0}^{\infty} a_l \, Y_{l0}(\theta) \, R_l(k,r)$$

Remark: In the absence of polarization $Y_{lm}(\theta,\varphi)$ contribute with only $m=0$.

For free particle:

$$R_l(k,r) \xrightarrow{\text{free}} J_l(k,r) \xrightarrow{r\to\infty} \frac{1}{kr} \sin\left(kr - \frac{l\pi}{2}\right)$$

$$k = \sqrt{\frac{2ME}{\hbar^2}}$$

For elastic scattering:

The probability current density = conserved.
in each partial wave channel

$$R_l(k,r) \xrightarrow[r\to\infty]{\text{scatt.}} \frac{1}{kr} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right)$$

$$f(\theta) = \frac{\sqrt{4\pi}}{k} \sum_{l=0}^{\infty} \sqrt{2l+1} \, e^{i\delta_l} \sin\delta_l \, Y_{l0}(\theta)$$

$$\frac{d\sigma}{d\Omega} = \frac{4\pi}{k^2} \left| \sum_{l=0}^{\infty} \sqrt{2l+1} \, e^{i\delta_l} \sin\delta_l \, Y_{l0}(\theta) \right|^2$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_\ell(k)$$

Nucleon-nucleon Scattering phase shifts:

In realistic N-N scattering

1) Nucl. pot. in general depends on total intrinsic spin (S)

$\longrightarrow$ $J = \ell + S$ is conserved rather than ℓ. (in scatt.)

For two nucleons $S = \begin{cases} 0 \\ 1 \end{cases}$ $\longrightarrow$ To determine S we need to detect the orientation (polarization) of spins of the nucleons involved.

2) With sufficient energy, scattering can excite the internal degree of freedom of nucleons;

For example: $p + p \longrightarrow \Delta^{++} + n$

or $p + p \longrightarrow p + n + \pi^+$

or $p + p \longrightarrow p + p + p + \bar{p}$

These are inelastic scatt..

We are dealing with identical particles:

$$\longrightarrow \quad \psi(1,2) = -\psi(2,1)$$

i) For PP-scatt. ; $T=1$ (sym), if $S=0$ (antisym)

$\longrightarrow$ Relative orbital wave func must be symmetric

$\longrightarrow$ l : even

For $S=0 \rightarrow J=l$

$\rightarrow$ Partial waves for the lowest two orders are $\begin{cases} {}^1S_0 \\ {}^1D_2 \end{cases}$

By the same token; for $S=1 \longrightarrow l$: odd

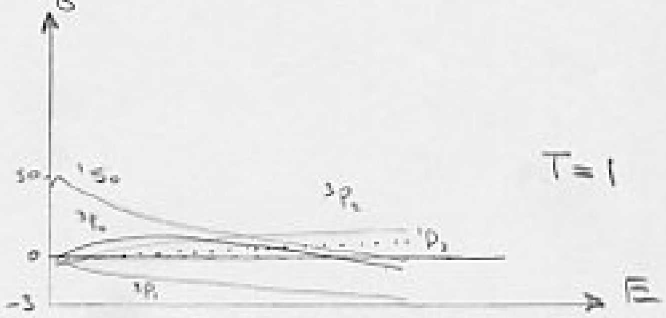
The lowest order $l=1$ P-wave

$$S=1\,,\ l=1 \longrightarrow J=0,1,2 \longrightarrow \begin{cases} {}^3P_0 \\ {}^3P_1 \\ {}^3P_2 \end{cases}$$

(three-triplet states)
$l=1$
$m=-1,0,1$
$S=1$
$m_s=-1,0,1$

Contribution from inelastic scatt. (imaginary part of δ) $\simeq 0$ for $E_{scatt} < 300$ MeV in the lab frame

There is no admixture between the two $J=0$ states $\begin{cases} {}^1S_0 \\ {}^3P_0 \end{cases}$, since they are of different parity. $\longrightarrow \begin{cases} S \\ l \end{cases}$ good quantum numbers



T=1

PP-scatt $\quad E_{scatt\,lab.} < 300$ MeV

(Coulomb effect removed)

ii) For np-scattering ; $T=0$ or $T=1$

For $T=0$ (antisym), $S=0$ (antisym) $\rightarrow$ l: odd

The lowest order $l=1$ ($l=3$ contribution in general $\simeq 0$)

For p-wave ($l=1$) np-scatt to be in $S=1$ $\longrightarrow$ $T=1$

Then, the phase shifts in this case $({}^3P_0, {}^3P_1, {}^3P_2)_{np}$ should be identical to those in pp-scatt. $({}^3P_0, {}^3P_1, {}^3P_2)$; if:

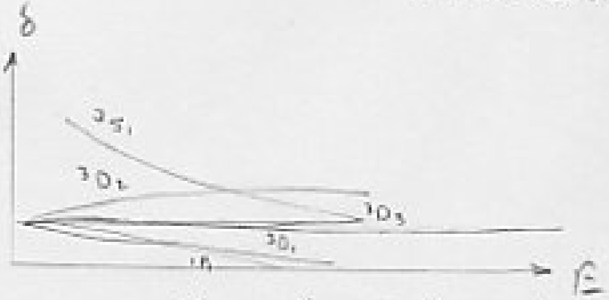
$\begin{cases} \text{1- Nucl. force is charge-indep.} \\ \text{2- Coulomb effects are removed.} \end{cases}$

The results are in good agreement.

But still there is a small difference, which is attributed to both:

$\begin{cases} \text{1- experiment errors} \\ \text{2- charge dep. effects} \end{cases}$

np-scatt. T = 1 (isovector)



np-scatt. T=0 (isoscalar)

For $T=0$ (antisym) $S=1$ (sym) $\longrightarrow$ l: even

ℓ = even ⟶ makes it possible to have mixing of different ℓ-partial waves (with the same J)

Up to now; each phase shift has been characterized by a definite ℓ-value (as well as J- and S-values).
(even though the ℓ is not fundamentally a good quantum number)

In general parity and other symmetry requirements restrict the mixing of different ℓ-partial waves.

As in the case of deuteron;

Tensor force $\xrightarrow{\text{can cause}}$ admixture between two triplet states of the same J but different in ℓ by two units ($\ell = J \pm 1$)

Now; for a given J, the scattering is now specified by two phase shifts;

$\delta_{J>}$ for $\ell = J-1$

$\delta_{J<}$ " $\ell = J+1$

and ϵ_J : which indicates the amount of mixing.

$\delta = \delta(k) \qquad \epsilon_J = \epsilon_J(k)$

Using the def. of ϵ_J (convention) given in Phys Rev. 105 (1957) 302, the scattering matrix for a given J is written in the form;

$$\{S\} = \begin{pmatrix} e^{2i\delta_{J>}} \cos 2\epsilon_J & i\, e^{i(\delta_{J>}+\delta_{J<})} \sin 2\epsilon_J \\ i\, e^{i(\delta_{J>}+\delta_{J<})} \sin 2\epsilon_J & e^{2i\delta_{J<}} \cos 2\epsilon_J \end{pmatrix}$$

For the scattering from $l = J+1$ channel to $l = J+1$ channel the connection is by

$$e^{2i\delta_J} = e^{2i\delta_{J>}} \cos 2\epsilon_J$$

From $l = J-1$ channel to $l = J-1$ channel;

$$e^{2i\delta_J} = e^{2i\delta_{J<}} \cos 2\epsilon_J$$

On the other hand; the scattering from $l = J-1$ to $l = J+1$ and vica versa is given by

$$e^{2i\delta_J} = e^{i(\delta_{J>}+\delta_{J<})} \sin 2\epsilon_J$$

These are generally reffered to as the <u>nuclear bar</u> phase shifts.

Spin polarization in nucleon – nucleon Scattering:

Scattering pot. may be $V = V(S)$

$\longrightarrow \sigma = \sigma(S) \qquad \longrightarrow \sigma(S=0) \neq \sigma(S=1)$

It is <u>not</u> easy to observe the <u>value</u> of S directly in the experiment.

The detected quantity in the experiment is the <u>orientation</u> of the nucleon.

There are 4-possibilities in the <u>initial</u> state and in the final state.

$|\tfrac{1}{2}, \tfrac{1}{2}\rangle, \; |-\tfrac{1}{2}, \tfrac{1}{2}\rangle, \; |\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \; |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

If $V = V(S) \longrightarrow$ the orientation of the nucleons may be changed as a result of scattering

$\left\{ \begin{array}{l} \text{4-different initial states} \\ \text{4- " final "} \end{array} \right. \longrightarrow$ 4×4 different polarization measurements (in principle)

$$\{S\} = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & {}_{44} \end{pmatrix}$$

16-quantities are not indep.

Time-reversal and other symmetries inherit in the scattering: leave only 5-indep. matrix elements.

$$f_1 = f_{++,++} = f_{--,--} \qquad f_2 = f_{++,--} = f_{--,++}$$

$$f_3 = f_{+-,+-} = f_{-+,-+} \qquad f_4 = f_{+-,-+} = f_{-+,+-}$$

$$f_5 = f_{++,+-} = f_{-+,--} = f_{--,+-} = f_{-+,++}$$

$$= f_{--,-+} = f_{+-,++} = f_{++,-+} = f_{+-,--}$$

By the use of Transition op, T:

$$\langle k' | V | \psi^+ \rangle = \langle k' | T | k \rangle$$

$$f(k,\theta) = f_{k,k'} = -\frac{\mu}{2\pi\hbar^2} \langle k' | V | \psi^+ \rangle = -\frac{\mu}{2\pi\hbar^2} \langle k' | T | k \rangle$$

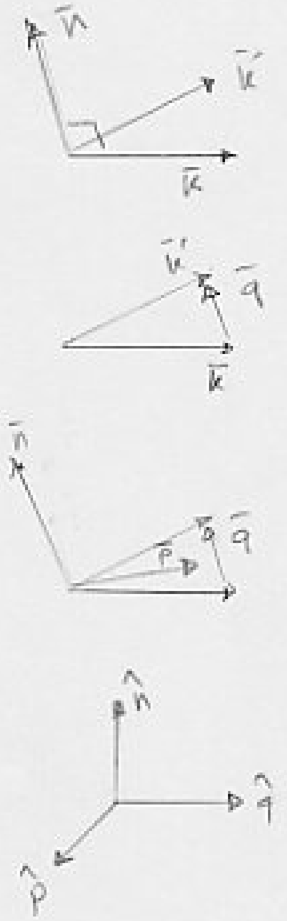
In place of f_1 to f_5, the T matrix element for nucleon-nucleon interaction is often written as a func. of five-coeffs., A, -- F in the following form:

$$T_{k',k}(1,2) = A + B\,\sigma_n(1)\,\sigma_n(2) + C\,\{\sigma_n(1) + \sigma_n(2)\}$$

$$+ E\,\sigma_q(1)\,\sigma_q(2) + F\,\sigma_p(1)\,\sigma_p(2)$$

$\left\{\begin{matrix}\hat{n}\\ \hat{p}\\ \hat{q}\end{matrix}\right.$ three vectors along which the nucleon spin components are taken.

$$\hat{n} = \frac{k \times k'}{|k \times k'|} \qquad \hat{q} = \frac{|k' - k|}{|k' - k|} \qquad \hat{p} = \hat{q} \times \hat{n}$$



Ref. : Bystricky, Lehar, and Winternitz
J. de Phys. 39 (1978) 1.

The amount of indep. information obtained from scattering is greatly increased with polarization measurements.

It is not difficult to obtain analyzing power (A_y) measurements whereby a polarized beam is scattered from an unpolarized target (and the polarization of the particles in the final state is not detected).

At a fixed θ:

$$\sigma(S_{\text{incident}} \text{ parallel to } \hat{n}) - \sigma(S_{\text{incident}} \text{ antiparallel to } \hat{n}) \neq 0$$
in general

Such a difference is characterized by the analyzing power.

This supplies One of the independent quantities in the scattering.

The sum of two differential cross section supplies the other.

For additional information the polarization of the scattered particle must be measured.

Valuable information can also be obtained using polarized targets.

However this requires low temperature techniques to freeze the spin orientation of the nucleons in the target.

Inelastic Scattering:

If $K_{cm} >$ Required energy for meson production $\left\{ \begin{array}{l} K_{cm}: \text{Kinetic} \\ \text{energy of} \\ \text{incident in} \\ \text{cm.} \end{array} \right.$

$\longrightarrow$ the inelastic scattering become possible.

For the lightest mesons;

$m_{\pi^\pm} \simeq 140\ \text{MeV}$

We expect pion production once the bombarding energy is above the threshold.

As the energy increases $\longrightarrow$ $\left\{ \begin{array}{l} \text{1- Excitation of the internal deg. of freedom of the nucleon} \\ \text{2- Particle production} \end{array} \right.$

becomes more probable.

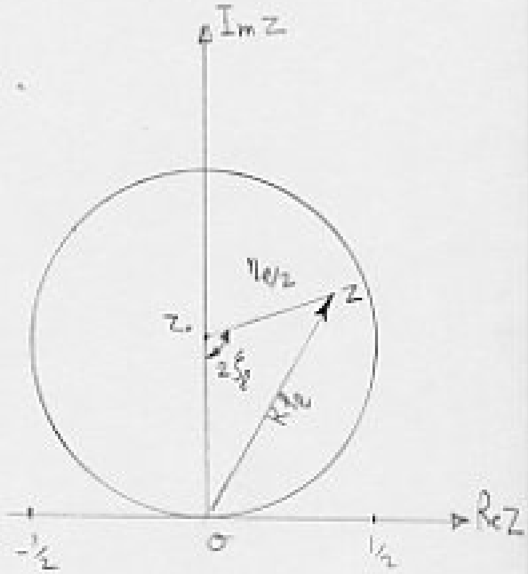
Inelastic scattering:

There is a loss in I (flux of the incident)

$\longrightarrow$ Probability is not conserved.

$\longrightarrow$ V: Complex

$\longrightarrow$ $\left\{\begin{array}{l}\text{scattering amplitude}\\ \text{phase shifts}\end{array}\right.$: complex



Argand diagram

$z_0 (0, \frac{i}{2})$

$Z = k f_\ell = \frac{1}{2i}\left(\eta_\ell e^{2i\delta_\ell} - 1\right)$

$$f(\theta) = \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)}\; f_\ell \; Y_{\ell 0}(\theta)$$

Since $S_\ell(k) \equiv 1 + 2ik f_\ell(k)$

$$\rightarrow \quad f_\ell = \frac{S_\ell - 1}{2ik}$$

For purely elastic scattering;

$$S_\ell = e^{2i\delta_\ell} \qquad \delta_\ell : \text{real} \qquad |S_\ell| = 1$$

$$f_\ell = \frac{e^{2i\delta_\ell} - 1}{2ik} = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = \frac{1}{k\cot\delta_\ell - ik}$$

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\left(\frac{e^{2i\delta_\ell} - 1}{2ik}\right) P_\ell(\cos\theta) \qquad Y_{\ell 0}(\theta) = \frac{P_\ell(\cos\theta)}{\sqrt{4\pi}}$$

$$= \frac{1}{k}\sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell \; P_\ell(\cos\theta)$$

In inelastic scattering:

$$S_\ell(k) \equiv \eta_\ell(k)\, e^{2i\delta(k)} \qquad |S_\ell(k)| \leqslant 1$$

$$0 \leqslant \eta_\ell(k) \leqslant 1$$

$$f_\ell(k) = \frac{S_\ell(k)-1}{2ik} = \frac{\eta_\ell(k)e^{2i\delta_\ell(k)}-1}{2ik} = \frac{\eta_\ell \sin 2\delta_\ell}{2k} + i\,\frac{1-\eta_\ell \cos 2\delta_\ell}{2k}$$

$$\sigma_{el} = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1)\, |f_\ell(k)|^2 = 4\pi \sum_\ell (2\ell+1)\, \frac{1+\eta_\ell^2 - 2\eta_\ell \cos 2\delta_\ell}{4k^2}$$

we remember;

$$\Psi(\bar r) \longrightarrow \frac{i}{2k} \sum_{\ell=0}^{\infty} (2\ell+1)\, i^\ell \Big[\underbrace{\frac{e^{-i(kr-\frac{\ell\pi}{2})}}{r}}_{\Psi_{in}} - \underbrace{S_\ell(k)\, \frac{e^{i(kr-\frac{\ell\pi}{2})}}{r}}_{\Psi_{out}} \Big] P_\ell(\cos\theta) \qquad (1)$$

$$\Psi_{in} \sim \frac{i}{2k}\, \frac{e^{-ikr}}{r}\, P_\ell(\cos\theta)$$

$$\bar J = \frac{\hbar}{2im}\left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*\right) \qquad \text{flux}$$

$$\text{Total flux} = \int J(\bar r)\cdot(d\Omega\, \bar i_r) = \frac{\hbar}{2im} \int d\Omega \left(\Psi^* \frac{\partial \Psi}{\partial r} - \frac{\partial \Psi^*}{\partial r}\Psi\right)$$

$$\underset{(in)}{\text{Total radial flux}} = \left(\frac{\hbar k}{m}\right)\left(\frac{4\pi}{(2k)^2}\right) \qquad (\text{inward})$$

Remark: inward ≠ incident

$$\underset{(out)}{\text{Total radial flux}} = \left(\frac{\hbar k}{m}\right) |S_\ell(k)|^2\, \frac{4\pi}{4k^2} \qquad \text{for each } \ell\text{-value}$$

(outward)

$\rightarrow$ The net flux $\underline{\text{lost}}$ $= TRL_{(in)} - TRL_{(out)}$

$$= \frac{\hbar k}{m}\left(\frac{\pi}{k^2}\right)\left(1-\eta_e^2(k)\right) \qquad \text{for each } l\text{-value}$$

Remark: Note that equ. (1) with the help of:

$$e^{ik\cdot r} \rightarrow \frac{i}{2k}\sum_{l=0}^{\infty}(2l+1)\,i^l\,[\underbrace{\frac{e^{-i(kr-\frac{l\pi}{2})}}{r}}_{\text{incoming}} - \underbrace{\frac{e^{i(kr-\frac{l\pi}{2})}}{r}}_{\text{outgoing}}]\,P_l(\cos\theta)$$

can be cast into:

$$\psi(\bar{r}) \rightarrow \underbrace{e^{ik\cdot r}}_{\text{incident}} + \underbrace{\left[\sum_{l=0}^{\infty}(2l+1)\frac{S_l(k)-1}{2ik}P_l(\cos\theta)\right]\frac{e^{ikr}}{r}}_{\text{outgoing}}$$

$$\text{Incident flux} = \frac{\hbar k}{m} \qquad (\text{only in one dir.})$$

Note that $\bar{J} = \frac{\hbar}{2im}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = \frac{\hbar\bar{k}}{m}$ for $\psi = e^{ik\cdot r}$

$$\frac{\text{The net flux lost}}{\text{Incident flux}} = \frac{\pi}{k^2}\left(1-\eta_e^2(k)\right) \qquad \text{for each } l\text{-value}$$

$$\rightarrow \sigma_{inel} = \frac{\pi}{k^2}\sum_{l}(2l+1)\left[1-\eta_e^2(k)\right]$$

$$\sigma_{tot} = \sigma_{el} + \sigma_{inel} = \frac{\pi}{k^2} \sum_{\ell} (2\ell+1)\left(1+\eta_\ell^2 - 2\eta_\ell \cos 2\delta_\ell + 1-\eta_\ell^2\right)$$

$$= \frac{2\pi}{k^2} \sum_{\ell} (2\ell+1)(1-\eta_\ell \cos 2\delta_\ell) \qquad (1)$$

Optical Theorem:

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(k) P_\ell(\cos\theta)$$

$$f(0) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(k)$$

$$\mathrm{Im}\, f(0) = \sum_{\ell=0}^{\infty} (2\ell+1)\, \mathrm{Im}\, f_\ell(k)$$

$$\mathrm{Im}\, f_\ell(k) = \mathrm{Im}\left(\frac{\eta_\ell \sin 2\delta_\ell}{2k} + i\,\frac{1-\eta_\ell \cos 2\delta_\ell}{2k} \right)$$

$$= \frac{1-\eta_\ell \cos 2\delta_\ell}{2k}$$

$$\mathrm{Im}\, f(0) = \sum_{\ell} (2\ell+1) \frac{1-\eta_\ell \cos 2\delta_\ell}{2k} \qquad (2)$$

$$(1)\,(2) \longrightarrow \mathrm{Im}\, f(0) = \frac{k}{4\pi}\sigma_{tot}$$

So the optical theo. is satisfied.

i) If $\eta_\ell(k) = 1 \longrightarrow$ no absorption $\sigma_{inel} = 0$ purely elastic scatt.

ii) If $\eta_\ell(k) = 0 \longrightarrow$ total absorption, however $\sigma_{el} \neq 0$

Consider a black disc;

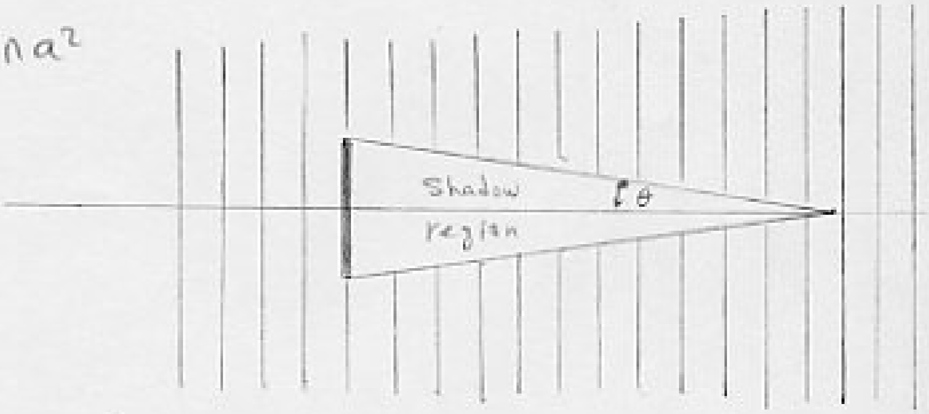
Black disc $\begin{cases} 1 - \text{It has well defined edge} \\ 2 - \text{It is totaly absorbing} \end{cases}$

Since $\begin{cases} L = \sqrt{\ell(\ell+1)}\,\hbar \simeq \ell\hbar \\ L = aP = a\hbar k \end{cases} \rightarrow \ell_{max} = ak$ $\quad$ a: the radius of disc

$\eta_\ell(k) = 0$ (total absorption) $\longrightarrow \sigma_{inel} = \frac{\pi}{k^2} \sum_{\ell=0}^{ka} (2\ell+1) = \frac{\pi}{k^2}(ka)^2 = \pi a^2$

$\eta_\ell(k) = 0$ $\quad$ " $\quad \longrightarrow \sigma_{el} = 4\pi \sum_{\ell=0}^{ka} (2\ell+1) \frac{1}{4k^2} = \pi a^2$

$\sigma_{tot} = \sigma_{el} + \sigma_{inel} = 2\pi a^2$



Classically: One expects no elastic scatt. when there is total absorption.

Quantum mechanically;

Disc takes flux $\sim \pi a^2$ out of the incident beam.

However; far away, shadow gets filled (by diffraction of incident wave at the edge of disc $\sim \pi a^2$, which is elastically).

This <u>elastic</u> scattering is also called shadow scattering

It is <u>strongly</u> peaked up forward.

The angle to which it is confined can be estimated from the uncertainty principle:

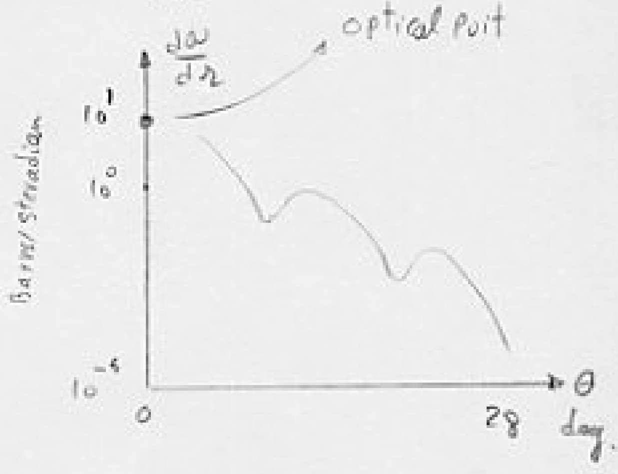
$\Delta x \, \Delta P \sim \hbar$

An uncertainty in the lateral dir of magnitude a

<u>will be accompanied by</u> $\rightarrow$ an uncontroled lateral momentum $P_\perp \sim \frac{\hbar}{a}$

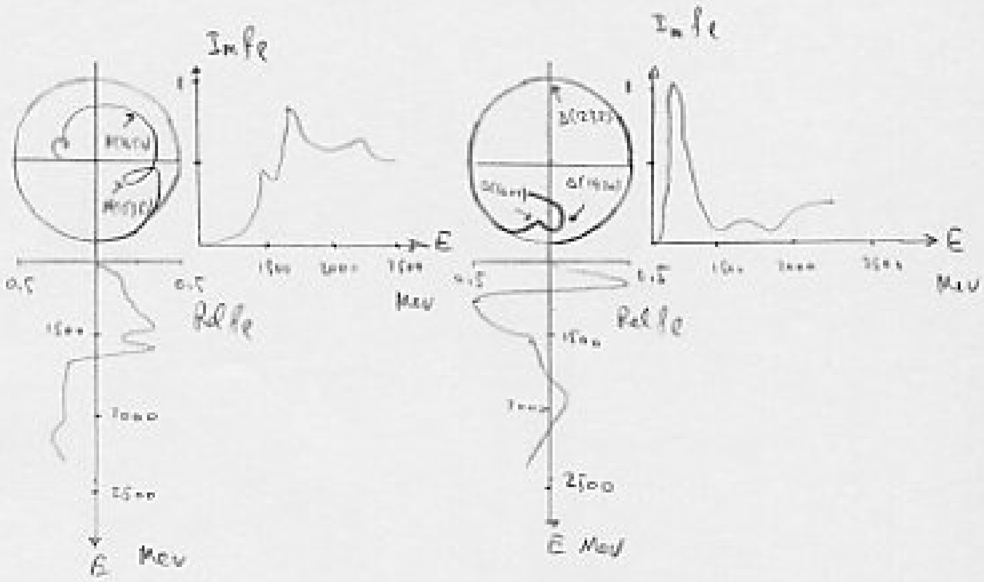
$P_\perp \simeq P\theta$ small θ $\longrightarrow \theta \sim \frac{\hbar}{ap} \sim \frac{1}{aK}$

This agrees with the optical result $\theta \sim \frac{\lambda}{a}$



scatt.; proton of ^{16}O nuclei

1000 MeV (1 BeV)



$\ell=0,\ J=\frac{1}{2},\ T=\frac{1}{2}$ $\qquad$ $\ell=1,\ J=\frac{3}{2},\ T=\frac{3}{2}$

(strong P_{33} resonance)

Pion-nucleon scatt.

Phase shifts $\simeq$ Real for $E_{scatt.} < 300$ MeV in the lab frame

~ 150 MeV " " CM "

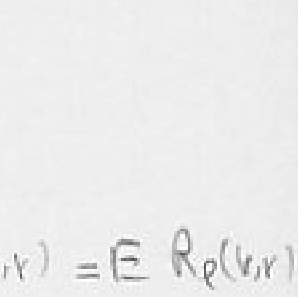
3-8) Low Energy Scattering Parameters:

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + (V-E)\psi = 0$$

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell Y_{\ell 0}(\theta) R_\ell(k,r)$$

$$-\frac{\hbar^2}{2\mu}\left\{\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2}\right\}R_\ell(k,r) + V(r)R_\ell(k,r) = E\,R_\ell(k,r)$$

$$\hat{V}(r) = V(r) + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2} \quad \text{effective pot.}$$



$$u_\ell(k,r) = r R_\ell(k,r)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}$$

$$\rightarrow \quad \frac{d^2}{dr^2} u_\ell + \left(k^2 - \frac{2\mu}{\hbar^2} V(r) - \frac{\ell(\ell+1)}{r^2}\right) u_\ell = 0$$

At low energies ($E < 10$ MeV), only $\ell = 0$ (s-wave) has important contribution.

Def.: $\lim_{E \to 0} \sigma = 4\pi a^2$ (remains finite) $\quad a$: scattering length

In low energy scatt. a is often used instead of δ.

They are related by

$$a = \lim_{k \to 0} \mathrm{Re}\left\{-\frac{e^{i\delta_0} \sin\delta_0}{k}\right\} \qquad \text{where } k^2 = \frac{2ME}{\hbar^2}$$

(convention here: for attractive V, $\rightarrow a > 0$ for bound state)

The energy dependence of δ_0 at low energy is given by <u>effective range parameter</u>, r_e, defined by:

$$k \cot \delta_0 = -\frac{1}{a} + \frac{1}{2} r_e k^2$$

$\left\{\begin{matrix} a \\ r_e \end{matrix}\right.$ provide a useful way to parametrize information on low energy scatt.

Effective Range:

The energy dep. of scatt. at low energies is given by effective range.

for $l = 0$

$$\frac{d^2 u_0(k,r)}{dr^2} - \left\{ \frac{2\mu}{\hbar^2} V(r) - k^2 \right\} u_0(k,r) = 0 \qquad (1)$$

$$\rightarrow \left\{ \begin{array}{l} u_0''(k_1,r) - \left\{ \frac{2\mu}{\hbar^2} V(r) - k_1^2 \right\} u_0(k_1,r) = 0 \\ u_0''(k_2,r) - \left\{ \frac{2\mu}{\hbar^2} V(r) - k_2^2 \right\} u_0(k_2,r) = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} E_1 = \frac{\hbar^2 k_1^2}{2\mu} \\ E_2 = \frac{\hbar^2 k_2^2}{2\mu} \end{array} \right.$$

$$\rightarrow \int_0^\infty \left\{ u_0(k_2,r)\, u_0''(k_1,r) - u_0(k_1,r)\, u_0''(k_2,r) \right\} dr$$
$$+ (k_1^2 - k_2^2) \int_0^\infty u_0(k_1,r)\, u_0(k_2,r)\, dr = 0$$

Integration by parts:

$$\left\{ u_0(k_2,r)\, u_0'(k_1,r) - u_0(k_1,r)\, u_0'(k_2,r) \right\} \Big|_0^\infty$$
$$= (k_2^2 - k_1^2) \int_0^\infty u_0(k_1,r)\, u_0(k_2,r)\, dr \qquad (2)$$

This is true for arbitrary $V(r)$ including $V(r) = 0$

Now consider $V(r) = 0$ with the sol. $V_0(k,r)$

$$\frac{d^2 V_0(k,r)}{dr^2} + k^2 V_0(k,r) = 0 \qquad (3)$$

Analogously:

$$\left\{ V_0(k_2,r) V_0'(k_1,r) - V_0(k_1,r) V_0'(k_2,r) \right\} \Big|_0^\infty$$
$$= (k_2^2 - k_1^2) \int_0^\infty V_0(k_1,r) V_0(k_2,r)\, dr \qquad (4)$$

If $V(r)$ in equ (1) is short in range; (1) and (3) are identical to each other in the asymptotic region.

$$\rightarrow \quad \underset{r\to\infty}{V_0(k,r)} = \underset{r\to\infty}{\mathcal{U}_0(k,r)} = A \sin(kr + \delta_0)$$

Remark: δ_0 is due to $V(r)$ for $\mathcal{U}(k,r)$, but for $V(k,r)$ is just a change in ref. point.

$$\begin{cases} \mathcal{U}_0(k,r) \xrightarrow[r\to 0]{} 0 \\ \mathcal{U}_0(k,r) \xrightarrow[r\to\infty]{} V_0(k,r) \end{cases}$$

$$\rightarrow \left\{ \mathcal{U}_0(k_2,r) \mathcal{U}_0'(k_1,r) - \mathcal{U}_0(k_1,r) \mathcal{U}_0'(k_2,r) \right\} \Big|_0^\infty$$
$$= \lim_{r\to\infty} \left\{ V_0(k_2,r) V_0'(k_1,r) - V_0(k_1,r) V_0'(k_2,r) \right\} \qquad (5)$$

(2), (4), (5) $\longrightarrow$

$$V_0(k_1,0)V_0'(k_2,0) - V_0(k_2,0)V_0'(k_1,0)$$
$$= (k_2^2-k_1^2)\int_0^\infty \{V_0(k_1,r)V_0(k_2,r) - U_0(k_1,r)U_0(k_2,r)\}\,dr \qquad (6)$$

Note that $V_0(k,0) \neq 0$

Assuming ; $V_0(k,0) = 1 \quad \longrightarrow V_0(k,r) = \dfrac{\sin(kr+\delta_0)}{\sin\delta_0}$

(6) $\longrightarrow V_0'(k_2,0) - V_0'(k_1,0)$
$$= (k_2^2-k_1^2)\int_0^\infty \{V_0(k_1,r)V_0(k_2,r) - U_0(k_1,r)U_0(k_2,r)\}\,dr$$

$$\frac{k_2\cot\delta_0(k_2) - k_1\cot\delta_0(k_1)}{k_2^2-k_1^2} = \int_0^\infty \{V_0(k_1,r)V_0(k_2,r) - U_0(k_1,r)U_0(k_2,r)\}\,dr$$

If we choose $E_1 \approx E_2 \approx E$

$$\frac{d}{d(k^2)}k\cot\delta_0 = \int_0^\infty \{V_0^2(k,r) - U_0^2(k,r)\}\,dr \qquad (7)$$

$$r_e \equiv 2\int_0^\infty \{V_0^2(k,r) - U_0^2(k,r)\}\,dr$$ effective range.

$(7) \rightarrow \quad K \cot \delta_0 = \int d(k^2)\, \frac{r_e}{2} + C$

$$K \cot \delta_0(k) = (K \cot \delta_0)_{k=0} + \frac{1}{2} r_e k^2 + \cdots$$

$r_e = r_e(k)$

Now using this def.;

$$a = \lim_{k\to 0} \mathrm{Re}\left\{-\frac{1}{k} e^{i\delta_0} \sin \delta_0\right\} = \lim_{k\to 0} \mathrm{Re}\left\{\frac{-1}{K\cot\delta_0(k) - ik}\right\}$$

$$a = \frac{1}{K\cot\delta_0(k) - \frac{1}{2} r_e k^2} \longrightarrow K\cot\delta_0(k) = -\frac{1}{a} + \frac{1}{2} r_e k^2$$

$$\sigma_0 = \frac{4\pi}{k^2} \sin^2 \delta_0(k) = \frac{4\pi}{k^2 + \left\{\frac{1}{2} r_e k^2 - \frac{1}{a}\right\}^2}$$

$$\sigma_0 \xrightarrow[k\to 0]{} 4\pi a^2$$

Neutron Scattering off hydrogen molecules:

In H — H molecule, (homonuclear molecule);

$$d_{H-H} = 7.8 \times 10^{-11} m >> \text{nucl. short range}$$

→ No nucl. int between two protons in H_2.

$$\psi_{H_2} = \psi(r)\,\chi_s \quad \text{antisym.}$$

For this reason → spin orientations of two protons are correlated with their relative orbital angular momentum.

→ Such a correlation may be exploited for neutron-proton scatt. length measurements.

There are two low-lying states for hydrogen molecule

lower one: para-hydrogen $\psi_{P-P}(r) = \text{sym.} \rightarrow S_H = 0$

higher one: Ortho- " $\psi_{P-P}(r) = \text{antisym.} \rightarrow S_H = 1$

$$S=1 \begin{cases} S_z = 1 \\ S_z = 0 \\ S_z = -1 \end{cases} \qquad S=0 \,\{ S_z = 0$$

→ Statistical weight of $(S=1) = 3 \cdot$ (statistical weight of $(S=0)=1$)

(in a sample in equilibrium at room temp.)

But at low temperatures $H_2 \xrightarrow{\text{tends}}$ to go into the lowest energy state

$\longrightarrow$ Almost completely in a para-hydrogen state

$\longrightarrow$ At temp. T, we have $\alpha(T)(H_2)_{para} + \beta(T)(H_2)_{ortho}$

Intense neutron flux is available at low energies;

$\longrightarrow$ high precesion measurements of neutron scatt. from H_2.

By lowering the energy; $\longrightarrow \lambda_n = \text{long} > \sigma \sim d$

$\xrightarrow{\text{so that}}$ scatt. off the two protons in a hydrogen molecule is a coherent one (interfrence can be seen)

If neutrons are low in energy $\longrightarrow$ little energy is transferred to hydrogen target.

Remark: Energy received by target may cause transitions from para $\xrightarrow{\text{to}}$ ortho-hydrogen states, and this decreases the accuracy.

For this reasons; the neutron energy is kept low ~ 10 meV (10^{-3} eV)

$\xrightarrow{\text{Corresponding}}$ to $T \approx 100°$K

At this limit:

$$K \cot \delta_0 \simeq -\frac{1}{a} + \left(\frac{1}{2} r_0 k^2\right)$$

And the scatt. is characterized by <u>two</u> np-scatt. length alone.

<u>For $l = 0$</u>; np-system is;

either $\begin{cases} S=0 & \text{(singlet state)} \\ T=1 \end{cases}$ or $\begin{cases} S=1 & \text{(triplet state)} \\ T=0 \end{cases}$

$$S = \frac{1}{2}(\sigma_n + \sigma_p)$$

$$\text{Eigenvalue of } \sigma_n \cdot \sigma_p = \begin{cases} 1 & \text{for } S=1 \\ -3 & \text{" } S=0 \end{cases}$$

Define the projection operators;

$$P_t = \frac{1}{4}(3 + \sigma_n \cdot \sigma_p) \qquad P_s = \frac{1}{4}(1 - \sigma_n \cdot \sigma_p)$$

$$\begin{cases} P_t \,|S=1\rangle = 1\,|S=1\rangle \\ P_t \,|S=0\rangle = 0\,|S=0\rangle \end{cases} \qquad \begin{cases} P_s \,|S=1\rangle = 0\,|S=1\rangle \\ P_s \,|S=0\rangle = 1\,|S=0\rangle \end{cases}$$

The scatt. length with spin-dep of the nucl. forces;

$$a = a_t P_t + a_s P_s = \frac{1}{4}(3a_t + a_s) + \frac{1}{4}(a_t - a_s)\,\sigma_n \cdot \sigma_p$$

For slow neutrons; ($\lambda \gg r_0$), we can take <u>Zero-range</u> approx. (contact int., Breit 1947, δ-func.)

$$V = \sum_i \left(\frac{2\pi\hbar^2}{\mu}\right) a^{(i)} \delta(r_n - r_i) \qquad (1)$$

i: the i-th proton

When we take the effects of E_b in H_2; different powers of $\frac{a}{d}$ or $\frac{a}{\lambda}$ share in V;

$\frac{a}{d}$, $\left(\frac{a}{d}\right)^2$, ... $\xrightarrow{\text{corresponds}}$ different orders of perturbations in (1)

or $\frac{a}{\lambda}$, $\left(\frac{a}{\lambda}\right)^2$, ...

$\Longrightarrow$ The cond. for applicability of (1) is;

$$a \ll d \quad , \quad a \ll \lambda$$

Moreover, in order to <u>disregard</u> <u>finite-range</u> of nucl. forces (i.e. $V \to \delta(r)$) we must have the conds.;

$$r_0 \ll d \quad \text{and} \quad r_0 \ll \lambda$$

r_0: effective range of nucl. force.

$$V = \frac{\Pi\hbar^2}{2\mu}\left[(3a_t+a_s)+(a_t-a_s)\,\sigma_n\cdot\sigma_1\right]\delta(r_n-r_1)$$
$$+\frac{\Pi\hbar^2}{2\mu}\left[(3a_t+a_s)+(a_t-a_s)\,\sigma_n\cdot\sigma_2\right]\delta(r_n-r_2) \qquad (2)$$

$\longrightarrow$ $V = V_S + V_A$

$$V_S = \left(\frac{\Pi\hbar^2}{2\mu}\right)\left[(3a_t+a_s)+(a_t-a_s)\,\sigma_n\cdot S_H\right]\left[\delta(r_n-r_1)+\delta(r_n-r_2)\right]$$
$$V_A = \left(\frac{\Pi\hbar^2}{4\mu}\right)\left[(a_t-a_s)\,\sigma_n\cdot(\sigma_1-\sigma_2)\right]\left[\delta(r_n-r_1)-\delta(r_n-r_2)\right] \qquad (3)$$

$(S_H = \frac{1}{2}(\sigma_1+\sigma_2))$

V_S : responsible for transitions from { Para-H to Para-H or Ortho-H to Ortho-H

(spin-sym of H_2 doesn't change)

V_A : responsible for transitions from { Para-H to Ortho-H and Vice Versa

For slow neutrons $\lambda \gg d$ $\longrightarrow$ Two protons can be considered at a single position $r_1 \simeq r_2 \equiv r'$

$\longrightarrow$ $[\delta(r_n-r_1)-\delta(r_n-r_2)] \simeq 0$

$\longrightarrow$ $V_A \simeq 0$

Having V_S, only elastic scatt. of neutrons is possible.

(because of orthogonality of wave funcs and since $V_S \neq dep(r_1-r_2)$)

protons

Comparing (1) and (2) $\longrightarrow$

$$a_H = \frac{1}{2}\left[(3a_t + a_s) + (a_t - a_s)\,\sigma_n \cdot S_H\right]$$

$$\langle a_H^2 \rangle = \frac{1}{4}\left\{(3a_t + a_s)^2 + 2(3a_t + a_s)(a_t - a_s)\langle \sigma_n \cdot S_H\rangle + (a_t - a_s)^2 \langle (\sigma_n \cdot S_H)^2\rangle\right\}$$

$$(\sigma_n \cdot S_H)^2 = \sigma_{nx}^2 S_{Hx}^2 + \sigma_{ny}^2 S_{Hy}^2 + \sigma_{nz}^2 S_{Hz}^2 + 2\sigma_{nx}\sigma_{ny} S_{Hx} S_{Hy} + 2\sigma_{ny}\sigma_{nz} S_{Hy} S_{Hz} + 2\sigma_{nz}\sigma_{nx} S_{Hz} S_{Hx}$$

For unpolarized neutron beam; $|n\rangle = \alpha \begin{pmatrix}1\\0\end{pmatrix}_s + \beta \begin{pmatrix}0\\1\end{pmatrix}_s \quad (\alpha = \beta)$

$\langle$ last 3-terms $\rangle = 0$

Since $\sigma_i^2 = 1$

$$\langle \sigma_{nx}^2 S_{Hx}^2 + \sigma_{ny}^2 S_{Hy}^2 + \sigma_{nz}^2 S_{Hz}^2 \rangle = \langle S_{Hx}^2 + S_{Hy}^2 + S_{Hz}^2 \rangle$$

$$= \langle S_H^2 \rangle = S_H(S_H + 1)$$

Also

$\langle \sigma_n \cdot S_H \rangle = 0$ for unpolarized beam.

$$\langle a_H^2 \rangle = \frac{1}{4}\left\{(3a_t + a_s)^2 + (a_t - a_s)^2 S_H(S_H + 1)\right\}$$

$$\sigma_H = 4\pi \langle a_H^2 \rangle = \pi\left\{(3a_t + a_s)^2 + (a_t - a_s)^2 S_H(S_H + 1)\right\}$$

For para-H; $S_H = 0$

$$\sigma_{para} = \pi (3a_t + a_s)^2$$

For ortho-H; $S_H = 1$

$$\sigma_{ortho} = \pi (3a_t + a_s)^2 + 2\pi (a_t - a_s)^2$$

Using the experimental data for σ_{para} and σ_{ortho} one may obtain,

$$|3a_t + a_s| \text{ and } |a_t - a_s| \longrightarrow |a_t|,\ |a_s| \text{ and the sign of } \frac{a_t}{a_s}$$

Neutron-Proton Scattering length:

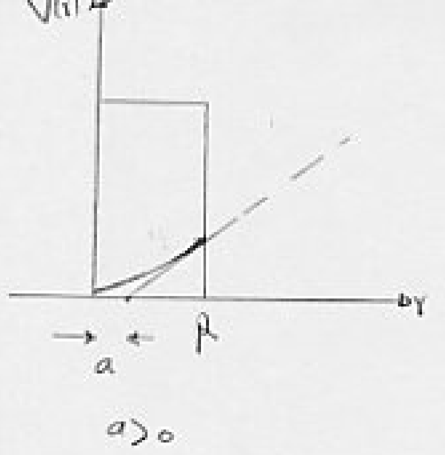
np system $\in T=1$

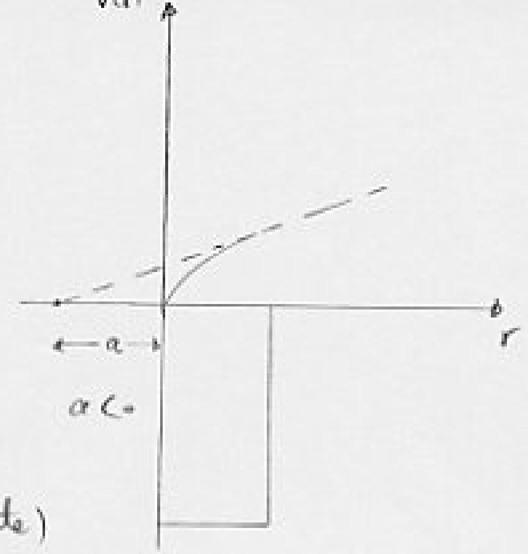
$$T=1 \begin{cases} T_z = 1 & pp\text{-system} \\ T_z = 0 & np\text{-} \text{"} \\ T_z = -1 & nn\text{-} \text{"} \end{cases}$$

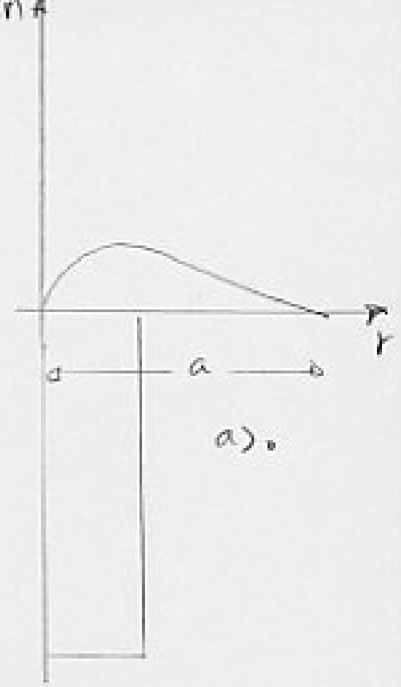
	$S=0,\ T=1$	$S=1,\ T=0$
a_{pp}	-17.1 ± 0.2 fm	—
r_e	2.794 ± 0.015 fm	—
a_{nn}	-16.6 ± 0.6 "	—
r_e	2.84 ± 0.03 "	—
a_{np}	-23.715 ± 0.015 "	5.423 ± 0.005 fm
r_e	2.73 ± 0.03 "	1.73 ± 0.02 fm

Then we can compare a_{np} with a_{nn} and a_{pp}.

$$a = \lim_{k \to 0} \mathrm{Re}\left\{ -\frac{1}{k} e^{i\delta_0} \sin\delta_0 \right\}$$







(convention: $a<0$ boundstate)

a: the intercept of outside wave-func. with r-axis.

The sign of all three scatt. lengths are negative (in T=1)

⟶ no boundstate (repulsive force) as we faced earlier.

a_{pp} is easily measured from low energy p-scatt. off H-Target

However; Since,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \left\{\left[\frac{1}{4\pi\epsilon_0}\right]\frac{zZe^2}{4T\sin^2\left(\frac{\theta}{2}\right)}\right\}^2 = \left\{\frac{\alpha\hbar c\, zZ}{4T}\;\frac{1}{\sin^2\left(\frac{\theta}{2}\right)}\right\}^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Coulomb}} \sim \frac{1}{T^2}$$

pp-scattering at low energies is dominated by electromagnetic effects.

⟶ The accuracy is limited

$$\underset{\substack{\downarrow \\ \text{small}}}{a_{\text{nucl.}}} = a_{\text{total}} - \underset{\substack{\downarrow \\ \text{large}}}{a_{\text{Coulomb}}}$$

$$a_{pp}^{\text{total}} = -7.823 \pm 0.01 \text{ fm} \qquad a_{pp}^{\text{nucl.}} = -17.1 \pm 0.2 \text{ fm}$$

Measurements of a_{nn} are complicated by the absence of fixed neutron targets.

$n + d \longrightarrow p + n + n$ (t: triton, d: deutrons)

$d + d \longrightarrow p + p + n + n$

$t + t \longrightarrow \alpha + n + n$

$n + t \longrightarrow d + n + n$

$t + d \longrightarrow {}^3He + n + n$

Also $\pi^- + d \longrightarrow \gamma + n + n$

due to the availability of good quality pion beams in the recent years is used to reduce the uncertainty of a_{nn}.

a_{nn} is deduced from the int. of two n in the final state.

Comparison of a-values in $T=1$ states:

$a_{np} = -23.715 \pm 0.015$ fm

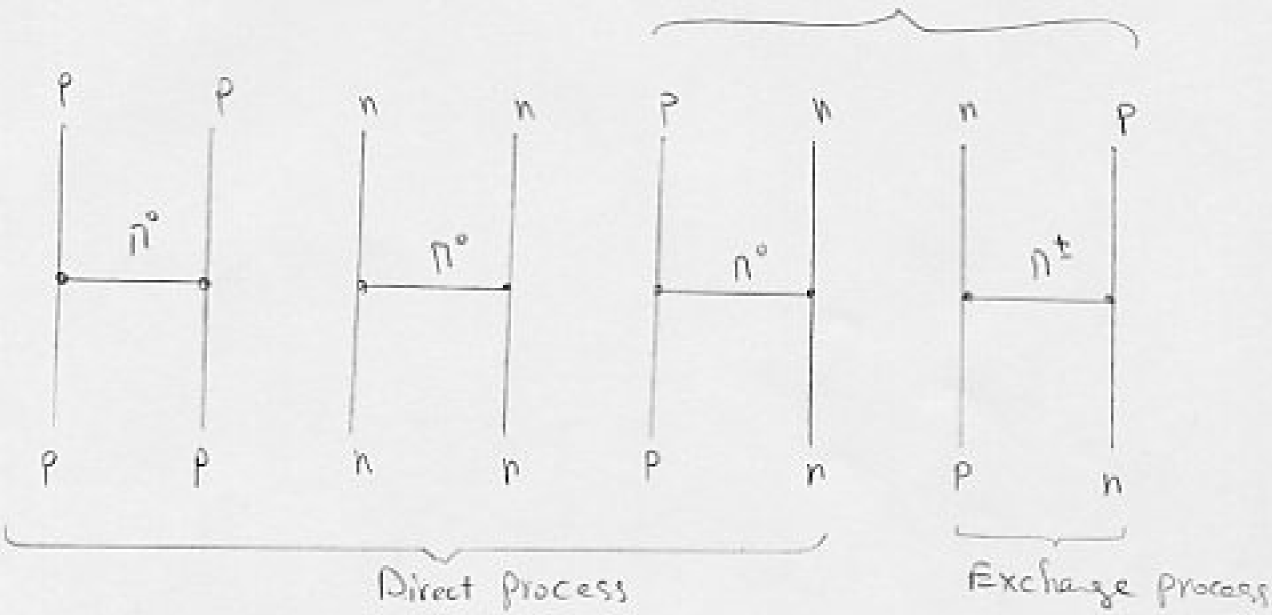
There is a big difference between a_{np} and the others.

This is due to:

1- Charge dependent effects.

2- But most of the difference is due to the mass difference of $\pi^{\pm}$ and π^{0}

$$m_{\pi^{\pm}} - m_{\pi^{0}} = 4.6 \text{ MeV}$$



$$\begin{cases} a_{pp} = -17.1 \pm 0.2 \text{ fm} \\ a_{nn} = -16.6 \pm 0.6 \text{ fm} \end{cases}$$ are <u>equal</u> within experimental errors.

But the last measurement for a_{nn} is -18.6 ± 0.5 fm from the reaction $\pi^- + d \longrightarrow \gamma + n + n$

<u>indicating</u> the charge-dep. of nucl. force.

Remark: At low energies the two nucleons can <u>not</u> come very close together, and the long <u>range</u> of nucl. forces is <u>dominated</u> by the exchange of <u>single</u> pion.

For $T=0 \longrightarrow S=1$

$a_{np} = 5.423 \pm 0.005$ fm $\qquad a_{np} > 0 \longrightarrow$ bound state

$a_{np_{T=1}} \neq a_{np_{T=0}} \longrightarrow$ Isospin-dep. of nucl. forces.

Comparing of r_e shows the isospin dependence.
But these values don't give any indication of charge-dependence within experimental errors.

3.9 - The Nucl. Potential:

One Pion Exchange Pot.: (OPEP)

Yukawa's idea of a simple one-pion exchange pot. can fit experimental data only for inter nucleon distances > 2 fm.

In retrospect this is not a surprise, since $M_\pi \sim 140$ MeV $\xrightarrow{\text{corresponds}}$ to a range of 1.4 fm

The experimental data from scatt. experiment show that the sources other than one-pion exchange are responsible.
For example, the values of δ_0 (Page 167) are $\left\{ \begin{array}{l} \text{large} \\ \text{positive} \end{array} \right.$
at low energies

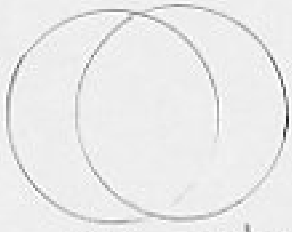
→ indicating that the force is an attractive one.

As the energy is increased $\begin{cases} \text{above } 250\,\text{MeV(lab)}, & {}^1S_0 \text{ becomes negative} \\ \text{〃} \quad 300\,\text{MeV(〃)}, & {}^3S_1 \quad \text{〃} \quad \text{〃} \end{cases}$

Showing → the forces now are repulsive (hard core evidence)

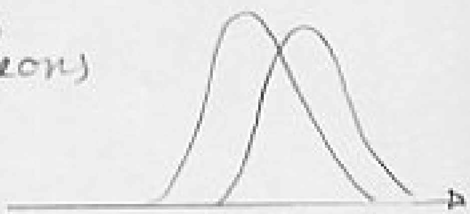
$d_{NN} \sim 1\,fm$.

This strong repulsion is expected:



N-N overlap

Three quarks (fermions) in each nucleon must occupy one of the three lowest available state.



In a collision Six quarks of two nucleons are no longer two separate groups of three quarks each (large overlap).

→ Acc to the Pauli exclusion principle, 3-of the 6-quarks must go $\xrightarrow{\text{to}}$ higher states

A large amount of energy is required to make this transition.

→ This phenomena prevents the nucleons come very close to each other.

Justification of existance of hard core in terms of quarks does not give a fair quantitative prediction.

OPEP explains the long-range behaviour of nuclear int.

But it has difficulties in, relating

$$g_{NN} \xrightarrow{\text{to}} g_{\pi N} \qquad g\text{: strength of int.}$$

$g_{NN} = g_{NN}$ (the probability of emitting and absorbing virtual π)

Such probabilities $= f\,(g_{\pi N})$

In OPEP the $g_{\pi N}$ can not be used directly to calculate g_{NN}.

i.e. There are some other processes effective in th int. other than OPEP.

$\longrightarrow$ g_{NN} is often treated as a parameter adjusted to fit the N-N data.

One Boson Exchange Pot. (OBEP):

Nucl. force may be divided into three parts:

i) Long-range part $r > 2fm$

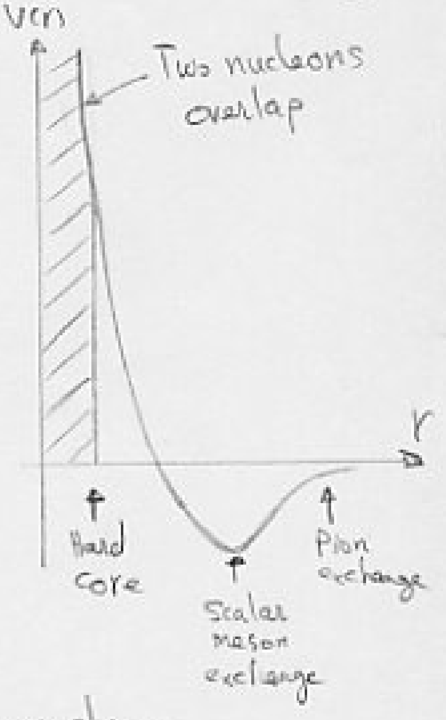
Dominated by one-pion exchange

ii) Intermediate-range part $1fm < r < 2fm$

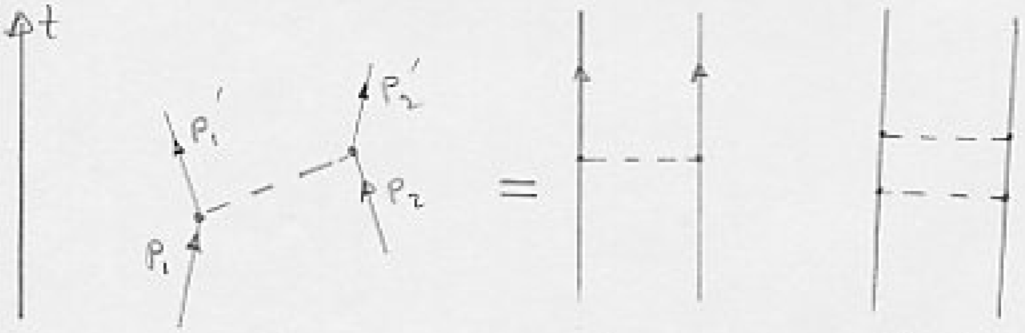
Dominated by two pions and heavier mesons

iii) Hard core int. $r \lesssim 1fm$

Dominated by, heavy meson exchange, multi-pion exchange as well as QCD effects.



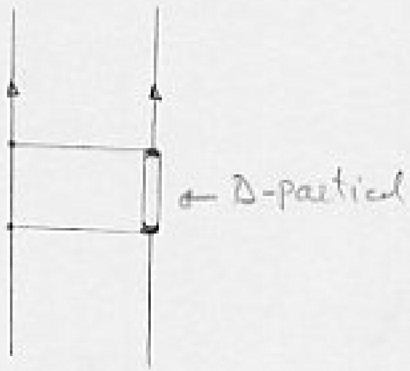
Feynman Diagrams:



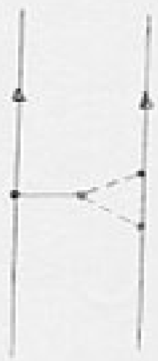
(a) OPE (b) (c) TPE

Two-pion exchange are emitted and absorbed one after another

For simplicity (a) is often shown in (b) form.



(d) TPE



(e)
ρ-meson decays into two pions
$t_{\mu} \sim 4\times10^{-24}$ s

(f)
TPE
Both pions are emitted before either one is absorbed

Nucleon-Nucleon Pot.:

There are two general approaches to constructing a pot. that has the correct for $\begin{cases} \text{long} \\ \text{intermediate} \\ \text{short} \end{cases}$ ranges.

i) Phenomenological pot.

Generalization of OPE picture $\xrightarrow{\text{to}}$ OBE picture

For simplification only a <u>single</u> boson is considered.
The <u>multi-meson</u> exchange is compensated, by letting the strength for each type of meson exchange to be determined by fitting from NN scatt. data (as a parameter).

Hard Core term is put by hand without any ref. to its source.

Some difficulties;

a) Lifetimes of many of the mesons are sufficiently short

$\longrightarrow$ in the exchange process their decay must be taken into account.

b) In order to fit experimental data, the range of each OBE term and consequently $\longrightarrow$ their mass $\xrightarrow{\text{become}}$ adjustable parameters

with little or no relation to real masses.

ii) To make use of our knowledge of hadrons as much as possible and treat phenomenologically only those aspects, mainly short-range ints., of which we have incomplete knowledge.

Nucleon-Nucleon interaction for bound nucleons:

One of the reasons for constructing a correct form of nucl. pot. is:

$\left\{ \begin{array}{l} \text{To investigate the nuclear structure} \\ \text{and nuclear reaction} \end{array} \right.$

We are faced with many-body system $\xrightarrow{\text{Can always be expressed}}$ in terms of two-body int.

i) However effective int. between two-nucleons inside the nucleus is different with that of bare (under the influence of all other nucleons).

ii) The second difficulty:

Can a nuclear pot. be specified completely within a two-body system?

In two-particle system;

$$E = \text{const} \quad , \quad \bar{P} = \text{Conserved}$$

$$\frac{P_1^2}{2\mu_1} + \frac{P_2^2}{2\mu_2} = E$$

$$\frac{p^2}{2\mu} = E \rightarrow p_x^2 + p_y^2 + p_z^2 = 2\mu E \qquad \sqrt{p_x^2 + p_y^2 + p_z^2} = \sqrt{2\mu E}$$

i.e. The sum of momenta of two nucleons is confined to <u>lie</u> on a spherical <u>shell</u> in momentum space with the radius of $\sqrt{2\mu E}$

E: energy in the center of mass

The two-body int (T-matrix elements) is said to be, <u>On the energy shell</u>, (<u>on shell</u> matrix elements).

In a nucleus with more than <u>two</u> nucleons;

$\left\{\begin{array}{l}\text{Energy conservation} \\ \text{Momentum} \quad "\end{array}\right.$ applies to the nucleus as a <u>whole</u>

and the momenta of a <u>pair</u> of nucleons are <u>no longer</u> restricted by $\frac{p_1^2}{2\mu_1} + \frac{p_2^2}{2\mu_2} = E$

$\longrightarrow$ off the energy shell (off shell).

Relation quark-quark int.:

Although, it is accepted the force between nucleons is a facet of q-q strong int., a <u>quantitative</u> connection between nucl. force and q-q int. is still <u>lacking</u>.

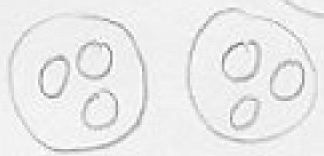
The difficulty arrises from QCD calculations at the low energies where nucl. phys. operates.

In order to study nucl. int. in quark model, we need a system of at least 6-quarks.

q-q force $\xrightarrow{\text{must give}}$ confinement cond.:

At close distances;

otherwise;



tightly clustered
(bag)

At large distances (compared with q-q distances);

The force between these two bags $\underset{\text{must be consistant}}{\approx}$ meson exchange

At intermediate distances:

the force = must be attractive and consistant with several pions and heavier mesons exchange.

Qualitatively: we can see how nucl. force may arise from q-q int. by making an analogy with the force between chemical molecules.

Molecules with spherical charge distribution excert no force between eachother.

But since there is a force between them ⟶ There must be a residual force present;

⟶ This is known as the Van der Waals force.

Suppose, a neutral molecule aquires an electric dipole moment $\bar{p}$ (say, as a result of fluctuation in its shape and consequently, its charge distribution)

$$\varphi(r) = \left[\frac{1}{4\pi\epsilon_0}\right] \frac{\hat{r} \cdot p}{r^2} = \left[\frac{1}{4\pi\epsilon_0}\right] \frac{p\cos\theta}{r^2} \qquad \text{due to dipole}$$

$$E(r,\theta) = -\nabla \varphi = -\left[\frac{1}{4\pi\epsilon_0}\right]\left\{\frac{\bar{P}}{r^3} - 3\left(\frac{P\cos\theta}{r^4}\right)\bar{r}\right\}$$ due to dipole

This electric field induces a dipole moment $\bar{P}'$ in another molecule as;

$$\bar{P}' = \chi \bar{E}$$ χ: polarizability of molecule

The dipole-dipole int. between these <u>two</u> molecule;

$$V(\bar{r}) = -\bar{P}'\cdot\bar{E} = -\left[\frac{1}{4\pi\epsilon_0}\right]^2 \chi\,(1+3\cos^2\theta)\,\frac{P^2}{r^6}$$

Interaction energy is always <u>negative</u>, regardless of the orientation of the first dipole.

$$\longrightarrow \quad F \sim \frac{1}{r^7}$$

For spherically symmetric molecule;

$$\langle \bar{P} \rangle = 0$$ ($\bar{P}$ on average is zero)

However, because of fluctuation, the instantaneous value of $\bar{P}$ may be $\neq 0$

i.e. $\longrightarrow \quad \langle P^2 \rangle \neq 0$

$\longrightarrow$ An attractive force between two molecules results from this fluctuation.

Analogusly:

Electrostatic force in molecules $\xrightarrow{\sim}$ Color force between quarks

Van der Vaals force between molecules $\xrightarrow{\sim}$ Color van der Vaals force between two nucleons