Chapter 8
Approximation Theory

8.1 Discrete least-Squares Approximation:

If the actual relationship between x and y is linear; a line $Y = ax + b$ that best fits the given data can be obtained from minimization of the least-squares error:

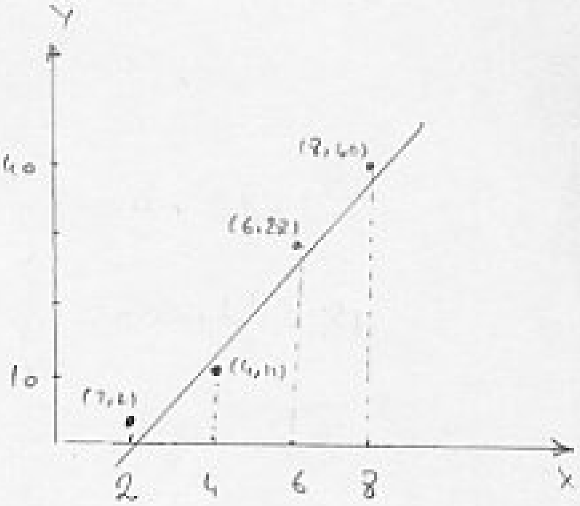
$$\sum_{i=1}^{m} [y_i - (ax_i + b)]^2$$

Ex. The given data shows a linear behaviour:

i	x_i	y_i
1	2	2
2	4	11
3	6	28
4	8	40

$$\sum_{i=1}^{4} [y_i - (ax_i + b)]^2 = [2 - (2a+b)]^2 + [11 - (4a+b)]^2$$

$$+ [28 - (6a+b)]^2 + [40 - (8a+b)]^2$$



$$\begin{cases} \dfrac{\partial}{\partial a} \displaystyle\sum_{i=1}^{4} [y_i - (ax_i + b)]^2 = 0 \\ \dfrac{\partial}{\partial b} \displaystyle\sum_{i=1}^{4} [y_i - (ax_i + b)]^2 = 0 \end{cases}$$

$$\begin{cases} 3a + 5b = 134 \\ 20a + 4b = 81 \end{cases}$$

$a = 6.55$

$b = -12.5$ $\Rightarrow Y = 6.55x - 12.5$

i	x_i	Y_i	$6.55x - 12.5$
1	2	2	0.6
2	4	11	13.7
3	6	28	26.8
4	8	40	39.9

with m arbitrary;

$$0 = \frac{\partial}{\partial a} \sum_{i=1}^{m} [y_i - (ax_i + b)]^2 = 2 \sum_{i=1}^{m} (y_i - ax_i - b)(-x_i)$$

$$0 = \frac{\partial}{\partial b} \sum_{i=1}^{m} [y_i - (ax_i + b)]^2 = 2 \sum_{i=1}^{m} (y_i - ax_i - b)(-1)$$

These equns. simplify to what is known as the <u>normal equns.</u>:

$$\begin{cases} a \sum_{i=1}^{m} x_i^2 + b \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} x_i y_i \\ a \sum_{i=1}^{m} x_i + b\, m = \sum_{i=1}^{m} y_i \end{cases}$$

Sol.;

$$a = \frac{m\left(\sum_{i=1}^{m} x_i y_i\right) - \left(\sum_{i=1}^{m} x_i\right)\left(\sum_{i=1}^{m} y_i\right)}{m\left(\sum_{i=1}^{m} x_i^2\right) - \left(\sum_{i=1}^{m} x_i\right)^2}$$

$$b = \frac{\left(\sum_{i=1}^{m} x_i^2\right)\left(\sum_{i=1}^{m} y_i\right) - \left(\sum_{i=1}^{m} x_i y_i\right)\left(\sum_{i=1}^{m} x_i\right)}{m\left(\sum_{i=1}^{m} x_i^2\right) - \left(\sum_{i=1}^{m} x_i\right)^2}$$

The General Prob.:

The general prob. of approximating a set of data, $\{(x_i, y_i) \mid i = 1, \ldots m\}$ with an algebraic polynomial

$P_n(x) = \sum_{k=0}^{n} a_k x^k$ of deg. $n < m-1$ using the least-squares procedure is handled in a similar manner and requires choosing the consts $a_0, a_1, \ldots a_n$ to minimize the least-squares error;

$$E = \sum_{i=1}^{m} (y_i - P_n(x_i))^2 = \sum_{i=1}^{m} y_i^2 - 2\sum_{i=1}^{m} P_n(x_i) y_i + \sum_{i=1}^{m} (P_n(x_i))^2$$

$$= \sum_{i=1}^{m} y_i^2 - 2 \sum_{i=1}^{m} \left(\sum_{j=0}^{n} a_j x_i^j \right) y_i + \sum_{i=1}^{m} \left(\sum_{j=0}^{n} a_j x_i^j \right)^2$$

$$= \sum_{i=1}^{m} y_i^2 - 2 \sum_{j=0}^{n} a_j \left(\sum_{i=1}^{m} y_i x_i^j \right) + \sum_{j=0}^{n} \sum_{k=0}^{n} a_j a_k \left(\sum_{i=1}^{m} x_i^{j+k} \right)$$

$$\frac{\partial E}{\partial a_j} = 0 \qquad j = 0, 1, \ldots n$$

$$0 = \frac{\partial E}{\partial a_j} = -2 \sum_{i=1}^{m} y_i x_i^j + 2 \sum_{k=0}^{n} a_k \sum_{i=1}^{m} x_i^{j+k}$$

This gives $n+1$ normal eqns in the $n+1$ unknowns a_j,

$$\sum_{k=0}^{n} a_k \sum_{i=1}^{m} x_i^{j+k} = \sum_{i=1}^{m} y_i x_i^j \qquad j = 0, 1, \ldots n$$

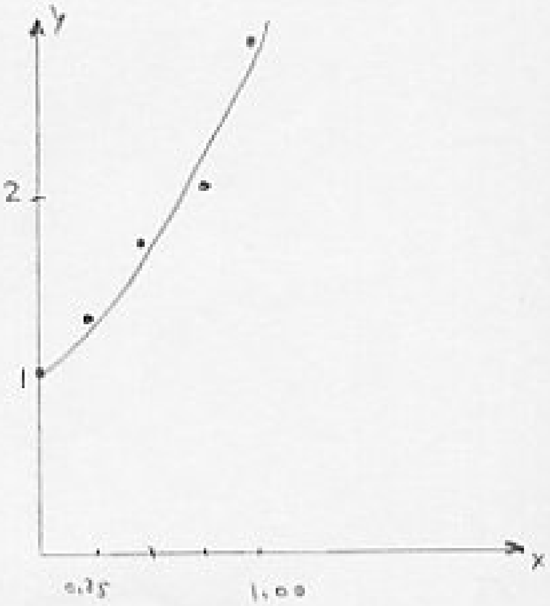
i.e.;

$$\begin{cases} a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 + a_2 \sum_{i=1}^{m} x_i^2 + \cdots + a_n \sum_{i=1}^{m} x_i^n = \sum_{i=1}^{m} y_i x_i^0 \\ a_0 \sum_{i=1}^{m} x_i^1 + a_1 \sum_{i=1}^{m} x_i^2 + a_2 \sum_{i=1}^{m} x_i^3 + \cdots + a_n \sum_{i=1}^{m} x_i^{n+1} = \sum_{i=1}^{m} y_i x_i^1 \\ \vdots \\ a_0 \sum_{i=1}^{m} x_i^n + a_1 \sum_{i=1}^{m} x_i^{n+1} + a_2 \sum_{i=1}^{m} x_i^{n+2} + \cdots + a_n \sum_{i=1}^{m} x_i^{2n} = \sum_{i=1}^{m} y_i x_i^n \end{cases}$$

Ex.

Fit the data in the Table with the discrete least-squares polynomial of deg. 2.

i	1	2	3	4	5
x_i	0	0.25	0.5	0.75	1.00
y_i	1.000	1.2840	1.6487	2.1170	2.7183



$n=2$, $m=5$

$$5a_0 + 2.5a_1 + 1.875a_2 = 8.7680$$

$$2.5a_0 + 1.875a_1 + 1.5625a_2 = 5.4514$$

$$1.875a_0 + 1.5625a_1 + 1.3828a_2 = 4.4015$$

$$\begin{cases} a_0 = 1.0052 \\ a_1 = 0.8641 \\ a_2 = 0.8437 \end{cases}$$

$$P_2(x) = 1.0052 + 0.8641x + 0.8437x^2$$

i	1	2	3	4	5
x_i	0	0.25	0.5	0.75	1.00
y_i	1.0000	1.2840	1.6487	2.1170	2.7183
$P(x_i)$	1.0052	1.2740	1.6482	2.1279	2.7130
$y_i - P(x_i)$	-0.0052	0.0100	0.0005	-0.0109	0.0053

$$E = \sum_{i=1}^{5} (y_i - P(x_i))^2 = 2.76 \times 10^{-4}$$

Factorization: (Doolittle algorithm)

Determine the elements of matrices L (lower) and U (upper) such $A = LU$.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ - & - & - & - & - \\ l_{n1} & l_{n2} & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ - & - & - & - & - \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$u_{1j} = a_{1j} \qquad j = 1, \dots n$$

$$l_{i1} = \frac{a_{i1}}{u_{11}} \qquad i = 2, \dots n$$

$$u_{2j} = a_{2j} - l_{21} u_{1j} \qquad j = 2, \dots n$$

$$l_{i2} = \frac{a_{i2} - l_{i1} u_{12}}{u_{22}} \qquad i = 3, \dots n$$

$$u_{rj} = a_{rj} - \sum_{k=1}^{r-1} l_{rk} u_{kj} \qquad j = r, \dots n$$

$$l_{ir} = \frac{a_{ir} - \sum_{k=1}^{r-1} l_{ik} u_{kr}}{u_{rr}} \qquad i = r+1, \dots n$$

Matrix Inverse;

$A = LU \qquad A^{-1} = ?$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \cdots & & & & \\ l_{n1} & l_{n2} & \cdots & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ c_{21} & 1 & 0 & \cdots & 0 \\ c_{31} & c_{32} & 1 & \cdots & 0 \\ \cdots & & & & \\ c_{n1} & c_{n2} & \cdots & & 1 \end{pmatrix} = L L^{-1} = I$$

$l_{21} + c_{21} = 0 \qquad c_{21} = -l_{21}$

$l_{31} + l_{32} c_{21} + c_{31} = 0 \qquad c_{31} = -(l_{31} + l_{32} c_{21})$

$l_{32} + c_{32} = 0 \qquad c_{32} = -l_{32}$

$l_{41} + l_{42} c_{21} + l_{43} c_{31} + c_{41} = 0 \qquad c_{41} = -(l_{41} + l_{42} c_{21} + l_{43} c_{31})$

$l_{42} + l_{43} c_{32} + c_{42} = 0 \qquad c_{42} = -(l_{42} + l_{43} c_{32})$

$l_{43} + c_{43} = 0 \qquad c_{43} = -l_{43}$

$$c_{ij} = -\sum_{k=j}^{i-1} l_{ik} c_{kj} \qquad \begin{array}{l} i = 2, \ldots n \\ j = 1, \ldots i-1 \end{array}$$

Similarly;

(d_{ij}: the elements of U^{-1})

$$d_{ii} = \frac{1}{u_{ii}} \qquad i = n, \ldots 1$$

$$d_{ij} = \frac{-1}{u_{ii}} \sum_{k=i+1}^{j} u_{ik} d_{kj} \qquad \begin{array}{l} i = n, \ldots 1 \\ j = n, \ldots i+1 \end{array}$$

$A^{-1} = U^{-1} L^{-1}$

23.7/2

Monte Carlo Methods:

The name Monte Carlo arises from the random or <u>chance</u> character of the method in the famous Casino in Monaco.

Systems with a large number of degrees of freedom are often of interest in physics.

A description of such systems often involves the evaluation of integrals of very high dimension.

<u>Ex.</u>

The classical partition func. for a gas of A atoms at $T = \frac{1}{\beta}$, interacting through a pair-wise potential V is proportional to the 3A-dimensional integral.

$$Z = \int d^3 r_1 \cdots d^3 r_A \, \exp\left[-\beta \sum_{i<j} V(r_{ij})\right]$$

Suppose a quadrature method for integration uses 10-different points for each coord.

$\xrightarrow{\text{So}}$ the integrand must be evaluated at 10^{3A} points

For $A = 20$, a fast computer with 10^7 evaluation/second would take $\longrightarrow$ 10^{53} second (more than 10^{34} times the age of the universe)

The essential idea is not to evaluate the integrand at every one of a large number of quadrature points, but rather at only representative random sampling of abscissae.

This is analogous to predicting the results of an election on the basis of a poll of a small number of voters.

The basic Monte Carlo Strategy:

A definite process is replaced by a random process that arrives at the same result.

Suppose we have to integrate:

$$I = \int_0^1 f(x)\,dx$$

$$I \simeq \underbrace{\frac{1}{N}}_{\text{width}} \sum_{i=1}^{N} f(x_i)$$

Remark:
$I = \int_a^b f(x)\,dx$
$I \simeq \frac{b-a}{N} \sum_{i=1}^{N} f(x_i)$
$x_i \in [a,b]$

x_i's are chosen randomly in $[0,1]$.

To estimate the uncertainty:

taking average

$$\sigma_I^2 = \frac{1}{N}\sigma_f^2 = \frac{1}{N}\left[\frac{1}{N}\sum_{i=1}^{N} f_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N} f_i\right)^2\right]$$

σ_f^2 : Variance in f in $\sum_{i=1}^{N} f(x_i)$

Remark: $\mu = \sum_j x_j P(x_j)$ P: probability
$\sigma^2 = \sum_j (x_j-\mu)^2 P(x_j) = \sum_j x_j^2 P(x_j) - \left[\sum_j x_j P(x_j)\right]^2$
$(\Delta A)^2 = \langle (A-\langle A\rangle)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$

σ_I decreases as $\frac{1}{\sqrt{N}}$

$\longrightarrow$ more points $\longrightarrow$ more precise answer

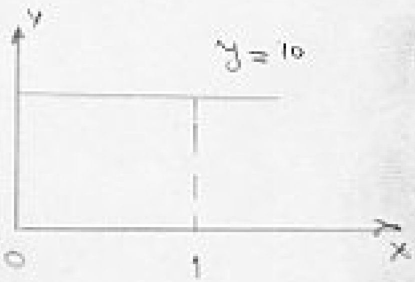
But it is very slowly converging.

The second important point:

The precision is greater if $\sigma_f \rightarrow$ small

σ_f is small if, f is smooth as possible.

Exterem examples:

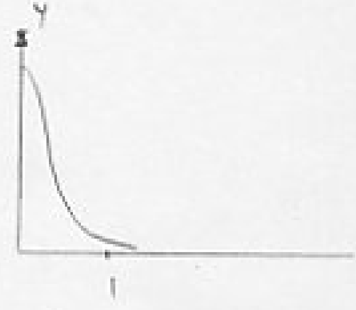


1) $I = \int_0^1 f(x)\,dx = \int_0^1 10\,dx = 10$

$I = \frac{1}{N}\sum_{i=1}^{N} f(x_i)$

Only one point is needed to give exact result.

$(\forall x_i \in [0,1])$. $(\sigma_I = 0)$



2) $I = \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} = 0.78540$

$f(x) = \frac{1}{1+x^2}$ everywhere ~ 0 except for a small region

$\longrightarrow$ more evaluation points are needed.

N	I ($\omega(x)=1$)	σ_I	I ($\omega(x)=\frac{1}{3}(4-2x)$)	σ_I
10	0.81441	0.04638	0.79982	0.00418
100	0.79513	0.01632	0.78832	0.00194
1000	0.78809	0.00508	0.78524	0.00066
5000	0.78963	0.00277	0.78530	0.00028

In order to improve the efficiency of the method (i.e. to decrease σ_I and to make the integrand smooth as possible). we multiply and divide the integrand by a weight func. with the properties;

$$\omega(x) \geq 0 \qquad \forall\, x \in [0,1]$$

$$\int_0^1 dx\, \omega(x) = 1 \qquad (1)$$

Thus; $I = \int_0^1 dx\, \omega(x) \dfrac{f(x)}{\omega(x)}$

Now we make a change of variable;

$$y(x) = \int_0^x dx'\, \omega(x') \qquad (2)$$

So that $\dfrac{dy}{dx} = \omega(x)$ $\quad y(x=0) = 0\ ,\ y(x=1) = 1$ (because of (1) and (2))

$$\rightarrow I = \int_0^1 dy\, \frac{f(y)}{\omega(y)} \qquad I \simeq \frac{1}{N}\sum_{i=1}^{N} \frac{f(y_i)}{\omega(y_i)}$$

$$y \in [0,1]$$

ω is chosen to behave like f (i.e., to be large where f is large, and to be small where f is small)

$\longrightarrow \dfrac{f}{\omega}$ can be made very smooth.

We return back to our example;

We choose $\omega(x) = A(4-2x) \geq 0 \quad \forall x \in [0,1]$

It decreases monotonically in $[0,1]$ (as does f), and

$$\int_0^1 A(4-2x)\,dx = 1 \quad \rightarrow \quad A = \frac{1}{3}$$

$$y(x) = \int_0^x dx' \left[\frac{1}{3}(4-2x')\right] = \frac{1}{3}(4x - x^2) \quad \longrightarrow x = 2-(4-3y)^{\frac{1}{2}}$$

$y(0) = 0 \qquad y(1) = 1$

$$f(x) = \frac{1}{1+x^2} \qquad \frac{f(x)}{\omega(x)} = \frac{3}{(1+x^2)(4-2x)}$$

$$\frac{f(1)}{\omega(1)} = \frac{3}{4} \qquad \frac{f(0)}{\omega(0)} = \frac{3}{4}$$

$$\frac{f(y)}{\omega(y)} = \frac{1}{1+\left[2-(4-3y)^{1/2}\right]^2}\;\frac{3}{4-2\left[2-(4-3y)^{1/2}\right]^2}$$

Random Numbers:

A typical mechomism is:

$$X_{n+1} = r X_n \pmod N$$

$$\begin{cases} X_{n+1} = 7^9 X_n \pmod{10^5} & \text{(decimal computers)} \leftarrow \text{length of sequ.} = 5\times10^{5-3} \\ X_{n+1} = (8t-3) X_n \pmod{2^5} & \text{(binary ") (t: large, to avoid long upward run)} \\ X_0 = 1 \end{cases}$$

$$X_{n+1} = (r X_n + C) \pmod N$$

$$X_{n+1} = (25173 X_n + 13849) \pmod{65536}$$

generates a well-scrambled arrangement of the integers from 0 to 65535.

To be convinced random, the sequence of X_n numbers must pass a test of <u>statistical</u> tests.

They must have properties like;

1 - evenly distributed over the interval $[0, N]$

2 - expected number of upward and downward - -.

- - -

These numbers are sometimes called pseudorandom numbers (not truly random) since they are produced by a deterministic mechanism.

Ex.

$$X_{n+1} = 13 X_n \pmod{100} \qquad X_0 = 1$$

01, 13, 69, 97, 61, 93, 09, 17, 21, 73, 49, 37, 81, 53, 89, 57, 41, 33, 29, 77

After 77 the sequence begins again at 01.

1) $\underbrace{01, \dots 73}_{\text{10 increasing}}, \underbrace{49 \dots 77}_{\text{10 decreasing}}$

2) They are evenly distributed.

⋮ . . .

Instead if we choose 5 (in place of 13);

$$X_{n+1} = 5 X_n \pmod{100} \quad x_0 = 1$$

01, 05, 25, 25, 25 .. (No randomness)

→ If r is chosen large → it may be best.

Ex.

$$X_{i+1} = \rho X_i \pmod{2^k} \qquad \rho = 8t \pm 3$$

For $k=5$ binary digit machine, we choose $\rho = 10101$

$X_0 = 10001 \rightarrow \rho X_0 = 101100101 \rightarrow \rho X_0 \pmod{2^5} = 00101$

binary	decimal
$X_0 = 10001$	17
$X_1 = 00101$	5
$X_2 = 01001$	9
$X_3 = 11101$	29
$X_4 = 00001$	1
$X_5 = 10101$	21
$X_6 = 11001$	25
$X_7 = 01101$	13
$X_8 = 10001$	17

Weighted Random Numbers:

We have faced with the weighted random numbers before.

The wieght func. introduced before played such a role;

i.e.:

uniform distribution of points in y $\xrightarrow{\text{implies}}$ the weighted distributions in x ($\sim W(x)$)

This means $\Rightarrow$ Points are concentrated about the most important values of x, where $W(x)$ (and hopefully $f(x)$) is large,

And little computing power is spent on calculating the integrand for unimportant values of x, when $W(x)$ and $f(x)$ are small.

Ex. $$I = \int_0^{\infty} dx\, e^{-x} g(x)$$

with $g(x) \sim$ smooth func.

It is sensible to generate x between 0 and ∞, with distribution e^{-x} and then to average $g(x)$ over these values.

To find random numbers x with <u>nonuniform</u> dist. of density $f(x)$, we generate them from a <u>uniform</u> dist. of random numbers Z by equating the cumulative distributions, that is,

$$\int_0^X 1\,dx = \int_0^Y f(y)\,dy = F(Y) = X$$

$$\longrightarrow \quad Y = F^{-1}F(Y) = F^{-1}(X)$$

<u>Ex.</u>

$$f(Y) = e^{-Y} \qquad F(Y) = \int_0^Y e^{-Y}\,dY = 1 - e^{-Y}$$

$$e^{-Y} = 1 - X \quad \rightarrow \quad Y = -\ln(1-X) \quad \text{or} \quad Y = -\ln X$$

<u>Ex.</u> $I \simeq \frac{1}{N}\sum_{i=1}^{N} g(x_i)$

Perform the integration for special cases $g(x) = x, x^2, x^3$, using the normally distributed random numbers.

<u>Sol.</u> $g(x) = x^{\cancel{3}\,2}$ case,

using the first ten random numbers of P244:

0.1, 1.3, 6.9, 9.7, 6.1, 9.3, 0.9, 1.7, 2.1, 7.3 (normalized to $x_i \in [0,10]$)

$$I = \int_0^\infty x^2 e^{-x}\,dx \simeq \int_0^{10} x^2 e^{-x}\,dx \approx \frac{(10-0)}{10}\sum_{i=1}^{10} x_i^2 e^{-x_i} \simeq 2.0480$$

$$\int_0^\infty x^2 e^{-\alpha x}\,dx = \frac{d^2}{d\alpha^2}\int_0^\infty e^{-\alpha x}\,dx = \frac{d^2}{d\alpha^2}\left[\frac{-1}{\alpha}e^{-\alpha x}\right]_0^\infty = \frac{d^2}{d\alpha^2}\left(\frac{1}{\alpha}\right)$$

$$= +\frac{2}{\alpha^3} \qquad \alpha = 1 \;\rightarrow\; \int_0^\infty = 2$$

X : 0.01 , 0.13 , 0.69 , 0.97 , 0.61 , 0.93 , 0.09 , 0.17 , 0.21 , 0.73

$\int_0^\infty e^{-x}dx \approx \int_0^{10} e^{-x}dx \qquad N'\int_0^{10} e^{-y}dy = 1 \rightarrow N' = 1.0000454$

$N'\int_0^{y} e^{-y}dy = x \qquad N'(1-e^{-y}) = x \qquad y = -\ln\left(1-\frac{x}{N'}\right)$

Y: 0.01, 0.14, 1.17, 3.51, 0.94 2.66 , 0.09, 0.19, 0.24, 1.31

$I \simeq \frac{(10-0)}{10} \sum_{i=1}^{10} y_i^2 e^{-y_i} \simeq 2.1960$

<u>Ex.</u> $I = \int_0^{25} x^2 = ? \qquad I \simeq \frac{b-a}{N} \sum_{i=1}^{N} f(x_i) \qquad x_i \in [a,b] = [0,25]$

Using the first 10-random number of P244 (i.e. $N=10$):

$0-100 \xrightarrow[\text{coeff}]{1/4} 0-25$

$\rightarrow x$: 0.25, 3.25, 17.25, 24.25, 15.25, 23.25, 2.25, 4.25, 5.25, 18.25

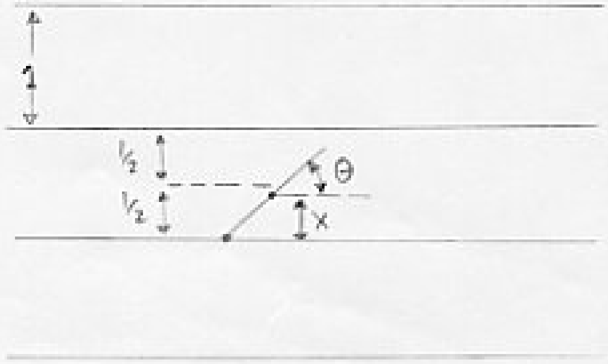
$I \simeq \frac{25-0}{10} \sum_{i=1}^{10} x_i^2 = 5133.8$

$\int_0^{25} x^2 = 5208.\bar{3}$ exact

Buffon needle:

In 1773 Buffen observed that if a needle of length $L \leq 1$ were tossed at random onto a horizontal surface ruled with equally spaced lines, say at unit spacing, then the probability of the needle crossing a line is

$$P = \frac{2L}{\pi}$$



Proof:

x: the position of the center of needle, that is randomly distributed in the interval

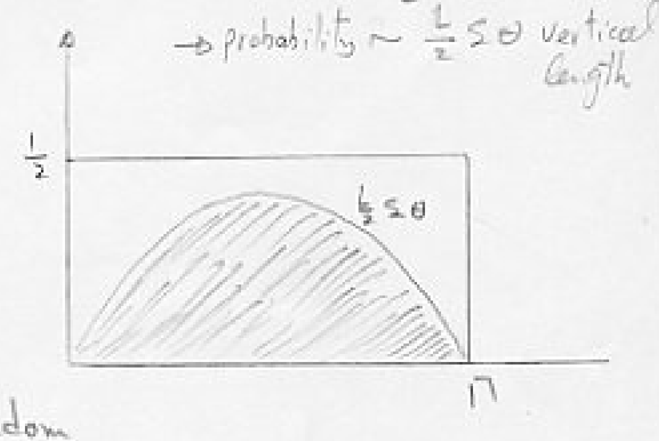
$0 \leq x \leq \frac{1}{2}$

i.e.: the distribution probability is uniform in this interval.

Also $0 \leq \theta \leq \pi$

A crossing will occur if and only if $x \leq \frac{L}{2} s\theta$

$$P = \frac{\int_0^{\pi} \frac{L}{2} s\theta \, d\theta}{\int_0^{\pi} \frac{1}{2} d\theta} = \frac{2L}{\pi}$$



By tossing the needle on the parallel lines (or using Monte Carlo random numbers) and calculating P, one may find the value of π.

{ Remark:
Crossing probability for a single tossing $= \frac{\text{vertical length of the needle}}{\text{Normalization (Maximum probability)}} = \frac{\frac{L}{2} s\theta}{\int_0^{\pi} \frac{1}{2} d\theta}$

→ $L=1$, $s\theta = 1$

Mean Value of distribution;

$$\mu = \sum_j x_j f(x_j) \qquad \text{discrete dist.}$$

$$\mu = \int_{-\infty}^{\infty} x f(x) dx \qquad \text{continuous} \ "$$

Variance;

$$\sigma^2 = \sum_j (x_j - \mu)^2 f(x_j)$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$

Remark:
$$(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$$

σ: standard deviation

$\sigma^2 \geq 0$

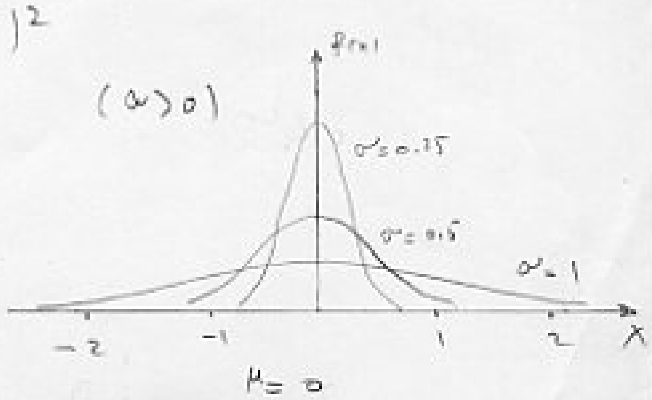
Theo. (Standard Variable)

If a random variable X has mean μ and variance σ^2, then the corresponding variable $Z = (X-\mu)/\sigma$ has mean $\underline{0}$ and variance $\underline{1}$.

Z is called <u>standard</u> variable corresponding to X.

Normal Distribution (Gauss dist.):

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad (\sigma > 0)$$



$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2} dv \quad \text{dist. func.}$$

$$P(a < X \leq b) = F(b) - F(a) = \frac{1}{\sigma\sqrt{2\pi}} \int_{a}^{b} e^{-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2} dv$$

$P(\mu - \sigma < X \leq \mu + \sigma) \approx 68\%$

$P(\mu - 2\sigma < X \leq \mu + 2\sigma) \approx 95.5\%$

$P(\mu - 3\sigma < X \leq \mu + 2\sigma) \approx 99.7\%$

Normal Distributed Random Numbers:

One easy way is to add 12 numbers from the flat distribution and then subtract 6 from the sum:

$$y = x_1 + x_2 + \dots x_{12} - 6$$

The dit. density of y is close to normal (Gaussian), having $\mu = 0$ and $\sigma^2 = 1$ -

The tails reach from -6 to $+6$, thus, "out of 6σ", and beyond that there are exactly zero occurences -

Remark: Flat dist.

$$\begin{cases} \mu = \dfrac{\int_0^1 x\,dx}{\int_0^1 dx} = 1/2 \\[2ex] \sigma_1^2 = \dfrac{\int_0^1 (x-\frac{1}{2})^2 dx}{\int_0^1 dx} = \frac{1}{2} \end{cases}$$

From the independence of the x_i:

Variance of sum = sum of variance

$$\sigma^2 = 12 \times \frac{1}{12} = 1$$