Chapter 6
Direct Methods for Solving Linear Systems

E_1: $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

E_2: $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$

$\vdots$

E_n: $a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$

6.1 Linear Systems of Equs.:

To solve a linear system like this one may perform the following operations:

1 - $(\lambda E_i) \longrightarrow (E_i)$

2 - $(E_i + \lambda E_j) \longrightarrow (E_i)$

3 - $(E_i) \longleftrightarrow (E_j)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$[A, b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$

6.2 Gaussian Elimination and Backward substitution:

$$\tilde{A} = [A, b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \cdots & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n,n+1} \end{bmatrix} \qquad \text{where } a_{i,n+1} = b_i \quad i = 1, 2, \cdots n$$

Provided $a_{11} \neq 0$, the operations corresponding to $(E_j - \frac{a_{j1}}{a_{11}} E_1) \to (E_j)$ are performed for each $j = 2, 3 \cdots n$ to eliminate the coeff. of x_1 in each of these rows.

We follow a sequential procedure for $i = 2, 3 \cdots n-1$ and perform the operation $(E_j - \frac{a_{ji}}{a_{ii}} E_i) \to (E_j)$ for $j = i+1, \cdots n$, provided $a_{ii} \neq 0$

This eliminates x_i in each row below the ith for all values of $i = 1, \cdots n-1$.

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \ddots & \ddots & & \\ 0 & \cdots & 0 & a_{nn} & a_{n,n+1} \end{bmatrix} \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= a_{1,n+1} \\ a_{22}x_2 + \cdots + a_{2n}x_n &= a_{2,n+1} \\ &\ddots \\ a_{nn}x_n &= a_{n,n+1} \end{aligned}$$

(new a_{ij}) (new a_{ij})

If necessary we use also make the interchange $(E_i) \longleftrightarrow (E_j)$

Since the equivalent linear system is triangular, backward substitution can be performed.

$$X_n = \frac{a_{n,n+1}}{a_{nn}} \qquad (n\text{th equ.})$$

$$X_{n-1} = \frac{a_{n-1,n+1} - a_{n-1,n} X_n}{a_{n-1,n-1}} \qquad ((n-1)\text{st equ.})$$

$$X_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}} \qquad i = n-1, \dots 2, 1$$

Ex.

$E_1: \; X_1 - X_2 + 2X_3 - X_4 = -8$

$E_2: \; 2X_1 - 2X_2 + 3X_3 - 3X_4 = -20$

$E_3: \; X_1 + X_2 + X_3 \qquad = -2$

$E_4: \; X_1 - X_2 + 4X_3 + 3X_4 = 4$

$$\tilde{A} = \tilde{A}^{(1)} = \begin{bmatrix} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 4 \end{bmatrix}$$

$$\begin{cases} (E_2 - 2E_1) \rightarrow (E_2) \\ (E_3 - E_1) \rightarrow (E_3) \\ (E_4 - E_1) \rightarrow (E_4) \end{cases} \qquad \tilde{A}^{(2)} = \begin{bmatrix} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{bmatrix}$$

Since the element $a_{22}^{(2)}$, called pivot element, is Zero, the Procedure cannot continue in its present form, so;

$$(E_2) \leftrightarrow (E_3) \qquad \tilde{A}^{(2)'} = \begin{bmatrix} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & -1 & 6 \\ 0 & 0 & -1 & 1 & -4 \\ 0 & 0 & 2 & 4 & 12 \end{bmatrix}$$

Since x_2 is already eliminated from E_3 and E_4;

$$\tilde{A}^{(3)} = \tilde{A}^{(2)'}$$

and the computation can continue with the operation;

$$(E_4 + 2E_3) \longrightarrow (E_4) \qquad \tilde{A}^{(4)} = \begin{bmatrix} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

$$x_4 = \frac{4}{2} = 2$$

$$x_3 = \frac{[-4 - (-1)x_4]}{-1} = 2$$

$$x_2 = \frac{[6 - x_4 - (-1)x_3]}{2} = 3$$

$$x_1 = \frac{[-8 - (-1)x_4 - 2x_3 - (-1)x_2]}{1} = -7$$

Gaussian Elimination with Backward Substitution Algorithm 6.1

Input: $A = (a_{ij})$ $\quad 1 \le i \le n, \quad 1 \le j \le n+1$

Output: $x_1, x_2 \ldots x_n$, or message that linear system has no sol.

S1 Do $i = 1, n-1$ (elimination process)

S2 Let p be the smallest integer with $i \le p \le n$ and $a_{pi} \ne 0$

If no integer p can be found,

then Output (no unique sol.)

Stop

S3 If $p \neq i$ then perform $(E_p) \leftrightarrow (E_i)$

S4 Do $j = i+1, n$

S5 $m_{ji} = \dfrac{a_{ji}}{a_{ii}}$

S6 $(E_j - m_{ji} E_i) \rightarrow (E_j)$

continue

S7 If $a_{nn} = 0$ then Output (no unique sol.)

stop

S8 $x_n = a_{n,n+1} / a_{nn}$

S9 Do $i = n-1, 1$

$$x_i = \left(a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j\right) / a_{ii}$$

S10 Output: $(x_1, \ldots x_n)$

stop