Chapter 5

Initial-Value Problems for Ordinary Differential Equs.

5.2 Euler's Method;

It is seldom used (it does not produce very accurate results). It is given for illustrating the method.

$$\frac{dy}{dt} = f(t,y) \qquad a \leq t \leq b \quad y(a) = \alpha \quad \text{(initial value Prob)}$$

In actuality, a continuous approx. to the sol. $y(t)$ will not be obtained; instead approx. to $y(t)$ will be generated at various values, called mesh points, in the interval $[a,b]$. Once the approximate sol. is obtained at the points, the approx. sol. at other points in the interval can be obtained by interpolation.

We use equally spaced mesh-points;

$\{t_0, t_1, t_2 -- t_N\}$ N: positive integer

$t_i = a + ih \qquad i = 0, 1, --N$

$h = \dfrac{b-a}{N}$ step size

We use Taylor's Theorem to derive Euler method.

Suppose $y \in C^2[a,b]$

$$y(t) = y(t_i) + (t-t_i)\,y'(t_i) + \frac{(t-t_i)^2}{2}\,y''(\xi_i)$$

ξ_i between t and t_i

Remark: $\frac{dy}{dt} = \frac{\partial y}{\partial t}$

At $t = t_{i+1}$

$$y(t_{i+1}) = y(t_i) + (t_{i+1}-t_i)\,y'(t_i) + \frac{(t_{i+1}-t_i)^2}{2}\,y''(\xi_i)$$

$t_i \le \xi_i \le t_{i+1}$ $\quad h = t_{i+1} - t_i$

$$\to \quad y(t_{i+1}) = y(t_i) + h\,y'(t_i) + \frac{h^2}{2}\,y''(\xi_i)$$

$$\downarrow$$

$$y(t_{i+1}) = y(t_i) + h\,f(t_i, y(t_i)) + \frac{h^2}{2}\,y''(\xi_i)$$

Euler's method constructs $w_i \approx y(t_i)$ $\quad i = 1, 2, \dots N$
(deleting the error term)

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h\,f(t_i, w_i) \quad i = 0, 1, \dots N \end{cases}$$

Difference equ.

Euler's Algorithm 5.1

To approximate the sol. of the initial-value prob.

$$y' = f(t,y) \quad a \le t \le b \quad , \; y(a) = \alpha$$

at $(N+1)$ equally spaced numbers in the interval $[a,b]$.

Input ; a, b, N, α

Output ; approx. w to y at the $(N+1)$ values of t.

S1 $\quad h = \dfrac{b-a}{N}$

$t = a$

$w = \alpha$

S2 $\quad$ Do $i = 1, N$

S3 $\quad w = w + hf(t, w)$ $\quad$ (compute w_i)

$t = a + ih$ $\quad$ (compute t_i)

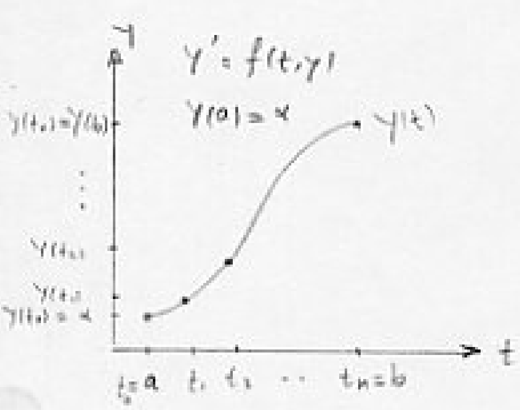
S4 $\quad$ Output (t, w)

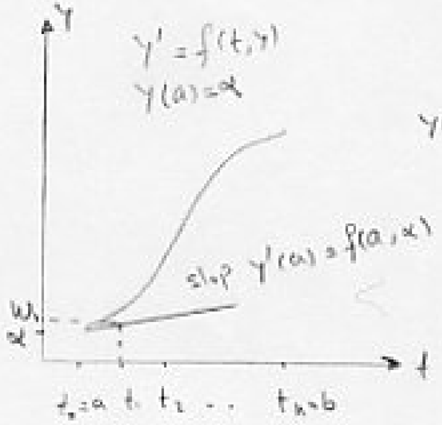
Continue

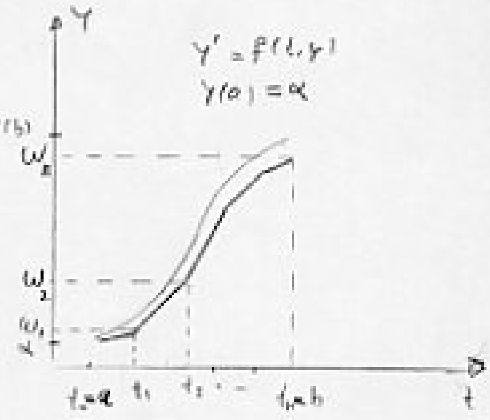
S5 $\quad$ stop

Geometrical interpretation:

When $\quad w_i \approx y(t_i) \longrightarrow f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$

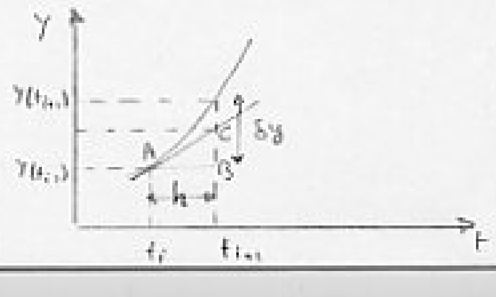






$$\left(\frac{dy}{dt}\right)_{t_i} = \frac{BC}{AB} \approx \frac{\delta y}{h} \quad \rightarrow \quad \delta y = h f(t_i, y(t_i))$$

$$\left(\frac{dy}{dt}\right)_{t_i} = f(t_i, y(t_i)) \qquad w_{i+1} = w_i + h f(t_i, y(t_i))$$



Ex. $y' = -y + t + 1 \qquad 0 \le t \le 1 \qquad y(0) = 1$

Suppose $N = 10 \rightarrow h = \frac{b-a}{N} \rightarrow h = 0.1 \qquad t_i = 0.1i$

$y' = f(t,y) \rightarrow f(t,y) = -y + t + 1$

$w_0 = 1$

$w_i = w_{i-1} + h\,f(t_{i-1}, w_{i-1}) = w_{i-1} + h(-w_{i-1} + t_{i-1} + 1)$

$= w_{i-1} + 0.1\left[-w_{i-1} + 0.1(i-1) + 1\right] = 0.9\,w_{i-1} + 0.01(i-1) + 0.1$

$i = 1, \ldots 10$

t_i	w_i	y_i	Error $= \lvert w_i - y_i \rvert$
0.0	1.000000	1.000000	0.0
0.1	1.000000	1.004837	0.004837
0.2	1.010000	1.018731	0.008731
⋮			
1.0	1.348678	1.367879	0.019201

The exact sol. is $y(t) = t + e^{-t}$

The error grows slightly as the value of t_i increases.

This controlled error growth is a consequence of the stability of the Euler's method, which implies that the errors due to rounding are expected to grow in no worse than a linear manner.

5.3 Higher-Order Taylor Methods:

Euler's method was derived by using Taylor's Theo. with n=2.

$$y' = f(t,y) \qquad a \le t \le b \quad , y(a) = \alpha$$

Suppose the sol. $y(t)$ to the mentioned initial-value prob. has $(n+1)$ continuous derivative. $y(t) \in C^{n+1}[a,b]$

Taylor expansion about t_i;

$$y(t) = y(t_i) + (t-t_i)y'(t_i) + \frac{(t-t_i)^2}{2}y''(t_i) + \cdots + \frac{(t-t_i)^n}{n!}y^{(n)}(t_i)$$

where ξ_i between t_i and t

$$+ \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

$$t_i < \xi_i < t_{i+1}$$

$$y'(t) = f(t, y(t))$$

$$y''(t) = f'(t, y(t))$$

$$\vdots$$

$$y^{(k)}(t) = f^{(k-1)}(t, y(t)) \qquad \left(\frac{d^k}{dt^k} y(t) = \frac{d^{k-1}}{dt^{k-1}} f(t, y(t)) \right)$$

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}f'(t_i, y(t_i)) + \cdots + \frac{h^n}{n!}f^{(n-1)}(t_i, y(t_i))$$

$$+ \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i))$$

The difference-equ. method is obtained by deleting the remainder term involving ξ_i.

$$W_0 = \alpha$$
$$W_{i+1} = W_i + h T^{(n)}(t_i, w_i) \qquad i = 0, 1, \dots N-1$$

where $T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \dots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$

Taylor method of order <u>n</u>

Note that <u>Euler's</u> method is <u>Taylor's</u> method of order <u>one</u>.

<u>Ex.</u> $\quad y' = -y + t + 1 \qquad 0 \leq t \leq 1 \quad y(0) = 1$

$$f(t, y(t)) = -y + t + 1$$
$$f'(t, y(t)) = \frac{d}{dt}(-y + t + 1) = -y' + 1 = -(-y + t + 1) + 1 = y - t$$
$$f''(t, y(t)) = -y + t$$
$$f'''(t, y(t)) = y - t$$

Remark: $f' = \frac{d}{dt} f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$

$$\Rightarrow T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) = -w_i + t_i + 1 + \frac{h}{2}(w_i - t_i)$$
$$= (1 - \frac{h}{2})(t_i - w_i) + 1$$

$$T^{(4)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i) + \frac{h^3}{24} f'''(t_i, w_i)$$
$$= -w_i + t_i + 1 + \frac{h}{2}(w_i - t_i) + \frac{h^2}{6}(-w_i + t_i) + \frac{h^3}{24}(w_i - t_i)$$
$$= (1 - \frac{h}{2} + \frac{h^2}{6} - \frac{h^3}{24})(t_i - w_i) + 1$$

Taylor methods of order 2 and 4 are;

$$\begin{cases} w_0 = 1 \\ w_{i+1} = w_i + h\left[\left(1-\frac{h}{2}\right)(t_i - w_i) + 1\right] \end{cases}$$

$$\begin{cases} w_0 = 1 \\ w_{i+1} = w_i + h\left[\left(1-\frac{h}{2}+\frac{h^2}{6}-\frac{h^3}{24}\right)(t_i - w_i) + 1\right] \end{cases}$$

$$i = 0, 1, \cdots n-1$$

$$\text{If} \quad h = 0.1 \longrightarrow \begin{cases} N = 10 \\ t_i = 0.1 i \end{cases} \qquad i = 1, 2, \cdots 10$$

$$n=2 \quad \begin{cases} w_0 = 1 \\ w_{i+1} = w_i + 0.1\left[\left(1-\frac{0.1}{2}\right)(0.1 i - w_i) + 1\right] \\ \quad = 0.905\, w_i + 0.0095 i + 0.1 \end{cases}$$

$$n=4 \quad \begin{cases} w_0 = 1 \\ w_{i+1} = w_i + 0.1\left[\left(1-\frac{0.1}{2}+\frac{0.01}{6}-\frac{0.001}{24}\right)(0.1 i - w_i) + 1\right] \\ \quad = 0.9048375\, w_i + 0.009516251 i + 0.1 \end{cases}$$

$$i = 0, 1, \cdots 9$$

t	Exact value $y(t) = t + e^{-t}$	Euler's Method	Error	Taylor's Method $n=2$	Error	Taylor's Method $n=4$	Error
0.1	1.0000000000	1.000000	0	1.000000	0	1.0000000000	0
⋮							
0.3	1.0408182707	1.029000	1.182×10^{-2}	1.041218	3.998×10^{-4}	1.0408184220	2.013×10^{-7}
⋮							
1.0	1.3678794412	1.348678	1.920×10^{-2}	1.368541	6.616×10^{-4}	1.3678797744	3.332×10^{-7}

5-4 Runge-Kutta Methods:

Taylor methods have high-local truncation error, but disadvantage of requiring the computation and evaluation of the derivatives of $f(t,y)$.

Theo 5.12

Suppose that $f(t,y)$ and all of partial derivatives of order $\leq n+1$ are continuous on $D = \{(t,y) \mid a \leq t \leq b,\ c \leq y \leq d\}$.

Let $(t_0, y_0) \in D$.

$\forall (t,y) \in D$ $\exists\ \xi$ between t and t_0, and η between y and y_0

with;

$$f(t,y) = P_n(t,y) + R_n(t,y)$$

where

$$P_n(t,y) = f(t_0,y_0) + \left[(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)\right]$$

$$+ \left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + (t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t\,\partial y}(t_0,y_0)\right.$$

$$\left. + \frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right] + \cdots$$

$$+ \left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j \frac{\partial^n f}{\partial^{n-j}\,\partial y^j}(t_0,y_0)\right]$$

$$\binom{n}{j} \equiv \frac{n!}{j!\,(n-j)!} \quad \text{Binomial coeff.}$$

$$R_n(t,y) = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t-t_0)^{n+1-j} (y-y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(\xi,\eta)$$

The func. P_n is called the nth Taylor polynomial in two-variables for f about (t_0,y_0), and $R_n(t,y)$ is the remainder term associated with $P_n(t,y)$.

Ex.

The second Taylor polynomial for $f(t,y) = \sqrt{4t+12y-t^2-2y^2-6}$ about $(2,3)$ is found from;

$$P_2(t,y) = f(t_0,y_0) + (t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)$$
$$+ \left[\frac{(t-t_0)^2}{2} \frac{\partial^2 f}{\partial t^2}(t_0,y_0) + (t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0,y_0) \right.$$
$$\left. + \frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0) \right]$$

at $(t_0,y_0) = (2,3)$

$$P_2(t,y) = 4 - \frac{1}{4}(t-2)^2 - \frac{1}{2}(y-3)^2$$

P_2 gives an accurate approx. to $f(t,y)$ when t is close to 2 and y is close to 3.

For example; $P_2(2.1, 3.1) = 3.9925$ and $f(2.1,3.1) = 3.9962$

First step in deriving Runge-Kutta method is to determine values for a_1, α_1 and β_1 with the property that;

$a_1 f(t+\alpha_1, y+\beta_1)$ approximates;

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2} f'(t,y)$$

with error no greater than $O(h^2)$, the local truncation error for the Taylor method of order 2.

Since; $f'(t,y) = \frac{df}{dt}(t,y) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y) \cdot y'(t)$

and $\quad y'(t) = f(t,y)$

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y) \cdot f(t,y) \qquad (1)$$

First order Taylor expansion of $f(t+\alpha_1, y+\beta_1)$ about (t,y);

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 R_1(t+\alpha_1, y+\beta_1) \qquad (2)$$

where

$$R_1(t+\alpha_1, y+\beta_1) = \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2}(\xi,\eta) + \alpha_1\beta_1 \frac{\partial^2 f}{\partial t\,\partial y}(\xi,\eta) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2}(\xi,\eta)$$

ξ between t and $t+\alpha_1$ $\qquad$ η between y and $y+\beta_1$

Matching the coeffs. of f and its derivatives in (1) and (2):

$$\begin{cases} f(t,y): & a_1 = 1 \\ \frac{\partial f}{\partial t}(t,y): & a_1\alpha_1 = \frac{h}{2} \\ \frac{\partial f}{\partial y}(t,y): & a_1\beta_1 = \frac{h}{2} f(t,y) \end{cases} \longrightarrow \begin{cases} a_1 = 1 \\ \alpha_1 = \frac{h}{2} \\ \beta_1 = \frac{h}{2} f(t,y) \end{cases}$$

$$\rightarrow T^{(2)}(t,y) = f\left(t+\frac{h}{2},\ y+\frac{h}{2}f(t,y)\right) - R_1\left(t+\frac{h}{2},\ y+\frac{h}{2}f(t,y)\right)$$

where

$$R_1\left(t+\frac{h}{2},\ y+\frac{h}{2}f(t,y)\right) = \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2}(\xi,\eta) + \frac{h^2}{4} f(t,y)\frac{\partial^2 f}{\partial t\,\partial y}(\xi,\eta) + \frac{h^2}{8}\left(f(t,y)\right)^2\frac{\partial^2 f}{\partial y^2}(\xi,\eta)$$

If $\frac{\partial^2 f}{\partial t^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial t\,\partial y}$ are bounded;

$R_1\left(t+\frac{h}{2},\ y+\frac{h}{2}f(t,y)\right)$ is $O(h^2)$, (the order of the local truncation error of Taylor's method of order 2).

Midpoint method:

The difference equ. method resulting from replacing $T^{(2)}(t,y)$ in Taylor's method of order $\underline{2}$ by $f\left(t+\frac{h}{2},\ y+\frac{h}{2}f(t,y)\right)$ is a specific Runge-Kutta method known as the Midpoint method.

$w_0 = \alpha$

$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right) \qquad i = 0, 1, \dots N-1$

To match $T^{(2)}$ and $a_1 f(t + \alpha_1, y + \beta_1) \longrightarrow$ 3-parameters are needed

$\Rightarrow$ We need more complicated form to satisfy the conds. required for any of the higher-order Taylor methods.

The most appropriate 4-parameter form for approximating

$$T^{(3)}(t, y) = f(t, y) + \frac{h}{2} f'(t, y) + \frac{h^2}{6} f''(t, y)$$

is $\quad a_1 f(t, y) + a_2 f(t + \alpha_2, y + \delta_2 f(t, y))$

Methods of local truncation error $O(h^2)$ can be obtained from this:

i) Modified Euler Method:

$$a_1 = a_2 = \frac{1}{2}, \quad \alpha_2 = \delta_2 = h$$

$w_0 = \alpha$

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i)) \right]$$

$$i = 0, 1, \dots N-1$$

ii) Heun's method;

$$a_1 = \frac{1}{4}, \quad a_2 = \frac{3}{4}, \quad \alpha_2 = \delta_2 = \frac{2}{3}h$$

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{4}\left[f(t_i, w_i) + 3f\left(t_i + \frac{2}{3}h,\ w_i + \frac{2}{3}h f(t_i, w_i)\right)\right]$$

$$i = 0, 1, \dots N-1$$

Ex. Apply any of the Runge-Kutta methods of order 2, to

$$y' = -y + t + 1 \qquad 0 \le t \le 1 \qquad y(0) = 1$$

Sol.

All the $O(h^2)$ methods give the following difference-equ. (the same as given in Taylor's method of order 2) because of the nature of the differential equ.

$$w_0 = 1$$

$$w_{i+1} = 0.905\, w_i + 0.0095\, i + 1$$

Ex.

$$y' = -y + t^2 + 1 \qquad 0 \le t < 1 \qquad y(0) = 1$$

Exact sol.: $y(t) = -2e^{-t} + t^2 - 2t + 3$

Choose $N = 10$, $h = 0.1$ $t_i = 0.1 i$

The difference equs.:

Midpoint method: $w_{i+1} = 0.905\, w_i + 0.00095\, i^2 + 0.001\, i + 0.09525$

Modified Euler's " : $w_{i+1} =$ " + " + " $+ 0.0955$

Heun's " : $w_{i+1} =$ " + " + " $+ 0.095333333$

$i = 0, 1, \ldots 9$

t_i	Exact	Midpoint	Error	Modified Euler	Error	Heun's	Error
0.0	1.0000000	1.0000000	0	1.0000000	0	1.0000000	0
⋮							
0.3	1.0083636	1.0082458	1.18×10^{-4}	1.0089268	5.63×10^{-4}	1.0084728	1.09×10^{-4}
⋮							
1.0	1.2642411	1.2645798	3.39×10^{-4}	1.2662416	2.00×10^{-3}	1.2651337	8.93×10^{-4}

$T^{(3)}(t,y)$ can be approximated with error $O(h^3)$ by an expression of the form:

$$f(t+\alpha_1,\ y+\delta_1 f(t+\alpha_2,\ y+\delta_2 f(t,y)))$$

But the algebra is complicated.

Runge-Kutta method of order-4:

$$w_0 = \alpha$$
$$K_1 = h f(t_i, w_i)$$
$$K_2 = h f(t_i + \frac{h}{2}, w_i + \frac{K_1}{2})$$
$$K_3 = h f(t_i + \frac{h}{2}, w_i + \frac{K_2}{2})$$
$$K_4 = h f(t_{i+1}, w_i + K_3)$$
$$w_{i+1} = w_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$i = 0, 1 \ldots N-1$

with the local truncation error $O(h^4)$, provided the $y(t)$ has 5-continuous derivatives.

Runge-Kutta (order-4) Algorithm 5.2

To appoximate of sol. of initial-value prob.

$$y' = f(t, y) \quad a \leqslant t \leqslant b, \quad y(a) = \alpha$$

at $(N+1)$ equally spaced numbers in $[a,b]$.

Input: a, b, N, α

Output: approx. w to y at $N+1$ values of t.

S1 $h = \dfrac{b-a}{N}$

$t = a$

$w = \alpha$

S2 $\quad$ Do $i = 1, N$

S3 $\quad K_1 = h\,(t, w)$

$K_2 = hf\left(t + \frac{h}{2}, w + \frac{K_1}{2}\right)$

$K_3 = hf\left(t + \frac{h}{2}, w + \frac{K_2}{2}\right)$

$K_4 = hf\left(t + h, w + K_3\right)$

S4 $\quad W = w + (K_1 + 2K_2 + 2K_3 + K_4)/6 \quad$ (compute w_i)

$t = a + ih$

Continue

S5 $\quad$ output (t, w)

S6 $\quad$ Stop

Ex. Runge-Kutta method of order 4, for the initial-value prob. $\quad y' = -y + t + 1 \quad 0 \le t \le 1 \quad y(0) = 1$

with $h = 0.1$, $N = 10$, $t_i = 0.1i$ gives the results:

t_i	Exact	Runge-Kutta order 4	Error
0.0	1.0000000000	1.0000000000	0
0.1	1.0048374180	1.0048375000	8.200×10^{-8}
⋮			
0.5	1.1065306597	1.1065309344	2.747×10^{-7}
⋮			
1.0	1.3678794412	1.3678797744	3.332×10^{-7}

5.9 Higher Order Equs. and Systems of Differential Equs.

An mth-order system of first-order initial value problems can be expressed in the form:

$$\begin{cases} \dfrac{du_1}{dt} = f_1(t, u_1, u_2 \ldots u_m) \\ \dfrac{du_2}{dt} = f_2(t, u_1, u_2, \ldots u_m) \\ \vdots \\ \dfrac{du_m}{dt} = f_m(t, u_1, u_2 \ldots u_m) \end{cases} \qquad \begin{cases} u_1(a) = \alpha_1 \\ u_2(a) = \alpha_2 \\ \vdots \\ u_m(a) = \alpha_m \end{cases}$$

$a \leq t \leq b$

Def 5.15

The func. $f(t, y_1 \ldots y_m)$ defined on the set:

$$D = \{(t, u_1, u_2 \ldots u_m) \mid a \leq t \leq b, \ -\infty < u_i < \infty \text{ for each } i = 1, \ldots m\}$$

is said to satisfy a Lipschitz cond. on D in the variables $u_1, u_2, \ldots u_m$, if a const. $L > 0$ exists with the property that

$$|f(t, u_1, \ldots u_m) - f(t, z_1, \ldots z_m)| \leq L \sum_{j=1}^{m} |u_j - z_j|$$

for all $(t, u_1 \ldots u_m)$ and $(t, z_1, \ldots z_m)$ in D.

By using the Mean Value Theo., it can be shown that if f and its first partial derivatives are continuous on D and if

$$\left| \frac{\partial f(t, u_1 \ldots u_m)}{\partial u_i} \right| \leq L$$

for each $i = 1, \ldots m$ and all $(t, u_1 \ldots u_m)$ in D
$\xrightarrow{\text{then}}$ f satisfies a Lipschitz cond. on D with Lipschitz const L.

Theo 5.16

suppose

$D = \{(t, u_1 \ldots u_m) \mid a \leq t \leq b, -\infty < u_i < \infty \text{ for each } i = 1 \ldots m\}$

and let $f_i(t, u_1 \ldots u_m)$ for each $i = 1, 2 \ldots m$, be continuous on D and satisfy a Lipschitz cond. then.
The system of first-order differential equs. (P 207) subject to the initial conds. (P207), has a unique sol. $u_1(t), u_2(t) \ldots u_m(t)$ for $a \leq t \leq b$.

Runge-Kutta for systems of diff. eqns.

We used:

$$w_0 = \alpha$$
$$K_1 = h f(t_i, w_i)$$
$$K_2 = h f\left(t_i + \frac{h}{2},\ w_i + \frac{K_1}{2}\right)$$
$$K_3 = h f\left(t_i + \frac{h}{2},\ w_i + \frac{K_2}{2}\right)$$
$$K_4 = h f(t_{i+1},\ w_i + K_3)$$
$$w_{i+1} = w_i + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right] \qquad i = 0, 1, \dots N-1$$

to solve $y' = f(t, y)$, $a \le t \le b$, $y(a) = \alpha$

It can be generalized as follows:

For a chosen $N > 0$, $h = \dfrac{b-a}{N}$

$$t_j = a + jh \quad j = 0, \dots N \qquad t_j \in [a, b]$$

Notation;

w_{ij} approximation for $u_i(t_j)$

$i = 0, 1 \dots m \qquad j = 0, 1, \dots N$

i.e. $w_{ij} \xrightarrow{\text{will approximate}}$ ith sol. $u_i(t)$ at the jth mesh point t_j.

$w_{10} = \alpha_1 \qquad w_{20} = \alpha_2 \quad \cdots \quad w_{m0} = \alpha_m$

If we assume that $w_{1j}, w_{2j} \cdots w_{mj}$ have been calculated, we obtain $w_{1,j+1}, w_{2,j+1} \cdots w_{m,j+1}$ by first calculating:

$$K_{1,i} = h f_i(t_j, w_{1j}, w_{2j} \ldots w_{mj}) \qquad \text{for each } i = 1, \ldots m$$

$$K_{2,i} = h f_i\left(t_j + \frac{h}{2}, w_{1j} + \frac{K_{11}}{2}, w_{2j} + \frac{K_{12}}{2}, \ldots w_{mj} + \frac{K_{1m}}{2}\right)$$

for each $i = 1 \cdots m$

$$K_{3,i} = h f_i\left(t_j + \frac{h}{2}, w_{1j} + \frac{K_{21}}{2}, w_{2j} + \frac{K_{22}}{2}, \ldots w_{mj} + \frac{K_{2m}}{2}\right)$$

for each $i = 1 \cdots m$

$$K_{4,i} = h f_i(t_j + h, w_{1j} + K_{31}, w_{2j} + K_{32}, \ldots w_{mj} + K_{3m})$$

for each $i = 1, \cdots m$

$$W_{i,j+1} = W_{i,j} + \frac{1}{6}\left[K_{1i} + 2K_{2i} + 2K_{3i} + K_{4i}\right]$$

$i = 1 \cdots m$

Note that $K_{11}, K_{12}, \ldots K_{1m}$ must all be computed before K_{21} can be determined.

Runge-Kutta for Systems of Diff. Equs.

Algorithm 5.7

To approximate the sol. of the mth-order system of first-order initial value probs.;

$$u_j' = f_j(t, u_1, u_2, \dots u_m) \quad j = 1, 2, \dots m$$

$$a \leq t \leq b \quad u_j(a) = \alpha_j \quad "$$

at (N+1) equally spaced numbers in the interval [a,b]

Input $a, b, m, N, \alpha_1 \dots \alpha_m$.

Output approxs. w_j to $u_j(t)$ at $N+1$ values of t.

S1 $\quad h = \dfrac{b-a}{N}$

$\quad t = a$

S2 $\quad$ Do $j = 1, m$

$\quad w_j = \alpha_j$

S3 $\quad$ Continue

S4 $\quad$ Do 10 $i = 1, N$

S5 $\quad$ Do 20 $j = 1, m$

$\quad K_{1j} = hf_j(t, w_1, w_2 \dots w_m)$

20 Continue

S6 Do 30 j=1, m

$$K_{2j} = hf_j\left(t+\frac{h}{2}, w_1+\frac{K_{11}}{2}, w_2+\frac{K_{12}}{2}, \dots w_m+\frac{K_{1m}}{2}\right)$$

30 Continue

S7 Do 40 j=1, m

$$K_{3j} = hf_j\left(t+\frac{h}{2}, w_1+\frac{K_{21}}{2}, w_2+\frac{K_{22}}{2}, \dots w_m+\frac{K_{2m}}{2}\right)$$

40 Continue

S8 Do 50 j=1, m

$$K_{4j} = hf_j\left(t+h, w_1+K_{31}, w_2+K_{32}, \dots w_m+K_{3m}\right)$$

50 Continue

S9 Do 60 j=1, m

$$W_{ij} = w_{ij} + (K_{1j}+2K_{2j}+2K_{3j}+K_{4j})/6$$

60 continue

S10 $t = a + ih$

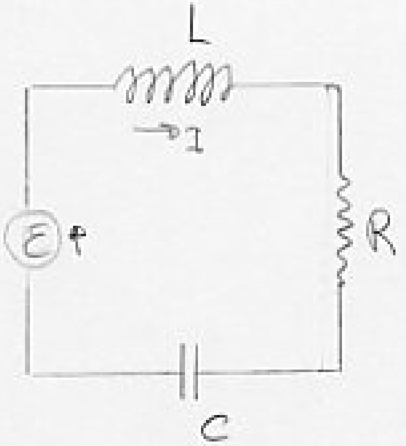
10 Continue

S11 Output (t, w_{ij}) $i=1,N$, $j=1,m$

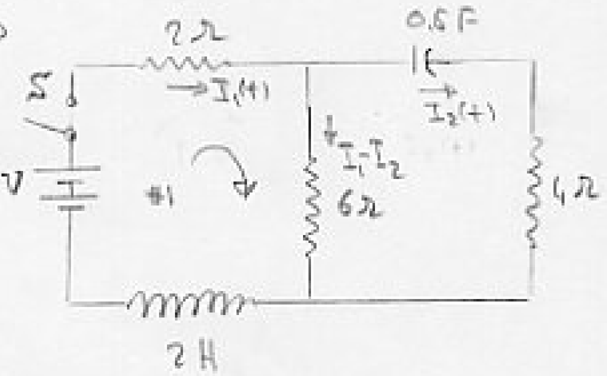
S12 Stop

Ex.

Acc. to Kirchhoff's Law;



$$-L\frac{dI(t)}{dt} - RI(t) - \frac{1}{C}\underbrace{\int I(t)dt}_{q(t)} + \mathcal{E}(t) = 0$$



$$\begin{cases} -2I_1(t) - 6\left(I_1(t) - I_2(t)\right) - 2\frac{dI_1(t)}{dt} + 12 = 0 & (1) \\ -\frac{1}{0.5}\int I_2(t)dt - 4I_2(t) - 6\left(I_1(t) - I_2(t)\right) = 0 & (2) \end{cases}$$

S is closed at $t=0$

$\longrightarrow I_1(0) = 0 \ , \ I_2(0) = 0$

$$(2) \rightarrow \frac{1}{0.5}I_2(t) + 4\frac{dI_2(t)}{dt} + 6\left(\frac{dI_1(t)}{dt} - \frac{dI_2(t)}{dt}\right) = 0 \qquad (3)$$

$$(1)(3) \rightarrow \begin{cases} \frac{dI_1}{dt} = f_1(t, I_1, I_2) = -4I_1 + 3I_2 + 6 \\ \frac{dI_2}{dt} = f_2(t, I_1, I_2) = -2.4I_1 + 1.6I_2 + 3.6 \end{cases} \qquad \begin{cases} I_1(0) = 0 \\ I_2(0) = 0 \end{cases}$$

$$\text{Exact sol.:} \begin{cases} I_1(t) = -3.375e^{-2t} + 1.875e^{-0.4t} + 1.5 \\ I_2(t) = -2.25e^{-2t} + 2.25e^{-0.4t} \end{cases}$$

Applying the Runge-Kutta 4th order with $h=0.1$

Since $W_{10} = I_1(0) = 0$

$W_{20} = I_2(0) = 0$

$$\begin{cases} K_{11} = h\, f_1(t_0, w_{10}, w_{20}) = 0.1\, f_1(0,0,0) \\ \quad = 0.1\,[-4(0) + 3(0) + 6] = 0.6 \\ K_{12} = h\, f_2(t_0, w_{10}, w_{20}) = 0.1\, f_2(0,0,0) \end{cases}$$

$$= 0.1\,[-2.4(0) + 1.6(0) + 3.6] = 0.36$$

$$\begin{cases} K_{21} = h\, f_1(t_0 + \frac{h}{2}, w_{10} + \frac{K_{11}}{2}, w_{20} + \frac{K_{12}}{2}) = 0.1\, f_1(0.05, 0.3, 0.18) \\ \quad = 0.1\,[-4(0.3) + 3(0.18) + 6] = 0.534 \\ K_{22} = h\, f_2(t_0 + \frac{h}{2}, w_{10} + \frac{K_{11}}{2}, w_{20} + \frac{K_{12}}{2}) = 0.1\, f_2(0.05, 0.3, 0.18) \end{cases}$$

$$= 0.1\,[-2.4(0.3) + 1.6(0.18) + 3.6] = 0.3183$$

Similarly:

$$\begin{cases} K_{31} = (0.1)\, f_1(0.05, 0.267, 0.1584) = 0.54072 \\ K_{32} = (0.1)\, f_2(0.05, 0.267, 0.1584) = 0.321264 \end{cases}$$

$$\begin{cases} K_{41} = (0.1)\, f_1(0.1, 0.54072, 0.321264) = 0.4800912 \\ K_{42} = (0.1)\, f_2(0.1, 0.54072, 0.321264) = 0.28162944 \end{cases}$$

$$I_1(0.1) \simeq w_{11} = w_{10} + \frac{1}{6}\left[K_{11} + 2K_{21} + 2K_{31} + K_{41}\right]$$

$$= 0 + \frac{1}{6}\left[0.6 + 2(0.534) + 2(0.54072) + 0.4800912\right]$$

$$= 0.5382552$$

$$I_2(0.1) \simeq w_{21} = w_{20} + \frac{1}{6}\left[K_{12} + 2K_{22} + K_{32} + K_{42}\right]$$

$$= 0.3196263$$

t_j	w_{1j}	w_{2j}	$\lvert I_1(t_j)-w_{1j}\rvert$	$\lvert I_2(t_j)-w_{2j}\rvert$
0	0	0	0	0
0.1	0.5382550	0.3196263	0.8285×10^{-5}	0.5803×10^{-5}
$\vdots$				
0.5	1.793505	1.014402	0.2193×10^{-4}	0.1240×10^{-4}

mth-Order Differential Equ.:

$$y^{(m)}(t) = f(t, y, y', \cdots y^{(m-1)}) \qquad a \leq t \leq b$$

with the initial conds.:

$$y(a)=\alpha_1 \quad , \; y'(a)=\alpha_2 , \cdots \; y^{(m-1)}(a)=\alpha_m$$

Let
$$\begin{cases} u_1(t) = y(t) \\ u_2(t) = y'(t) \\ \vdots \\ u_m(t) = y^{(m-1)}(t) \end{cases}$$

$$\rightarrow \begin{cases} \dfrac{du_1}{dt} = \dfrac{dy}{dt} = u_2 \\ \dfrac{du_2}{dt} = \dfrac{dy'}{dt} = u_3 \\ \vdots \\ \dfrac{du_{m-1}}{dt} = \dfrac{dy^{(m-2)}}{dt} = u_m \end{cases}$$

and

$$\frac{du_m}{dt} = \frac{dY^{(m-1)}}{dt} = Y^{(m)} = f(t, Y, Y', \dots Y^{(m-1)}) = f(t, u_1, u_2 \dots u_m)$$

with initial conds. :

$$u_1(a) = \alpha_1 \quad , \quad u_2(a) = \alpha_2 \quad - - \quad u_m(a) = Y^{(m-1)}(a) = \alpha_m$$

Ex.

$$Y'' - 2Y' + 2Y = e^{2t} \sin t \qquad 0 \leq t < 1 \qquad \begin{cases} Y(0) = -0.4 \\ Y'(0) = -0.6 \end{cases}$$

Sol.

$$\begin{matrix} u_1(t) = Y(t) \\ u_2(t) = Y'(t) \end{matrix} \longrightarrow \begin{cases} u_1'(t) = u_2(t) \\ u_2'(t) = e^{2t} \sin t - 2u_1(t) + 2u_2(t) \end{cases}$$

$$\text{with} \begin{cases} u_1(0) = -0.4 \\ u_2(0) = -0.6 \end{cases}$$

Ex.
$$\begin{cases} \dot{x} = -x - y \\ \dot{y} = x - y \end{cases}$$

$\rightarrow \quad x(t) = e^{-t}\cos t \qquad y(t) = e^{-t}\sin t \qquad$ exact sols.

Ex.

$$\begin{cases} \ddot{x} = x - y - 9\dot{x}^2 + \dot{y}^2 + 6\ddot{y} + 2t \\ \dddot{y} = \ddot{y} - \dot{x} + e^{x} - t \end{cases}$$

$x(1) = 2 \;,\; \dot{x}(1) = -4 \;,\; y(1) = -2 \;,\; \dot{y}(1) = 7 \;,\; \ddot{y}(1) = 6$

t	x_1	1	$\dot{x}_1 = 1$
x	x_2	2	$\dot{x}_2 = x_3$
$\dot{x}$	x_3	-4	$\dot{x}_3 = x_2 - x_4 - 9x_3^2 + x_5^2 + 6x_6 + 2x_1$
y	x_4	-2	$\dot{x}_4 = x_5$
$\dot{y}$	x_5	7	$\dot{x}_5 = x_6$
$\ddot{y}$	x_6	6	$\dot{x}_6 = x_6 - x_3 + e^{x_2} - x_1$

Ex.

$\ddot{x} = 3\cos^2 t + 2$

$x(0) = 0 \quad \dot{x}(0) = 0$

$x(t) = \frac{1}{4}t^2 + \frac{3}{8}\cos(2t) + c_1 t + c_2$

$c_1 = 0 \qquad c_2 = -\frac{3}{8}$

Ex. $\dot{x} = x - y + 2t - t^2 - t^3$
$\dot{y} = x + y - 4t^2 + t^3$

$x(0) = 1 \qquad y(0) = 0$

$x(t) = e^t \cos t + t^2$
$y(t) = e^t \sin t - t^3$

Ex. $y'' + 101y' + 100y = 0 \qquad y(0) = 1\ ,\ y'(0) = -1$

$$\to \begin{cases} y' = p \\ p' = -100y - 101p \end{cases} \qquad \begin{matrix} y(0) = 1 \\ p(0) = -1 \end{matrix}$$

$\longrightarrow y = e^{-x}$

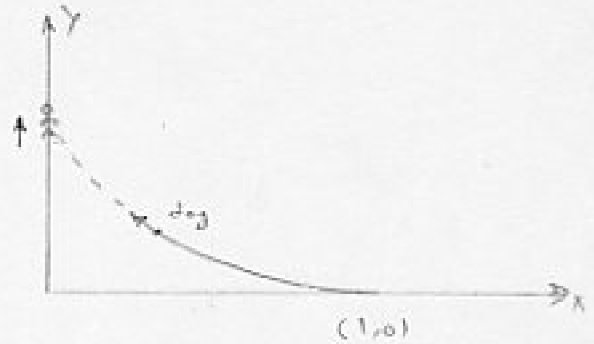
Ex. Dog's path (following his master moving along y-axis)

$x y'' = c\sqrt{1 + y'^2}$ c: ratio of the man's speed to the dog's

$$y = \frac{1}{2}\left(\frac{x^{1+c}}{1+c} - \frac{x^{1-c}}{1-c}\right) + \frac{c}{1-c^2} \qquad \text{for } 0 < c < 1$$

$$\begin{cases} y' = p \\ p' = \dfrac{c\sqrt{1+p^2}}{x} \end{cases}$$

$y(1) = 0$
$p(1) = 0$



Ex.

$$\begin{cases} r'' = \dfrac{9}{r^3} - \dfrac{2}{r^2} \\ \theta' = \dfrac{3}{r^2} \end{cases}$$

Newtons Orbit of a particle in an inverse square gravitational field

$r(0) = 3 \qquad \theta(0) = 0 \qquad r'(0) = 0$

$$r = \frac{9}{2 + G\theta}$$

$$\begin{cases} r' = p \\ p' = \dfrac{9}{r^3} - \dfrac{2}{r^2} \\ \theta' = \dfrac{3}{r^2} \end{cases}$$