Bisection Algorithm 2.1

Step 1 $\quad i = 1$

" 2 $\quad$ While $i \leq N_0$ Do steps 3–6 $\qquad$ N_0: Max. iteration No.

" 3 $\quad P = a + (b-a)/2$

" 4 $\quad$ If $f(P) = 0$ or, $(b-a)/2 <$ Tol. then

Output (P)

Stop

" 5 $\quad i = i+1$

" 6 $\quad$ If $f(a) f(P) > 0$ then $a = P$

else $b = P$

" 7 $\quad$ Output, 'Method failed after N_0 iterations'

Stop

Some other stopping procedures:

$$|P_n - P_{n-1}| < \epsilon$$

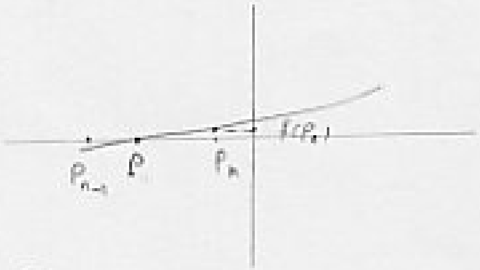
$$\frac{|P_n - P_{n-1}|}{|P_n|} < \epsilon \qquad P_n \neq 0$$

$$|f(P_n)| < \epsilon$$

Difficulties:

i) $|P_n - P_{n-1}| < \epsilon$ but $f(P_n)$ = large

ii) $f(P_n) \to \epsilon$ $|P_n - P_{n-1}|$ = large



Without additional knowledge about f or p
using the relative error for stopping criterion is best way.

Ex. $f(x) = x^3 + 4x^2 - 10$ has a root in $[1,2]$

$f(1) = -5$, $f(2) = 14$

n	a_n	b_n	P_n	$f(P_n)$
1	1.0	2.0	1.5	2.375
⋮				
9	1.36328125	1.3671875	1.365234375	0.000072
⋮				
13	1.364990235	1.365234375	1.365112305	-0.00194

Theorem 2.1

Let $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$
The Bisection method generates a sequence $\{P_n\}$ approximating p
with the property:

$$|P_n - P| \leqslant \frac{b-a}{2^n} \qquad n \geqslant 1$$

Acc. to Def. 1.17, this inequality implies that $\{P_n\}_{n=1}^{\infty}$ converges to P. and is bounded by a sequence that converges to Zero with $O(\frac{1}{2^n})$ rate of convergence.

In the last example:

$|P-P_9| \leqslant \frac{2-1}{2^9} \approx 2\times10^{-3}$ $\qquad |P-P_9| \simeq 4.4\times10^{-6}$

$|P-P_{13}| \leqslant \frac{2-1}{2^{13}} \simeq 1.22\times10^{-4}$ $\qquad |P-P_{13}| \simeq 1.18\times10^{-4}$

Bound of errors $\qquad$ actual errors (using their values)

Ex.

N=? Number of iterations to solve $f(x) = x^3+4x^2-10=0$ with the accuracy $\epsilon = 10^{-5}$, using $a_1=1$, $b_1=2$.

$|P_N - P| \leqslant \frac{b-a}{2^N} = \frac{2-1}{2^N} < 10^{-5}$

$\log_{10} 2^{-N} < \log_{10} 10^{-5} = -5$ $\qquad -N\log_{10} 2 < -5$

$N > \frac{5}{\log_{10} 2} \approx 16.6$ $\qquad N=17$

2.2 Fixed-Point Iteration

$$g(x) = x \qquad (\text{e.g. } x = \sqrt{x^3 - x + 1}\ )$$

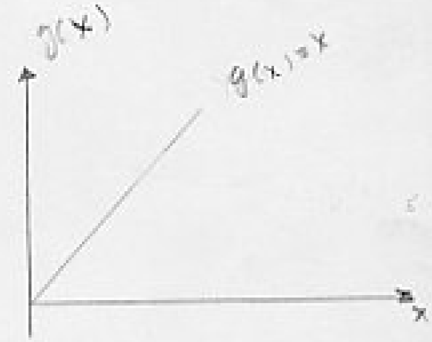
A sol. to such an equ. is said to be a fixed point of the func. g (Def.).

If a fixed point is found for any given g

$\longrightarrow$ every root-finding prob. could also be solved.

For example: $f(x) = 0$ has sols., that corresponds $\rightarrow$ to the fixed points of $g(x) = x$ when $g(x) = x - f(x)$.

Ex. a) $g(x) = x$, $0 \le x \le 1$, has a fixed point at each x in $[0,1]$.



remark $\begin{cases} Y = g(x) \\ Y = x \end{cases} \xrightarrow{\text{gives}}$ Fixed Point $\rightarrow \begin{cases} Y = x \\ Y = x \end{cases}$

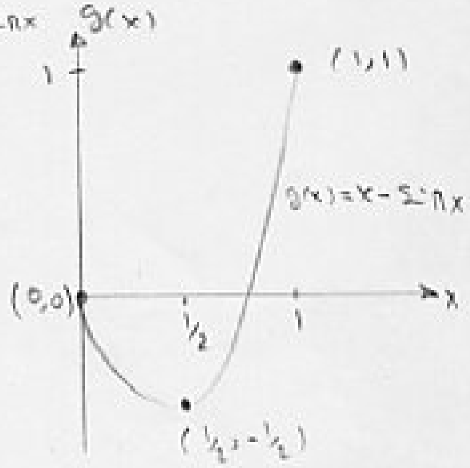
b) $g(x) = x - \sin x$ has exactly two fixed point in $[0,1]$ $\rightarrow \begin{cases} Y = x - \sin x \\ Y = x \end{cases}$

$X = 0$, $X = 1$

At $x = 0, x = 1$ $\quad g \rightarrow g = x$

i.e. $g(0) = 0$

$g(1) = 1$



Theorem 2.2

If $g \in C[a,b]$ and $g(x) \in [a,b]$ $\forall$ all $x \in [a,b]$

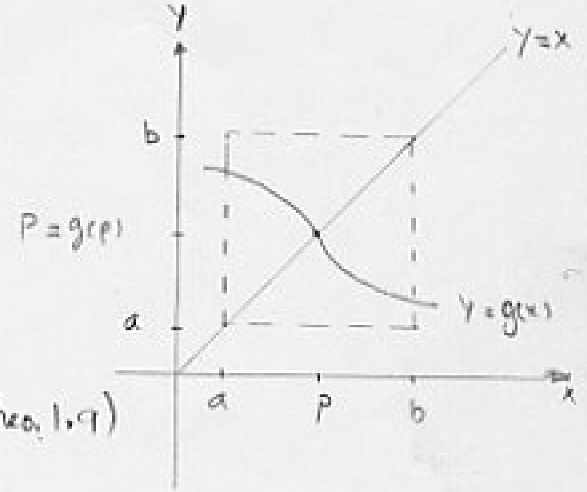
$\xrightarrow{\text{then}}$ $\exists$ a fixed point for g in $[a,b]$

Further, suppose $g'(x)$ exists on (a,b)

and a positive const. $K<1$ exists with

$$|g'(x)| \leq K < 1 \quad \forall \text{ all } x \in (a,b)$$

$\xrightarrow{\text{then}}$ g has a unique fixed point P in $[a,b]$



Ex.

Let $g(x) = (x^2-1)/3$ on $[-1,1]$

Abs Min of g at $x=0 \rightarrow g(0) = -\frac{1}{3}$ (Theo. 1.9)

Abs Max " " " $x = \pm 1 \rightarrow g(\pm 1) = 0$

Moreover g: continuous and $|g'(x)| = \left|\frac{2x}{3}\right| \leq \frac{2}{3}$ $\forall$ $x \in [-1,1]$

So g satisfies the hypotheses of Theorem 2.2 and has a unique fixed point in $[-1,1]$.

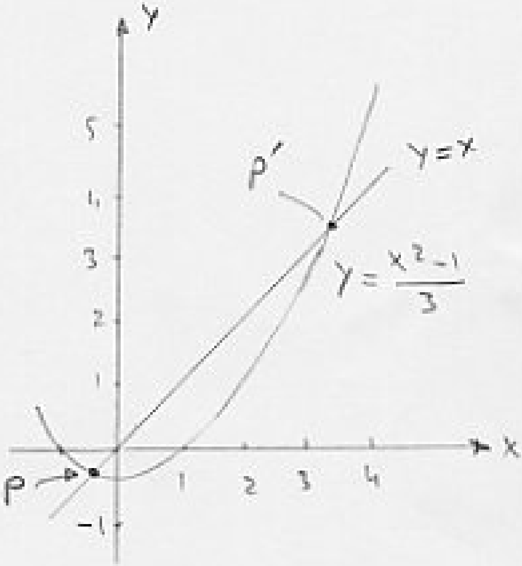
Fixed Point = ?

$P = g(P) = \frac{P^2-1}{3} \rightarrow P^2 - 3P - 1 = 0 \qquad P = \frac{3-\sqrt{13}}{2}$

Note that: $p' = \frac{3+\sqrt{13}}{2}$

for interval $[3,4]$

$\begin{cases} g(4) = 5 \notin [3,4] \\ g'(4) = \frac{8}{3} > 1 \end{cases}$



$\longrightarrow$ g does not satisfy the hypotheses of Theorem 2.2

This shows $\rightarrow$ The hypotheses of Theorem 2.2 are sufficient to gaurantee a unique fixed point but are not necessary.

Ex. $g(x) = 3^{-x}$

Since $g'(x) = -3^{-x}\ln 3 < 0$ on $[0,1]$ $\longrightarrow$ g is decreasing on $[0,1]$

Hence $\rightarrow$ $g(1) = \frac{1}{3} \leq g(x) \leq 1 = g(0)$ for $0 \leq x \leq 1$

Thus $\rightarrow$ $\forall x \in [0,1]$, $g(x) \in [0,1]$

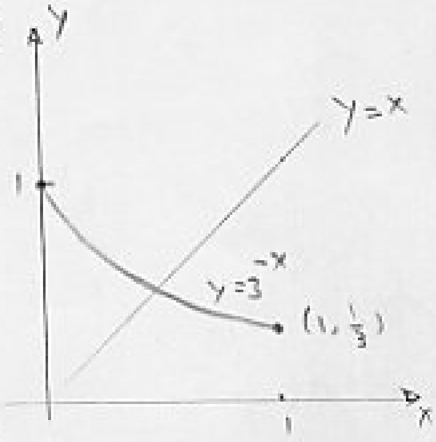
Therefore $\rightarrow$ g has a fixed point in $[0,1]$

But;

Since, $g'(0) = -\ln 3 = -1.098612289$

$g'(0.05) = -3^{-0.05}\ln 3 = -1.0398924$

$|g'(x)| \not\leq 1$ on $[0,1]$



and $\rightarrow$ Theorem 2.2 cannot be used to determine uniqueness.

However g is decreasing $\longrightarrow$ fixed point must be unique.

Approximation of the fixed-point of a func. g:

1) choose P_0 : initial value

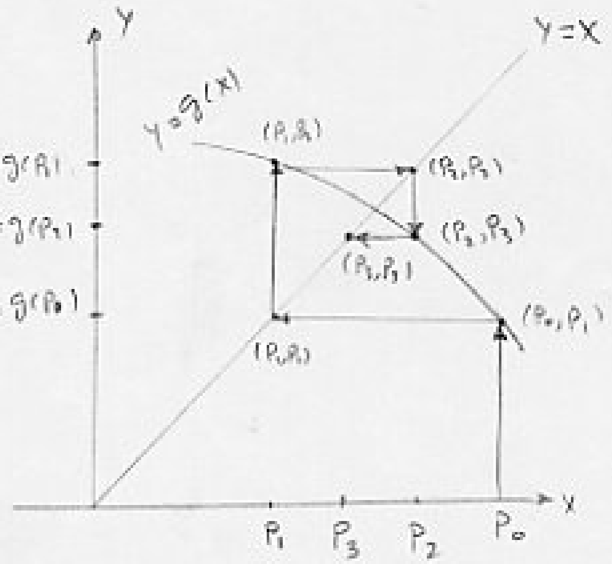
2) Generate $\{P_n\}_{n=0}^{\infty}$ by letting $P_n = g(P_{n-1})$ $n \geq 1$

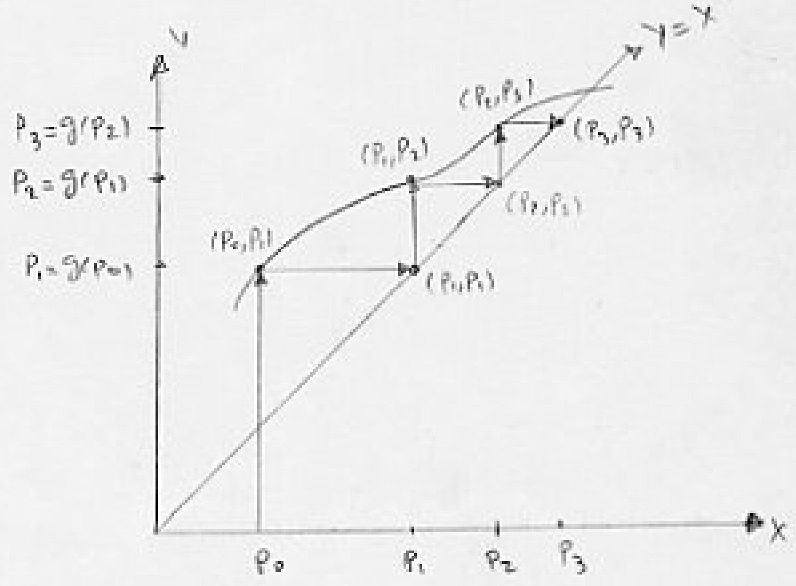
3) If $\begin{cases} 1\text{- Sequence} \xrightarrow{\text{converges to}} p \\ 2\text{- } g \text{ continuous} \end{cases}$

by $\xrightarrow{\text{Theo } 1.4}$ $p = \lim_{n \to \infty} P_n = \lim_{n \to \infty} g(P_{n-1}) = g\left(\lim_{n \to \infty} P_{n-1}\right) = g(p)$

and a sol. to $x = g(x)$ is obtained.

This technique is called fixed-point or functional iteration.





Fixed Point Algorithm: 2.2

S1 $i = 1$

S2 while $i \leq N_0$ do steps 3-6 — N_0: number of iterations

S3 $P = g(P_0)$

S4 If $|P - P_0| < Tol$, then

Output P

stop

S5 $i = i + 1$

S6 $P_0 = P$

S7 Output 'Method failed after N_0 iteration'

stop

Ex. The equ $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$.

There are many ways to change the equ $\xrightarrow{\text{to the form}}$ $x = g(x)$

a) $g(x) = x$ $\quad g(x) = x - f(x)$ $\quad (f(x) = 0)$

$x = g_1(x) = x - x^3 - 4x^2 + 10$

b) $x^3 + 4x^2 - 10 = 0 \longrightarrow x^2 = \frac{10}{x} - 4x$

$x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{\frac{1}{2}}$

c) $x^3+4x^2-10=0 \longrightarrow x^2=\frac{1}{4}(10-x^3)$

$x=\pm\frac{1}{2}(10-x^3)^{\frac{1}{2}}$

$x=g_3(x)=\frac{1}{2}(10-x^3)^{\frac{1}{2}}$ (Positive sol.)

d) $x^3+4x^2-10=0 \longrightarrow x^2=\frac{10}{4+x}$

$x=g_4(x)=\left(\frac{10}{4+x}\right)^{\frac{1}{2}}$

e) $x=g_5(x)=x-\frac{x^3-4x^2-10}{\underbrace{3x^2+8x}_{f'}}$ (because $f'\neq 0$)

with $P_0=1.5$

n	a	b	c	d	e
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165			
2	6.732	2.9969			
3	-469.7	$(-8.65)^{1/2}$			
4	1.03×10^8				
5					1.365230013
15				1.365230013	
30			1.365230013		

So, we need a criterion for g:

1- to converge to a sol. of $x=g(x)$

2- to converge as rapid as possible

Theorem 2.3

Let $g \in C[a,b]$ and suppose $g(x) \in [a,b]$ $\forall x [a,b]$.

Further, suppose g' exists on (a,b)

with $|g'(x)| \leq K < 1$ $\forall x \in (a,b)$

If P_0 is any number in $[a,b]$, the sequence defined by

$$P_n = g(P_{n-1}) \quad n \geq 1$$

converges to unique point $P \in [a,b]$

(difference with Theo. 2.2)

Corollary 2.4

If g satisfies the hypothesis of Theo. 2.3, a bound for the error involved in using P_n to approximate P is given by:

$$|P_n - P| \leq K^n \cdot \text{Max}\{P_0 - a, b - P_0\}$$

Corollary 2.5

If g satisfies the hypothesis of Theo. 2.3, then

$$|P_n - P| \leq \frac{K^n}{1-K} |P_0 - P_1| \qquad n \geq 1$$

Result:

Both corollaries show:

smaller K $\longrightarrow$ faster convergence.

Ex.

a) $g_1(x) = x - x^3 - 4x^2 + 10$ $\rightarrow$ $g_1'(x) = 1 - 3x^2 - 8x$

$|g_1'(x)| \not< 1$ for no interval $[a,b]$ containing p

Although Theo. 2.3 does not guarantee that the method must fail for this choice of g, there is no reason to expect convergence.

b) $g_2(x) = \left(\frac{10}{x} - 4x\right)^{\frac{1}{2}}$

g_2 does not map $[1,2] \xrightarrow{\text{into}} [1,2]$

$\longrightarrow$ $\{p_n\}_{n=0}^{\infty}$ is not defined with $p_0 = 1.5$ ($p_1 = 0.816\ldots$ out of range)

Moreover; $|g_2'(x)| \not< 1$ for no interval

(since $|g_2'(p)| \approx 3.4$)

$\underset{1.5}{\downarrow}$ ≈ 5.17

c) $g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

$g_3'(x) = -\frac{3}{4}x^2(10-x^3)^{-1/2} < 0$ on $[1,2]$

$\longrightarrow$ g decreasing on $[1,2]$

$|g'(1.99)| \approx 2.0404441 \not< 1$

However $|g_3'(2)| \approx 2.12 \not< 1$ on $[1,2]$

Now if we consider: $[1, 1.5]$;

still $g_3'(x) < 0$ on $[1, 1.5]$

but $1 < 1.28 \approx g_3(1.5) \leq g_3(x) \leq g_3(1) = 1.5$

$\forall \in x[1, 1.5]$

Then

g_3 maps $[1, 1.5] \xrightarrow{\text{into}} [1, 1.5]$

Since also $|g_3'(x)| \leq |g_3'(1.5)| \approx 0.66$ on $[1, 1.5]$

$\longrightarrow$ Theo. 2.3 confirms the convergence.

d) $g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$

$|g_4'(x)| = \left| \frac{-5}{\sqrt{10}(4+x)^{3/2}} \right| \leq \frac{5}{\sqrt{10}(5)^{3/2}} < 0.15 \quad \forall x \in [1,2]$

$\longrightarrow$ Since the bound on $|g_4'(x)|$ is much smaller than the bound on $|g_3'(x)|$ $\longrightarrow$ g_4 gives more rapid convergence.

2.3) The Newton-Raphson Method:

One of the most <u>powerful</u> methods for solving $f(x)=0$

Suppose f is <u>Twice</u> continuously differentiable on the interval $[a,b]$ (i.e. $f \in C^2[a,b]$).

Let $\bar{x} \in [a,b]$: an approx. to p such that $f'(\bar{x}) \neq 0$

and $|\bar{x}-p| =$ small

<u>First</u> Taylor polynomial expansion about $\bar{x}$:

$$f(x) = f(\bar{x}) + (x-\bar{x}) f'(\bar{x}) + \frac{(x-\bar{x})^2}{2} f''(\xi(x))$$

where $\xi(x)$: between x and $\bar{x}$

Since $f(P)=0$

$$\rightarrow \quad 0 = f(\bar{x}) + (P-\bar{x}) f'(\bar{x}) + \frac{(P-\bar{x})^2}{2} f''(\xi(P))$$

Newton's method is derived by assuming that $|P-\bar{x}| =$ small

$$\longrightarrow (P-\bar{x})^2 \approx 0$$

$$\longrightarrow \quad 0 \simeq f(\bar{x}) + (P-\bar{x}) f'(\bar{x})$$

$$\longrightarrow \quad P \simeq \bar{x} - \frac{f(\bar{x})}{f'(\bar{x})}$$

Remark:

$y_0 = a x_0 + b \quad \rightarrow b = y_0 - a x_0$

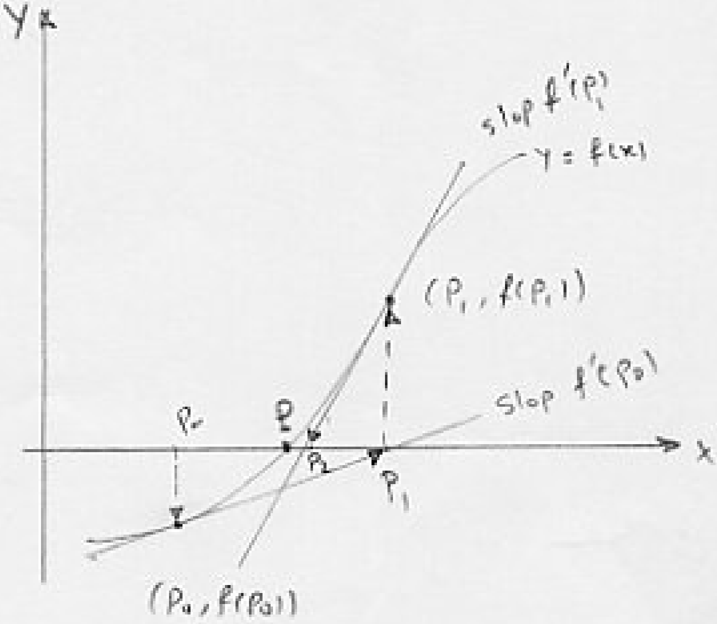
$y = ax+b \quad y = ax + (y_0 - a x_0)$

$y = 0 \rightarrow x_1 = x_0 - \frac{y_0}{a}$

(a: Known $= f'(x_0)$)

This sets the stage for the Newton method, which starts with initial approximation P_0 and generates the sequence $\{P_n\}$, defined by

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})} \qquad n \geq 1$$



Newton-Raphson Algorithm 2.3

S1 $i = 1$

S2 While $i \leq N_0$ do steps 3-6

S3 $P = P_0 - \dfrac{f(P_0)}{f'(P_0)}$

S4 If $|P - P_0| < Tol$, then

Output (P)
stop

S5 $i = i + 1$

S6 $P_0 = P$

S7 Output 'Method failed after N_0 iterations'
stop

Newton's method is a functional iteration techinque of the form $P_n = g(P_{n-1})$

$$g(P_{n-1}) = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})} \qquad n \geq 1$$

Newton's method **cannot** be continued if:

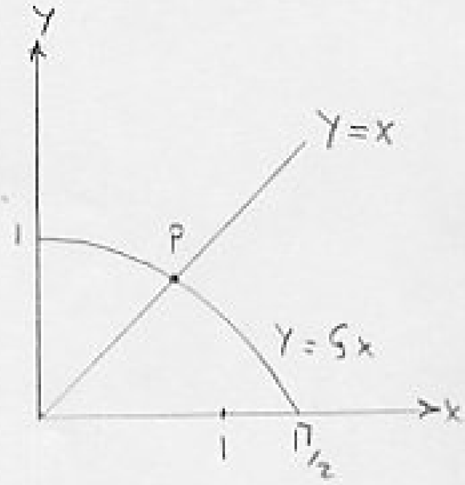
$$f'(P_{n-1}) = 0 \quad \text{for some } n$$

Ex. $\quad x = \cos x$

Let, $f(x) = \cos x - x \;\longrightarrow\; f(\pi/2) = -\frac{\pi}{2} < 0 < 1 = f(0)$

Theo. 1.12 $\longrightarrow$ there exists a zero of f in $[0, \pi/2]$

The intersection of $y = x$ and $y = \cos x$ is fixed point of $g(x) = \cos x$



P: fixed point

From the graph $\longrightarrow$ $f(x) = 0$ has a unique sol. in $[0, \pi/2]$

Newton's method has the form:

$$P_n = P_{n-1} - \frac{\cos(P_{n-1}) - P_{n-1}}{-\sin(P_{n-1}) - 1} \qquad n \geq 1$$

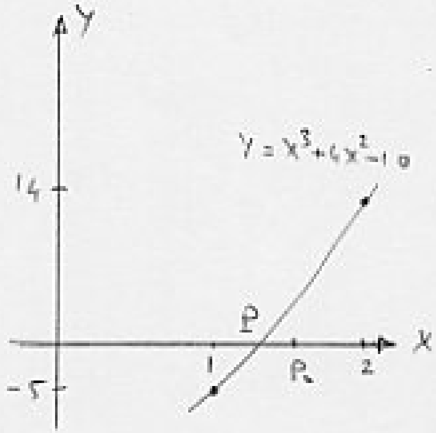
We have to choose a good initial point P_0

From the graph $\longrightarrow$ $P_0 = \pi/4$

n	P_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

Ex.

To obtain the unique sol. to $x^3+4x^2-10=0$ on the interval $[1,2]$, by Newton's method, generate the sequence $\{P_n\}_{n=1}^{\infty}$ by



$$P_n = P_{n-1} - \frac{(P_{n-1})^3+4(P_{n-1})^2-10}{3(P_{n-1})^2+8(P_{n-1})} \quad n \geq 1$$

From the Fig. $\longrightarrow P_0 = 1.5$

n	P_n
0	1.5
1	1.373333333
2	1.365262015
3	1.365230014
4	1.365230013

In Newton's method the initial point p_0 must be close to the actual root (remember $(P-\bar{x}) \approx 0$ assumption).
If it is not, the method may not converge to the root (but this is not always the case)

Theo. 2.6 (Convergence theo. for Newton's method)
Let $f \in C^2[a,b]$
If $P \in [a,b]$ such that $f(P)=0$ and $f'(P) \neq 0$

$\longrightarrow \exists\ \delta > 0$ such that Newton's method generates

a sequence $\{P_n\}_{n=1}^{\infty} \xrightarrow{\text{converging to}} P$

$\forall$ any initial $P_0 \in [P-\delta, P+\delta]$

Secant Method:

Sometimes it is tiresome to evaluate the $f'(x)$.

To circumvent the problem;

$$f'(P_{n-1}) = \lim_{x \to P_{n-1}} \frac{f(x) - f(P_{n-1})}{x - P_{n-1}}$$

letting $x = P_{n-2} \longrightarrow f'(P_{n-1}) \simeq \dfrac{f(P_{n-2}) - f(P_{n-1})}{P_{n-2} - P_{n-1}}$

$$\longrightarrow f'(P_{n-1}) \simeq \frac{f(P_{n-1}) - f(P_{n-2})}{P_{n-1} - P_{n-2}}$$

Using this approx. in Newton's method:

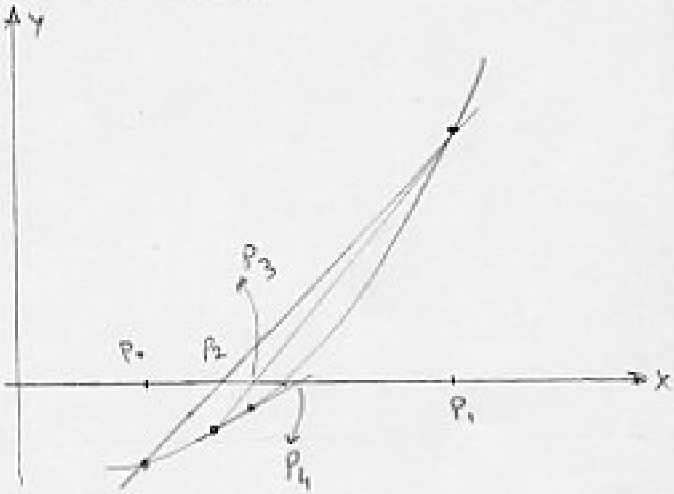
$$P_n = P_{n-1} - \frac{f(P_{n-1})(P_{n-1} - P_{n-2})}{f(P_{n-1}) - f(P_{n-2})} \qquad n \geqslant 2$$

Intersection of P_0P_1 with $y=0 \longrightarrow P_2$

" " P_1P_2 " " $\longrightarrow P_3$

" " P_2P_3 " " $\longrightarrow P_4$

- ---



Remark:

$\begin{cases} y = ax+b \\ \begin{cases} (x_0, y_0) \\ (x_1, y_1) \end{cases} \text{given points} \end{cases} \Longrightarrow \begin{cases} a = \dfrac{y_1 - y_0}{x_1 - x_0} \\ b = y_0 - \dfrac{y_1 - y_0}{x_1 - x_0} x_0 \end{cases}$

$y = \dfrac{y_1 - y_0}{x_1 - x_0} x + \left(y_0 - \dfrac{y_1 - y_0}{x_1 - x_0}\right) x_0$

$y = 0 \longrightarrow x = x_0 - \dfrac{y_0(x_1 - x_0)}{y_1 - y_0}$

Secant Algorithm 2.4

S1 $i=2$

$q_0 = f(P_0)$

$q_1 = f(P_1)$

S2 While $i \leq N_0$ do Steps 3-6

S3 $P = P_1 - \dfrac{q_1(P_1 - P_0)}{q_1 - q_0}$

S4 If $|P - P_1| \leq Tol$, then

Output (P)

stop

S5 $i = i+1$

S6 $P_0 = P_1$, $q_0 = q_1$

$P_1 = P$, $q_1 = f(P)$

S7 Output 'Method failed after N_0 iteration'

stop

Ex. Find a zero of $f(x) = \cos x - x$, using the Secant method.

we choose $P_0 = 0.5$ $P_1 = \frac{\pi}{4}$

$$P_n = P_{n-1} - \frac{(P_{n-1} - P_{n-2})(\cos(P_{n-1}) - P_{n-1})}{(\cos(P_{n-1}) - P_{n-1}) - (\cos(P_{n-2}) - P_{n-2})} \quad n \geq 2$$

n	P_n
0	0.5
1	0.7853981635
2	0.7363841390
3	0.7390581394
4	0.7390851492
5	0.7390851334

Note: The secant method is often used to refine an answer obtained by another technique.

2.4 Error Analysis for Iterative Methods:

Def 2.7

Suppose $\{P_n\}_{n=0}^{\infty}$ is a sequence $\xrightarrow{\text{Converging to}}$ P

and $e_n = P_n - P \quad \forall\, n \geq 0$

If $\lambda > 0$ and $\alpha > 0$ (const.) exisit with

$$\lim_{n\to\infty} \frac{|P_{n+1} - P|}{|P_n - P|^{\alpha}} = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^{\alpha}} = \lambda$$

then $\{P_n\}_{n=0}^{\infty}$ is said to converge to P of order α, with

asymptotic error const. λ.

An iterative technique of the form $P_n = g(P_{n-1})$ is said to be of order α if the sequence $\{P_n\}$ $\xrightarrow{\text{Converges to}}$ $P = g(P)$ sol., of order α

In general;

A sequence with high order of convergence $\xrightarrow{\text{Coverges}}$ more rapidly than a sequence with lower order.

1. If $\alpha = 1$, the method is called linear.
2. " $\alpha = 2$ " " " " quadratic.

Theo. 2.8

Let $g \in C[a,b]$ and suppose that: $g(x) \in [a,b]$ $\forall x \in [a,b]$.

Further, suppose g': continuous on (a,b) with:

$$|g'(x)| \leq K < 1 \qquad \forall x \in (a,b)$$

If $g'(p) \neq 0$ $\xrightarrow{\text{then}}$ $\forall$ p_0 in $[a,b]$

the sequence $p_n = g(p_{n-1})$ $n \geq 1$ $\xrightarrow{\text{converges}}$ only linearly

to the unique fixed point p in $[a,b]$

This Theo implies that:

Higher order convergence for fixed point methods can occur only when $g'(p) = 0$.

Ex

Suppose two convergent iterative schemes are described by:

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|} = 0.75 \qquad \text{linear}$$

$$\lim_{n \to \infty} \frac{|\tilde{e}_{n+1}|}{|\tilde{e}_n|^2} = 0.75 \qquad \text{quadratic}$$

suppose also for simplicity;

$$\frac{|e_{n+1}|}{|e_n|} \approx 0.75 \quad \text{and} \quad \frac{|\tilde{e}_{n+1}|}{|\tilde{e}_n|^2} \approx 0.75$$

For the first one;

$$|e_n| \approx 0.75\,|e_{n-1}| \approx (0.75)^2 |e_{n-2}| \approx \cdots \approx (0.75)^n |e_0|$$

while for the second one;

$$|\tilde{e}_n| \approx 0.75\,|\tilde{e}_{n-1}|^2 \approx (0.75)\left[(0.75)|\tilde{e}_{n-2}|^2\right]^2 \approx (0.75)^3\left[(0.75)|\tilde{e}_{n-3}|^2\right]^4$$

$$\approx \cdots \approx (0.75)^{2^n-1}\,|\tilde{e}_0|^{2^n}$$

To compare the speed of convergence, assume $|e_0| = |\tilde{e}_0| = 0.5$

$n_{min} = ?$ to obtain an error $\leqslant 10^{-8}$

$$|e_n| = (0.75)^n (0.5) \leqslant 10^{-8} \longrightarrow n \geqslant \frac{\log_{10} 2 - 8}{\log_{10} 0.75} \approx \underline{62}$$

$$|\tilde{e}_n| = (0.75)^{2^n-1}(0.5)^{2^n}$$

$$|\tilde{e}_n| = (0.75)^{-1}(0.375)^{2^n} < 10^{-8}$$

$$2^n \geqslant \frac{\log_{10} 0.75 - 8}{\log_{10} 0.375} \approx 10 \qquad \underline{n \geqslant 5}$$

Theo. 2.9

Let p be a sol. of $x = g(x)$,

suppose $g'(p) = 0$ and g'': $\begin{cases} 1\text{- Continuous} \\ 2\text{- Strictly bounded by } M \text{ on} \end{cases}$

(e.g. $x = x^2$) an open interval I containing p.

$\xrightarrow{\text{then}}$ $\exists\ \delta > 0$ such that for $p_0 \in [p-\delta, p+\delta]$

The sequence $p_n = g(p_{n-1})$ $(n \geq 1)$ $\xrightarrow{\text{Converges}}$ at least quadratically to p.

Moreover, for sufficiently large n

$$|p_{n+1} - p| \leq \left(\frac{M}{2}\right) |p_n - p|^2$$

Solving $f(x) = 0$ (by the use of mentioned Theo.):

suppose p: a sol. of $f(x) = 0$

and $f'(p) \neq 0$

Consider the fixed point scheme

$$p_n = g(p_{n-1}) \quad n \geq 1$$

with g of the form;

$$g(x) = x - \Phi(x) f(x)$$

φ: arbitrary func. to be chosen later;

If $\varphi(x)$ is bounded $\xrightarrow{\text{Then}}$ $g(P) = P$ $\left(g(P) = P - \underset{\text{bounded}}{\varphi(P)} \underset{0}{f(P)} \right)$

and for the iterative procedure derived from g to be quadratically convergent $\xrightarrow{\text{it suffices}}$ $g'(P) = 0$

But $\quad g'(x) = 1 - \varphi'(x) f(x) - f'(x)\varphi(x)$

and $\quad g'(P) = 1 - f'(P)\varphi(P) \qquad\qquad (f(P)=0)$

$\xrightarrow{\text{consequently}}$ $g'(P) = 0$ if and only if $\varphi(P) = \dfrac{1}{f'(P)}$

In particular, quadratic convergence holds for the scheme

$$P_n = g(P_{n-1}) = P_{n-1} - \left(\frac{1}{f'(P)}\right) f(P_{n-1})$$

$$\left(g(x) = x - \varphi(x) f(x) \right) \text{ compare}$$

under suitable conds. on f

However, P and consequently $f'(P)$ are <u>unknown</u>.

A reasonable approach:

if $\varphi(x) = \dfrac{1}{f'(x)}$ $\xrightarrow{\text{gaurantees}}$ $\varphi(P) = \dfrac{1}{f'(P)}$

$$P_n = g(P_{n-1}) = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$$

This is just Newton's method.

Conclusion: Newton's method converges to p rapidly.

In the proceeding discussion we had the restriction of:

$f'(p) \neq 0$ where p: sol. of $f(x) = 0$

Difficulties might occur if $\begin{cases} f'(P_n) \to 0 \\ f(P_n) \to 0 \end{cases}$ simultaneously

(from the def. of Newton's method)

Newton's and secant methods generally give problems if $f'(p) = 0$ when $f(p) = 0$

Def 2.10

A sol. p of $f(x) = 0$ is said to be a zero of multiplicity m of f if $f(x)$ can be written as

$$f(x) = (x-p)^m q(x) \quad \forall x \neq p \quad \text{where } \lim_{x \to p} q(x) \neq 0$$

Theo. 2.11

$f \in C^1[a,b]$ has a simple zero at p in (a,b) if and only if $f(p) = 0$, but $f'(p) \neq 0$

Theo. 2.12

The func. $f \in C^m[a,b]$ has a zero of multiplicity <u>m</u> at p if and only if

$$0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p) \quad \text{but } f^{(m)}(p) \neq 0$$

Conclusion:

The result in Theo. 2.11 implies that an interval about p exists, such that <u>Newton's</u> method converges quadratically to p for any initial approximation, provided that p is a <u>simple</u> zero.

<u>Ex.</u>

This example shows that the quadratic convergence <u>need not</u> occur if the zero <u>is not</u> simple.

$f(x) = e^x - x - 1$ has a zero of multiplicity $m = 2$ at $p = 0$

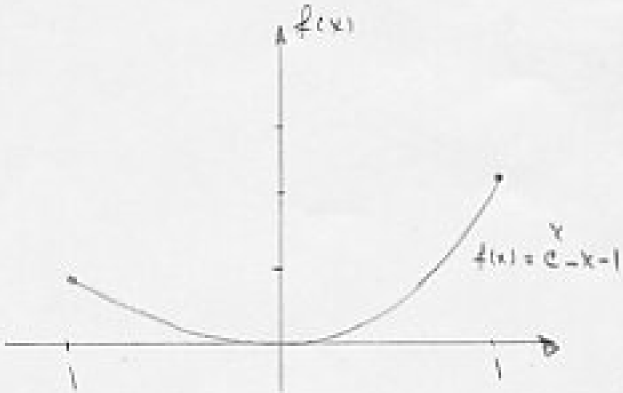
Reason:

$$f(0) = e^0 - 0 - 1 = 0 \;,\; f'(0) = e^0 - 1 = 0 \;\text{ but }\; f''(0) = e^0 = 1$$

In fact $f(x)$ can be expressed:

$$f(x) = (x-0)^2 \, \frac{e^x - x - 1}{x^2}$$

by Hôpital's rule; $\lim\limits_{x \to 0} \dfrac{e^x - x - 1}{x^2} = \lim\limits_{x \to 0} \dfrac{e^x - 1}{2x} = \lim \dfrac{e^x}{2} = \dfrac{1}{2} \neq 0$



n	p_n
0	1.
1	0.58198
2	0.31906
3	0.16800
⋮	
15	4.2610×10^{-5}
16	1.9142×10^{-5}

Not quadratically converges to zero.

Remedy:

One method of handling the prob. of multiple root is to define a func. μ by:

$$\mu(x) = \frac{f(x)}{f'(x)}$$

If p is a root of multiplicity $m \geq 1$ and $f(x) = (x-p)^m q(x)$ then

$$\mu(x) = \frac{(x-p)^m q(x)}{m(x-p)^{m-1} q(x) + (x-p)^m q'(x)}$$

$$\mu(x) = \frac{(x-p)\,q(x)}{m\,q(x) + (x-p)\,q'(x)}$$

has a root at p, but of multiplicity one.

Newton's method can then be applied to the func. $\mu(x)$, to give:

$$g(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{f(x)/f'(x)}{\{[f'(x)]^2 - [f(x)][f''(x)]\}/[f'(x)]^2}$$

$$g(x) = x - \frac{f(x)\,f'(x)}{[f'(x)]^2 - [f(x)][f''(x)]}$$ (modified Newton's method)

If $g(x)$ has the required continuity conds.;

functional iteration applied to g, will be quadratically convergent, regardless of the multiplicity of the root of f.

Note: The denominator in $g(x)$ may cause round off errors (difference of two small numbers).

Ex.

$g(x)$ given above is used to find the double root at $x=0$ of $f(x) = e^x - x - 1$ with 10-digit precesion. ($P_0 = 1$)

n	P_n
1	-2.3421061×10^{-1}
2	-8.4582788×10^{-3}
⋮	
5	-2.8085217×10^{-7}

No improvement to the root approx. P_5 will occur, in the subsequent computation since both numerator and denomerator approach zero.

Ex

We found before the root of $f(x)=x^3+4x^2-10=0$, to be $P=1.36523001$ (multiplicity one).

(i) $$P_n = P_{n-1} - \frac{(P_{n-1})^3+4(P_{n-1})^2-10}{3(P_{n-1})^2+8(P_{n-1})}$$ Newton's method

(ii) $$P_n = P_{n-1} - \frac{(P_{n-1}^3+4P_{n-1}^2-10)(3P_{n-1}^2+8P_{n-1})}{(3P_{n-1}^2+8P_{n-1})^2-(P_{n-1}^3+4P_{n-1}^2-10)(6P_{n-1}+8)}$$

Modified Newton's method.

P_n	(i)	(ii)
P_1	1.37333333	1.35689898
P_2	1.36526201	1.36519585
P_3	1.36523001	1.36523001

$P_0=1.5$

Both have the same convergence.

2.5 Accelerating Convergence:

Aitken's Δ^2-method:

This method is used to accelarate the convergence of a sequence that is linearly convergent, regardless of its origin.

Suppose $\{P_n\}_{n=0}^{\infty}$ is linearly convergent sequence with limit P,

and $\lim \frac{|P_{n+1}-P|}{|P_n-P|} = \lambda < 1$ λ: asymptotic error const.

Now we form $\{\hat{P}_n\}$ that converges more rapidly to P than does $\{P_n\}$.

Assume the sign of P_n-P, $P_{n+1}-P$ and $P_{n+2}-P$ agree and n is sufficiently large that;

$$\frac{P_{n+1}-P}{P_n-P} \simeq \lambda \simeq \frac{P_{n+2}-P}{P_{n+1}-P}$$

$$\longrightarrow (P_{n+1}-P)^2 \simeq (P_{n+2}-P)(P_n-P)$$

So $P_{n+1}^2 - 2P_{n+1}P + P^2 \simeq P_{n+2}P_n - (P_n+P_{n+2})P + P^2$

and $(P_{n+2}+P_n-2P_{n+1})P \simeq P_{n+2}P_n - P_{n+1}^2$

$$\longrightarrow P \simeq \frac{P_{n+2}P_n - P_{n+1}^2}{P_{n+2}-2P_{n+1}+P_n}$$

$$P = \frac{P_n^2 + P_n P_{n+2} + 2P_n P_{n+1} - 2P_n P_{n+1} - P_n^2 - P_{n+1}^2}{P_{n+2} - 2P_{n+1} + P_n}$$

$$= \frac{(P_n^2 + P_n P_{n+2} - 2P_n P_{n+1}) - (P_n^2 - 2P_n P_{n+1} + P_{n+1}^2)}{P_{n+2} - 2P_{n+1} + P_n}$$

$$= P_n - \frac{(P_{n+1} - P_n)^2}{P_{n+2} - 2P_{n+1} + P_n}$$

Aitken's Δ^2 method is based on the assumption that the sequence $\{\hat{P}_n\}_{n=0}^{\infty}$ defined by

$$\hat{P}_n = P_n - \frac{(P_{n+1} - P_n)^2}{P_{n+2} - 2P_{n+1} + P_n}$$

converges more rapidly than does the original sequence $\{P_n\}_{n=0}^{\infty}$.

Ex.

The sequence $\{P_n\}_{n=1}^{\infty}$, where $P_n = \cos(\frac{1}{n})$, converges linearly to $P=1$. The first few terms of $\{P_n\}_{n=1}^{\infty}$ and $\{\hat{P}_n\}_{n=1}^{\infty}$ are given below;

n	P_n	$\hat{P}_n$
1	0.54030	0.96178
2	0.87758	0.98213
3	0.94496	0.98979
4	0.96891	0.99342
5	0.98007	0.99541
6	0.98614	
7	0.98981	

$P_n = \cos(\frac{1}{n})$

$$\hat{P}_n = \cos(\tfrac{1}{n}) - \frac{[\cos(\frac{1}{n+1}) - \cos(\frac{1}{n})]^2}{\cos(\frac{1}{n+2}) - 2\cos(\frac{1}{n+1}) - \cos(\frac{1}{n})}$$

Def. 2.13

Given the sequ. $\{P_n\}_{n=0}^{\infty}$, define the following difference ΔP_n by

$$\Delta P_n = P_{n+1} - P_n \qquad n \geq 0$$

Higher powers $\Delta^k P_n$ are defined recursively by

$$\Delta^k P_n = \Delta^{k-1}(\Delta P_n) \qquad k \geq 2$$

Now;

$$\Delta^2 P_n = \Delta(P_{n+1} - P_n) = \Delta P_{n+1} + \Delta P_n = (P_{n+2} - P_{n+1}) - (P_{n+1} - P_n)$$

$$\Delta^2 P_n = P_{n+2} - 2P_{n+1} + P_n$$

$$\longrightarrow \quad \hat{P}_n = P_n - \frac{(\Delta P_n)^2}{\Delta^2 P_n} \qquad n \geq 0$$

This is the origin of Δ^2 notation.

Theo. 2.14

Let $\{P_n\}$ be a sequ. that converges <u>linearly</u> to the limit P with asymptotic const. $\lambda < 1$ and $P_n - P \neq 0$ $\forall$ $n \geq 0$. Then the sequ. $\{\hat{P}_n\}_{n=0}^{\infty}$ converges to P <u>faster</u> than $\{P_n\}_{n=0}^{\infty}$ in the sence that

$$\lim_{n \to \infty} \frac{\hat{P}_n - P}{P_n - P} = 0$$

- Steffensen's Method:

By applying a modified Aitken's Δ^2 method to a linearly convergent sequ. obtained from fixed-point iteration, we can accelerate the convergence to quadratic.

$$P_0, \quad P_1 = g(P_0), \quad P_2 = g(P_1), \quad \hat{P}_0 = \text{Aitken-}\Delta^2(P_0, P_1, P_2)$$

$$P_1' = g(\hat{P}_0), \quad P_2' = g(P_1') \quad \hat{P}_0' = \text{Aitken-}\Delta^2(\hat{P}_0, P_1', P_2')$$

$$P_1'' = g(\hat{P}_0'), \ \ldots$$

Note: Steffensen's method is slightly different from applying Aitken's Δ^2 method directly to linearly convergent fixed point iteration sequ.

Steffensen's Algorithm 2.5 (sol. of $P = g(P)$)

Input P_0

S1 $i = 1$

S2 while $i \leq N_0$ do Steps 3-6

S3 $P_1 = g(P_0)$

$P_2 = g(P_1)$

$P = P_0 - (P_1 - P_0)^2 / (P_2 - 2P_1 + P_0)$

S4 If $|P - P_0| < Tol.$ then

output (P)

stop

S5 $i = i+1$

S6 $P_0 = P$

S7 Output 'Method failed after N_0 iterations'

Note: If $\Delta^2 P_n = 0$ then we choose P_2 in the step $n-1$ as the answer.

Ex.

Solve $x^3 + 4x^2 - 10 = 0$ by Steffensen's method:

$$g(x) = \left(\frac{10}{x+4} \right)^{1/2}$$

Then $g(x) = x$ implies $x^3 + 4x^2 - 10 = 0$

This method gives the same convergence as Newton's method (quadratic)

K	$P_0^{(k)}$	$P_1^{(k)}$	$P_2^{(k)}$
0	1.5	1.348399725	1.367376372
1	1.365265224	1.365225534	1.365230583
2	1.365230013		

Theo. 2.15

Suppose that $x = g(x)$ has a sol. P with $g'(x) \neq 1$.

If there exists a $\delta > 0$ such that $g \in C^3[P-\delta, P+\delta]$, then Stephensen's method gives quadratic convergence for any $P_0 \in [P-\delta, P+\delta]$.

2.6 Zeros of Polynomials and Müller's Methods

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + a_0$ Polynomial of degree n
$a_n \neq 0$

The Zero func. $P(x) = 0$ for all values of x, is considered a Plynomial but is assigned no degree.

Theo 2.16 (Fundamental Theo. of Algebra)

If P is a polynomial of deg. $n \geq 1$, $\xrightarrow{\text{then}}$ $P(x) = 0$ has at least one (possibly complex) root.

Corollary 2.17

If $P(x) = a_n x^n + \dots a_1 x + a_0$ is a polynomial of deg. $n \geq 1$,
$\xrightarrow{\text{then}}$ $\exists$ unique consts. $x_1, x_2 \dots x_k$, possibly complex
and positive integers $m_1, m_2 \dots m_k$

such that $\sum_{i=1}^{k} m_i = n$

and $P(x) = a_n (x - x_1)^{m_1} (x - x_2)^{m_2} \dots (x - x_k)^{m_k}$

This corollary states that:

i) The zeros of a polynomial are unique and
ii) if each zero x_i is counted as many times as its multiplicity m_i, a polynomial of deg. n has exactly n zeros.

Corollary 2.18

Let P and Q be polynomials of deg. at most n. If $x_1, x_2 \ldots x_k$, $k > n$, are distinct numbers with $P(x_i) = Q(x_i)$ for $i = 1, 2 \ldots k$ $\xrightarrow{\text{Then}}$ $P(x) = Q(x)$ $\forall x$.

Theo 2.19 (Horner's Method)

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_0$ and $b_n = a_n$

If $b_k = a_k + b_{k+1} x_0$ for $k = n-1, n-2, \ldots 1, 0$

$\xrightarrow{\text{then}}$ $b_0 = P(x_0)$

Moreover, if $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \ldots b_2 x + b_1$

$\xrightarrow{\text{then}}$ $P(x) = (x - x_0) Q(x) + b_0$

Ex. Evaluate $P(x) = 2x^4 - 3x^2 + 3x - 4$ at $x_0 = -2$ using Horner's method.

$b_n = a_n \rightarrow b_4 = 2$ $\quad b_k = a_k + b_{k+1} x_0$

$b_3 = 0 + 2(-2) = -4$ $\quad b_2 = -3 + (-4)(-2) = 5$

$b_1 = 3 + (5)(-2) = -7$ $\quad b_0 = -4 + (-7)(-2) = 10$

Also since $P(x) = (x - x_0) Q(x) + b_0 \rightarrow b_0 = P(-2)$

$P(x) = (x+2)(2x^3 - 4x^2 + 5x - 7) + 10$

An additional advantage of using Horner's (or synthetic-division) procedure is that, since,

$$P(x) = (x - x_0)\,Q(x) + b_0$$

where $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots\ b_2 x + b_1$

$$\rightarrow\ P'(x) = Q(x) + (x - x_0)\,Q'(x) \ \rightarrow\ P'(x_0) = Q(x_0)$$

Therefore, in Newton-Raphson method, since <u>both</u> $P(x)$ and $P'(x)$ are evaluvated in each step, Horner's method is useful since it evaluvates both $P(x)$ and $P'(x)$ in the same manner.

Note: We use Horner's method to evaluvate a polynomial instead of directly putting x_0 in the polynomial because of computational efficiency (discussed in Setion 1.2).

Horner's Algorithm; 2.6 (to evaluvate $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$) derivative at x_0

Input $x_0, a_1, a_2 \cdots a_n$

output $PX0 = P(x_0)$, $PPX0 = P'(x_0)$

Remark: $b(k)$: the coeffs. of $Q_1(x)$
$bP(k)$: the coeffs of $Q_2(x)$
$P(x) = (x - x_0)\,Q_1(x) + b_0$
$Q_1(x) = (x - x_0)\,Q_2(x) + bP_0$

S1 $\quad b(n) = a_n$, $\quad bP(n) = a_n$

S2 $\quad$ Do $k = n-1, 0$

$\quad b(k) = a_k + b(k+1) * x_0$

S3 $\quad bP(k) = b(k) + bP(k+1) * x_0$ (exclude $k = 0$)

S4 $\quad PX0 = b(0)$; $PPX0 = bP(0)$

output (PX0, PPX0)

Stop

Finding the other roots: (Deflation Procedure)

Suppose we have found one of the roots, say x_1 by Newton-Raphson method:

Since $P(x) = (x - x_0)\, Q(x) + b_0$ — $P(x)$: Polynomial of deg. n

here $x_0 = x_1 \longrightarrow b_0 \approx 0$

$P(x) = (x - x_1)\, Q(x)$ — $Q(x)$: Polynomial of deg $n-1$

let $Q(x) \equiv Q_1(x)$

$P(x) = (x - x_1)\, Q_1(x)$

If $P(x)$ has n real root, we can apply the method repeatedly and find $n-2$ approximate zeros, and an approximate quadratic factor $Q_{n-2}(x)$.

At this stage $Q_{n-2}(x) = 0$ can be solved by the quadratic formula.

Since this method depends on repeated appox., it leads to very inaccurate approxs.

i.e.

$$P(x) \approx (x - x_1)(x - x_2) \cdots (x - x_k)\, Q_k(x)$$

An approximate zero x_{k+1} is the root of $Q_k(x)$, not $P(x)$.

One way to overcome this problem:

To consider x_i as an approximate sol., and improve it by applying the Newton-Raphson method to the original polynomial $P(x)$.

Note: The success of Newton's method often depends on obtaining a good initial approx..

The basic idea;

If $P(x_i)P(x_j) < 0 \longrightarrow P(x)$ has a zero between x_i and x_j

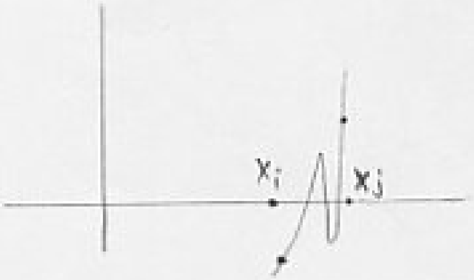
The prob. becomes a matter of choosing the interval so that the chance of missing a change of sign is minimized (Fig.)

Ex.

$$p(x) = 16x^4 - 40x^3 + 5x^2 + 20x + 6$$

It can be shown $P(x_i) > 0 \quad \forall x_i = \text{integer}$

$\longrightarrow$ evaluating $P(x)$ at these points one can not find any interval containing zeros. However, $P(x)$ has real roots.



$x_j - x_i$ = small
but still three roots inside this interval

Theo. 2.20

If $Z = a+bi$ is a complex zero of multiplicity $\underline{m}$ of the polynomial $P(x)$, ($P(x)$ with real coeffs.)

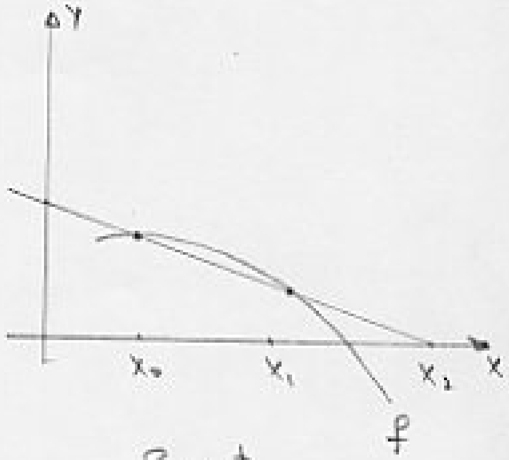
$\xrightarrow{\text{then}}$ $Z^* = a-bi$ is also a zero of multiplicity $\underline{m}$ of the polynomial $P(x)$ and $(x^2 - 2ax + a^2 + b^2)^m$ is a factor of $P(x)$.

$$\overset{\uparrow}{(x-z)(x-z^*)}$$

This method for finding complex roots will not be discussed here.

Müller's Method:

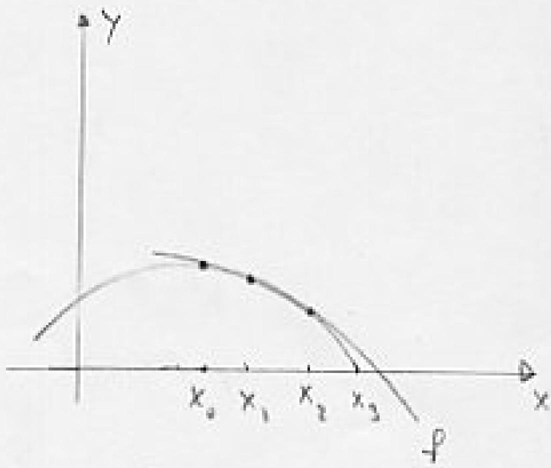
Müller's method is a generalization of Secant method



Secant method

In Secant method:

with x_0 and x_1 approx. $\longrightarrow$ x_2 is found by intersection of x-axis with the line through $(x_0, f(x_0))$ and $(x_1, f(x_1))$.



Müller's method

In Müller's method:

with x_0, x_1 and x_2 approx. $\rightarrow$ x_3 is found by intersection of x-axis with the parabola passing through $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

$$P(x) = a(x-x_2)^2 + b(x-x_2) + c$$

This quadratic polynomial passes through $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

$$f(x_0) = a(x_0-x_2)^2 + b(x_0-x_2) + c$$
$$f(x_1) = a(x_1-x_2)^2 + b(x_1-x_2) + c$$
$$f(x_2) = a(0)^2 + b(0) + c$$

$\rightarrow$ $$c = f(x_2)$$

$$b = \frac{(x_0-x_2)^2\,[f(x_1)-f(x_2)] - (x_1-x_2)^2\,[f(x_0)-f(x_2)]}{(x_0-x_2)(x_1-x_2)(x_0-x_1)}$$

$$a = \frac{(x_1-x_2)\,[f(x_0)-f(x_2)] - (x_0-x_2)\,[f(x_1)-f(x_2)]}{(x_0-x_2)(x_1-x_2)(x_0-x_1)}$$

We use quadratic formula to find x_3 (the zero of $P(x)$);

$$x_3 - x_2 = \frac{-2c}{b \pm \sqrt{b^2-4ac}}$$

Remark: $P(u) = au^2 + bu + c$, $u = x - x_2$

In Müller's method we choose the sign of radical to agree with the sign of b.

$\longrightarrow$ denominator large $\longrightarrow$ x_3 close to x_2

Now we repeat with the three values of x_1, x_2 and x_3 to find x_4 and so, up to convergence.

Since at each step, the method involves the radical $\sqrt{b^2-4ac}$, the method will appoximate complex roots when it is appropriate.

Müller.s Algorithm:

Remark:
$C \equiv f(x_2)$
$a \equiv d$
$b \equiv b$
$D \equiv \sqrt{\ }$

S1 $h_1 = x_1 - x_0$
$h_2 = x_2 - x_1$
$\delta_1 = (f(x_1) - f(x_0))/h_1$
$\delta_2 = (f(x_2) - f(x_1))/h_2$
$d = (\delta_2 - \delta_1)/(h_2 + h_1)$
$i = 2$

S2 while $i \leq N_0$ do steps 3-7

S3 $b = \delta_2 + h_2 d$
$D = (b^2 - 4 f(x_2) d)^{1/2}$ (May be complex arithmetic)

S4 If $|b-D| < |b+D|$ then set $E = b+D$
else set $E = b-D$

S5 $h = -2 f(x_2)/E$
$P = x_2 + h$

S6 If $|h| < Tol$ then
output (P)
stop

S7 $x_0 = x_1$, $x_1 = x_2$, $x_2 = P$

$h_1 = x_1 - x_0$

$h_2 = x_2 - x_1$

$\delta_1 = (f(x_1) - f(x_0))/h_1$

$\delta_2 = (f(x_2) - f(x_1))/h_2$

$d = (\delta_2 - \delta_1)/(h_2 + h_1)$

$i = i+1$

58 output 'Method failed after N_0 iteration'

Ex

Solve $f(x) = 16x^4 - 40x^3 + 5x^2 + 20x + 6$ using Muller's method.

$x_0 = 0.5,\ x_1 = -0.5,\ x_2 = 0$

i	x_i	$f(x_i)$
3	$-0.555556 + 0.598352i$	$-29.4007 - 3.89872i$
4	$-0.435450 + 0.102101i$	$1.33223 - 1.19309i$
⋮		
8	$-0.356062 + 0.162758i$	$0.286102\times10^{-5} + 0.953674\times10^{-6}i$

$x_0 = 0.5,\ x_1 = 1.0,\ x_2 = 1.5$

i	x_i	$f(x_i)$
3	1.28785	-1.37624
4	1.23746	0.126941
⋮		
7	1.24168	0.257492×10^{-4}

$x_0 = 2.5$, $x_1 = 2.0$, $x_2 = 2.25$

i	x_i	$f(x_i)$
3	1.96059	-0.611255
4	1.97056	0.748825×10^{-2}
⋮		
6	1.97044	-0.259639×10^{-4}

The actual values for roots are:

1.241677, 1.970446, $-0.356062 \pm 0.162758i$

The importance of Müller method is that it converges with any initial approximation choice. (Although for example if $f(x_i) = f(x_{i+1}) = f(x_{i+2})$, it does not converge)

Order of convergence:

$\alpha = 1.84$ Müller's method

$\alpha = 2$ Newton's "

$\alpha = 1.62$ Secant "