Chapter 10
Hamilton - Jacobi Theory

We mentioned that the canonical tr. can be used to provide a general procedure for solving mecanical Probs.

Two methods have been suggested.

I) If $H = \text{const.}$

then we make a canonical tr. in the following way:

$$\text{old} \xrightarrow{\text{tr.}} \text{New (the coords. are cyclic)}$$

Then the integration becomes trivial.

II - In this tr. we seek a tr. in the following way:

$$q, P \text{ at } t \xrightarrow{\text{tr.}} \text{new const. quantities}$$

which may be " $\longrightarrow$ q_0, P_0 at t_0

new set of const. quantities

Then we have exactly the desiered sol.

$$q = q(q_0, P_0, t)$$

$$P = P(q_0, P_0, t)$$

This is used specially when $H = H(t)$.

We shall begin our discussion by considering how such a tr. may be found.

10-1 The Hamilton-Jacobi Equ. for Hamilton's Principal Func.

The new variables are const. in time.

The requirement is

For $\begin{cases} \dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \\ \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \end{cases}$ if $K=0$, we can be sure $\begin{cases} \dot{Q}_i = 0 \\ \dot{P}_i = 0 \end{cases}$

As we have seen; $K = H + \frac{\partial F}{\partial t}$

$K = 0 \longrightarrow H(q,P,t) + \frac{\partial F}{\partial t} = 0$

It is convenient to take $F = F(q, P, t)$

$\longrightarrow F = F_2(q, P, t)$

But $p_i = \frac{\partial F_2}{\partial q_i}$ (also $Q_i = \frac{\partial F_2}{\partial P_i}$)

$$\rightarrow H(q_1, \ldots q_n, \frac{\partial F_2}{\partial q_1}, \ldots \frac{\partial F_2}{\partial q_n}; t) + \frac{\partial F_2}{\partial t} = 0$$

$(H = H(q,p,t), F_2 = F_2(q,P,t))$ Hamilton-Jacobi Equ.

This is a first order partial equ. in $(n+1)$ variables; $q_1 \cdots q_n; t$ for F_2. $(F_2 = F_2(q_i, \underbrace{P_i}_{=\text{const}}, t))$

It is customary to show the sol. by S

$F_2 \equiv S$ Hamilton's principal func.

Of course: $F_2 = S(q, t)$

The new momenta P_i have not yet been specified except that we know;

$$P_i = \text{const.}$$

Suppose there exists a sol. of the form (by $(n+1)$ integration)

$$F_2 \equiv S = S(q_1 \cdots q_n ; \alpha_1 \cdots \alpha_{n+1} ; t)$$

where $\alpha_1 \cdots \alpha_{n+1}$ are indep. consts. of integration.

This is known as Complete Sol.

Remark: The most general sol. involves <u>one or more</u> arbitrary funs. rather than arbitrary consts.

If $S_0 \neq S_0(q_1 \cdots q_n, t)$ $\quad S_0 = \text{const.}$

$\underrightarrow{\text{we observe}}$ $S \longrightarrow S + S_0$ leaves the H-J equ. unchanged.

and we choose to write S in the form

$$S = S(q_1 \cdots q_n ; \alpha_1 \cdots \alpha_n ; t) + S_0$$

α_{n+1}

S_0: additive const. of int.

$\alpha_1 \cdots \alpha_n$: non-additive indep. const. of int.

We are therefore at liberty to take n-consts. (new ones) of integration to be the new (const.) momenta.

$$\underline{P_i = \alpha_i}$$

(1) (n-equs) $P_i = \dfrac{\partial S(q,\alpha,t)}{\partial q_i} \longrightarrow p$ No contradiction (They are diff.)

$\longrightarrow$ At $t=0$ $\alpha_i = \alpha_i(q_0, p_0)$

α_i's can be obtained in terms of the specific initial cond. of the prob..

The other half of the equs. of tr., which provide new const. coords., appear as:

(2) (n-equs) $Q_i = \beta_i = \dfrac{\partial S(q,\alpha,t)}{\partial \alpha_i}$

$\longrightarrow$ At $t=0$ $\beta_i = \beta_i(q_0, p_0)$

(1), (2) $\rightarrow$ $\left\{\begin{array}{l} p_i = p_i(q,\alpha,t) \\ Q_i = Q_i(q,\alpha,t) \end{array}\right. \longrightarrow q_i = q_i(\alpha,\beta,t)$

$\longrightarrow p_i = p_i(\alpha,\beta,t)$

$\left\{\begin{array}{l} \alpha_i = \alpha_i(q_0, p_0) \\ \beta_i = \beta_i(q_0, p_0) \end{array}\right.$

$$\longrightarrow \begin{cases} q_i = q_i(q_0, P_0, t) \\ P_i = P_i(q_0, P_0, t) \end{cases}$$ complete set of Hamilton's eqns of motion

Thus; S (Hamilton's principal func.) is the generator of canonical tr. to const. coords and momenta.

Note:

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t}$$

But $\begin{cases} H + \dfrac{\partial F_2}{\partial t} = 0 \\ P_i = \dfrac{\partial S}{\partial q_i} \end{cases} \longrightarrow \dfrac{dS}{dt} = P_i\dot{q}_i - H = L$

$$\longrightarrow S = \int L\, dt + \text{const.} \qquad \text{(not helpful)}$$

One can not integrate the lagrangian with respect to time until q_i and P_i are known.

10-2 The Harmonic Osc. Prob. (Ex.) –

Consider one-dim. Harmonic Osc. Hamiltonian;

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = E$$

where $\omega = \sqrt{\frac{k}{m}}$

The Hamilton-Jacobi equ. is

$$H\left(q, \frac{\partial S}{\partial q}, \cancel{t}\right) + \frac{\partial S}{\partial t} = 0$$

setting $p = \frac{\partial S}{\partial q}$

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0$$

let $S = \ln v \rightarrow \frac{1}{2m}\left[\frac{\left(\frac{\partial v}{\partial q}\right)^2}{v^2} + m^2\omega^2 q^2\right] + \frac{\frac{\partial v}{\partial t}}{v} = 0$

try $v(q,t) = f(q)\,g(t)$

$$\underbrace{\frac{1}{2m}\left[\left(\frac{f'}{f}\right)^2 + m^2\omega^2 q^2\right]}_{\text{func. of } q} = \underbrace{-\frac{g'}{g}}_{\text{func of } t}$$

$\rightarrow -\frac{g'}{g} = \alpha$ const

$$\frac{1}{2m}\left[\left(\frac{f'}{f}\right)^2 + m^2\omega^2 q^2\right] = \alpha$$

try $g(t) = e^{-\alpha t}$

$S = \ln v = \ln(fg) = \ln f + \ln g = \ln f - \alpha t$

$\longrightarrow S(q,\alpha,t) = W(q,\alpha) - \alpha t$

sub. in H-J equ.

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha \quad (1) \text{ (total energy)}$$

Because $\frac{\partial S}{\partial t} + H = 0 \;\rightarrow -\alpha + H = 0 \;\rightarrow H = \alpha$

$$(1) \rightarrow \quad W = \sqrt{2m\alpha}\int dq \sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}$$

$$\rightarrow \quad S = \sqrt{2m\alpha}\int dq \sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}} \;-\alpha t$$

Int. is not diff., but we don't integrate, because we need partial derivative of S.

Now, sol. of q can be obtained from;

$$\beta = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{\alpha}}\int \frac{dq}{\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}} \;-t$$

$$t + \beta = \frac{1}{\omega}\sin^{-1} q\sqrt{\frac{m\omega^2}{2\alpha}}$$

$$\rightarrow q = \sqrt{\frac{2\alpha}{m\omega^2}}\,\sin\omega(t+\beta) \quad (2)$$ a familiar sol. for H.O. (the equ. of motion)

The sol. for P;

$$P = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2}$$

$$P = \sqrt{2m\alpha(1 - \sin^2\omega(t+\beta))}$$

$$P = \sqrt{2m\alpha}\,\cos\omega(t+\beta) \qquad \text{tr. equ. } (3)$$

This result checks, the

$$P = mv = m\dot{q}$$

$(2)(3) \longrightarrow 2m\alpha = P_0^2 + m^2\omega^2 q_0^2$ at $t = t_0 = 0$

Of course this result can be obtained from

$\alpha = H = \frac{1}{2m}(P^2 + m^2\omega^2 q^2)$ at $t = 0$

Also

$(2)(3) \longrightarrow \tan(\omega\beta) = m\omega \frac{q_0}{P_0}$ at $t = t_0 = 0$

Thus, Hamilton's principal func. is the generator of a canonical tr. to a new coord. that measures the phase angle of the oscillation and to a new canonical momentum identified as the total energy.

$$S = \sqrt{2m\alpha}\int (\dot{q}\,dt)\sqrt{1-\frac{m\omega^2 q^2}{2\alpha}} - \gamma t$$

$$S = \sqrt{2m\alpha}\int \omega\sqrt{\frac{2\alpha}{m\omega^2}}\, C_s\,\omega(t+B)\sqrt{1-\frac{m\omega^2}{2\alpha}\left(\sqrt{\frac{2\alpha}{m\omega^2}}\right)^2 S^2\,\omega(t+B)} - \alpha t$$

$$S = 2\alpha\int C_s^2\,\omega(t+B)\,dt - \gamma t = 2\alpha\int \left(C_s^2\,\omega(t+B) - \tfrac{1}{2}\right)dt$$

Now, the Lagrangian;

$$L = \frac{1}{2m}\left(P^2 - m^2\omega^2 q^2\right) = \alpha\left[C_s^2\,\omega(t+B) - S^2\,\omega(t+B)\right]$$

$$= 2\alpha\left(C_s^2\,\omega(t+B) - \tfrac{1}{2}\right)$$

which is in agreement with;

$$S = \int L\,dt + C.$$

10-3 The Hamilton - Jacobi Equ. for Hamilton's Characterstic Func.

Hamilton - Jacobi equ. was integrable for Simple Harmonic Oscillator. This was due to S could be seperated into two parts $\begin{cases} \text{one involving } q \\ \text{the other only } t \end{cases}$

This is possible whenever $\frac{\partial H}{\partial t} = 0$

When $\frac{\partial H}{\partial t} = 0 \xrightarrow{\text{Hamilton - Jacobi equ.}} \underbrace{\frac{\partial S}{\partial t}}_{\text{dep. of } S \text{ on } t} + \underbrace{H(q_i, \frac{\partial S}{\partial q_i})}_{\text{dep. of } S \text{ on } q_i} = 0$

Assuming a sol.;

$$S(q_i, \alpha_i, t) = W(q_i, \alpha_i) - \alpha_1 t$$

$$\longrightarrow H(q_i \frac{\partial W}{\partial q_i}) = \alpha_1 \qquad (1)$$

Time-indep func. W appears as a part of generating func. S, when $H = \alpha_1$ is const.

Consider a canonical tr. $\longrightarrow \begin{cases} P_i = \alpha_i & \text{const of motions} \\ H = \alpha_1 \end{cases}$

Denote the generating func. by $W(q,P)$

$$\rightarrow P_i = \frac{\partial W}{\partial q_i} \quad , \; Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial \alpha_i}$$

While these are similar to

$$p_i = \frac{\partial S(q,\alpha,t)}{\partial q_i} \quad \text{and} \quad Q_i = \beta_i = \frac{\partial S(q,\alpha,t)}{\partial \alpha_i}$$

we have a cond. on determining W, that is;

$$H(q_i, p_i) = \alpha_1$$

$$\longrightarrow H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1 \qquad (2) \quad \text{partial differential equ.}$$

which is identical to (1).

Since $W \neq W(t) \longrightarrow H = K = \alpha_1$

<u>W</u> is known Hamilton's characteristic func.

In a tr. generated by W all coord. are <u>cyclic</u>

We noted in chap <u>9</u>, when H = const. of motion, a tr. of this nature, <u>leads</u> → to sol. of the prob. with <u>trivial</u> integrations.

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \quad (\text{cyclic}) \rightarrow P_i = \alpha_i$$

Since $H = K = \alpha_1$

$$\rightarrow \dot{Q}_i = \frac{\partial K}{\partial \alpha_i} = 1 \qquad \text{for } i = 1$$
$$= 0 \qquad i \neq 1$$

$$\longrightarrow Q_1 = t + \beta_1 \equiv \frac{\partial W}{\partial \alpha_1}$$
$$Q_i = \beta_i \equiv \frac{\partial W}{\alpha_i} \qquad i \neq 1$$

The only that is not const. of motion is Q_1 -

The dep. of W on old coords. q_i are given by:

$$H(q_i, \frac{\partial W}{\partial q_i}) = \alpha_1$$ (referred to Hamilton-Jacobi) equ.

$\longrightarrow$ n const. of integration, but one of them is an additive const.

$\longrightarrow$ n-1 remaining indep. consts $\alpha_2 .. \alpha_n$ together with α_1 may be taken as the new const. momenta.

The first half of the equs.:

$$p_i = \frac{\partial W}{\partial q_i} \qquad Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial \alpha_i} \qquad (1)$$

evaluated at t_0 relates (n-number) $\alpha_i \xrightarrow{\text{to}} q_i$ and p_i

The other half;

$$Q_1 = t + \beta_1 \equiv \frac{\partial W}{\partial \alpha_1}$$

$$Q_i = \beta_i \equiv \frac{\partial W}{\partial \alpha_i} \quad i \neq 1 \qquad (2)$$

$\longrightarrow q_i = q_i(\alpha_i, \beta_i, t)$

completing the sol. of the problem.

$(n-1)$ of the equs. (2) don't involve t at all.

One of the q_i's can be chosen as an independent variable, and the remaining coords can then be expressed in terms of it by solving only these time-indep. equs..

$\longrightarrow$ we are led to orbit equs. of motion

Ex. In central problem;

$r = r(\theta)$

without solving explicitly $r = r(t)$ or $\theta = \theta(t)$.

It is not always necessary to take $\begin{cases} \alpha_i \\ \text{and the consts. of integration in } W \\ \text{as the new cond. momenta} \end{cases}$

Rather it may be desirable to use some particular set of n-indep funcs. of the α_i's as the transfored momenta.

Designate then consts. by γ_i:

$$W = W(q_i, \gamma_i)$$

$$\dot{Q}_i = \frac{\partial K}{\partial \gamma_i} = \nu_i \qquad \text{when } \nu_i = \nu_i(\gamma)$$

$$Q_i = \nu_i t + \beta_i$$

Since $\frac{\partial W}{\partial t} = 0 \longrightarrow \frac{dW}{dt} = \frac{\partial W}{\partial q_i} \dot{q}_i = p_i \dot{q}_i$

$\rightarrow \quad W = \int p_i \dot{q}_i \, dt = \int p_i \, dq_i$ abbreviated action

This information is of little practical help.

The form of W cannot be found priori without obtaining a complete integral of the Hamilton-Jacobi equ.

The procedure involved in solving a mechanical problem by either <u>Hamilton's principal</u> or <u>characteristic func</u> may now be summarized in the following tabular form:

The two methods of sols. are applicable when the Hamiltonian

$H = H(q, p, t)$	$H(q, p) = \text{const.}$

We seek canonical trs to new variables such that

All $Q_i = \text{const}$ $P_i = \text{const}$	All $P_i = \text{const.}$

To meet these requirements it is sufficient to demand that the new Hamiltonian

$K = 0$	$K = H(P_i) = \alpha_1$

Under these conds. the new equs. of motion become:

$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0$	$\dot{Q}_i = \frac{\partial K}{\partial P_i} = \nu_i$
$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$	$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$

with the immadiate sols.:

$Q_i = \beta_i$ $P_i = \gamma_i$	$Q_i = \nu_i t + \beta_i$ $P_i = \gamma_i$

which satisfies the stipulated requirements.

The generating fun. is Hamilton's

Principal Func. $S(q, P, t)$	characteristic Func. $W(q, P)$

Satisfying the Hamilton-Jacobi differential equ.

$H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0$	$H(q, \frac{\partial W}{\partial q}) - \alpha_1 = 0$

A complete sol. to the equ contains;

n- nontrivial consts of integration, $\alpha_1 \dots \alpha_n$	n-1 nontrivial consts of integration together with α_1 form a set of n-indep consts, $\alpha_1 \dots \alpha_n$

The new const momenta;

$P_i = \gamma_i(\alpha_1 \dots \alpha_n)$	$P_i = \gamma_i(\alpha_1 \dots \alpha_n)$

So that the complete sols. to the Hamilton-Jacobi equ. may be considered as funcs. of the new momenta;

$$S = S(q_i, \gamma_i, t) \qquad\qquad W = W(q_i, \gamma_i)$$

In particular, the γ_i's may be chosen to be the α_i's themselves. One half of the tr. equs.

$$p_i = \frac{\partial S}{\partial q_i} \qquad\qquad p_i = \frac{\partial W}{\partial q_i}$$

are fulfilled automatically, since the have been used in constructing the Hamilton-Jacobi equ.. The other half.

$$Q_i = \frac{\partial S}{\partial \gamma_i} = \beta_i \qquad\qquad Q_i = \frac{\partial W}{\partial \gamma_i} = \nu_i(\gamma_j)t + \beta_i$$

can be solved for q_i in terms of t and $2n$ consts. β_i, γ_i. The sol. to the prob. is then completed by evaluating these $2n$ consts. in terms of the initial values (q_{i0}, p_{i0}).

When $\frac{\partial H}{\partial t} = 0 \longrightarrow$ Both methods are suitable, and the generating funcs are related to each other acc. to;

$$S(q, P, t) = W(q, P) - \alpha_1 t$$