Chapter 9

Canonical Transformation

9-1 The equs. of canonical transformation

The sol. of Hamilton's equs. is trivial in the following situation;

$$\begin{cases} \frac{\partial H}{\partial t} = 0 \\ \quad \text{and} \\ \frac{\partial H}{\partial q_i} = 0 \quad (i=1,\dots n) \end{cases} \quad \text{Since } \dot{p}_i = -\frac{\partial H}{\partial q_i} \longrightarrow \dot{p}_i = \text{cont} \rightarrow p_i = \alpha_i$$

In general $H = H(q, p, t)$

In this case $H = H(p) \longrightarrow H = H(\alpha_1, \dots \alpha_2)$

And the Hamilton's equs. for $\dot{q}_i$ are;

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \omega_i$$

Since $H = H(\alpha_1 \dots \alpha_n) \longrightarrow \omega_i = \omega_i(\alpha_1 \dots \alpha_n)$

$\rightarrow \omega_i = \text{const}$

$\rightarrow q_i = \omega_i t + \beta_i$

Ex. - In central force problem we may choose two different set of coords.

$$\begin{cases} q_1 = x \\ q_2 = y \end{cases} \qquad \begin{cases} q_1 = r \\ q_2 = \theta \end{cases}$$

For central force neither x nor y is cyclic, while the second set does contain a cyclic coord. in the angle θ.

Thus the number of cyclic coords. depends on the choice of generalized coords.

For each problem there may be one particular choice for which all coords are cyclic.

We try to make transformation from one set of generalized coords. to some other set that may be more suitable.

The transformation $Q_i = Q_i(q,t)$ is called Point tr.. (tr. of configuration space)

In Hamiltonian formulation $\left\{ \begin{matrix} q_i\text{'s} \\ p_i\text{'s} \end{matrix} \right.$ are indep-variables

In this case the concept of tr. must be widened to include the simultaneous tr. of independent coords. and momenta.

$$Q_i = Q_i(q,p,t) \qquad \text{(tr. of phase space)}$$
$$P_i = P_i(q,p,t)$$

In developing Hamiltonian mechanics, only canonical trs. are of interest.

This requirement is satisfied, if there exists some func. $K(Q,P)$, such that the equs. of motion in the new set are in the Hamiltonian form

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = \frac{-\partial K}{\partial Q_i}$$

(Q,P) must be canonical for all mechanical systems of the same number of degs. of freedom (Problem-indep.).

If Q_i and P_i are to be canonical coords., they must satisfy a modified Hamilton's principal;

$$\delta \int_{t_1}^{t_2} \sum_i (P_i \dot{Q}_i - K(Q,P,t)) dt = 0$$

at the same time;

$$\delta \int_{t_1}^{t_2} \sum_i (p_i \dot{q}_i - H(q,p,t)) dt = 0$$

Both eqns. must be satisfied simultaneously, if the tr. is canonical

Since the general form of the modified Hamilton's principle has Zero variation at the end points, both statements will be satisfied if the integrands are connected by a relation of the form;

$$\lambda (p_i \dot{q}_i - H(q_i, p_i, t)) = P_i \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF}{dt}$$

when

$F = F(q, p, Q, P, t)$ with continuous 2^{nd} derivative

λ = const.

$\lambda \neq 1$ for extended canonical tr.

$\lambda = 1$ Canonical tr.

Scale transformation;

$$Q_i' = \mu q_i \qquad P_i' = \nu p_i$$

The $\begin{cases} \dot{Q}_i = \dfrac{\partial K}{\partial P_i} \\ \dot{P}_i = -\dfrac{\partial K}{\partial Q_i} \end{cases}$ will be satisfied if $K'(Q', P') = \mu\nu H(q,p)$

$$\longrightarrow \quad \mu\nu(p_i\dot{q}_i - H) = P_i'\dot{Q}_i' - K'$$

$$\longrightarrow \quad \lambda = \mu\nu$$

If in $(q,p) \xrightarrow{tr.} (Q,P)$, $\quad \lambda \neq 1$

one may make two trs. as below

$$(q,p) \xrightarrow{tr.} (Q', P') \qquad \lambda \neq 1$$

$$(Q', P') \xrightarrow{tr.} (Q, P) \qquad \lambda = 1$$

Thus we will discuss only for trs. with $\lambda = 1$,

Def.:

$$\begin{cases} Q_i = Q_i(q,p,t) = Q_i(q,p) \\ P_i = P_i(q,p,t) = P_i(q,p) \end{cases}$$ restricted canonical tr.

Through $\begin{cases} Q_i = Q_i(q_i, p_i, t) \\ P_i = P_i(q_i, p_i, t) \end{cases}$ and their inverses

F can be expressed in terms partly of the old set of variables and partly of the new.

F is useful for specifying the exact form of the canonical tr only when:

and $\begin{cases} \text{half of the variables (beside } t\text{) are from the old set} \\ \text{half are from the new} \end{cases}$

F_i is called Generating func. of tr.

I- $F = F_1(q, Q, t)$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF_1}{dt}$$

$$= P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i$$

$$\longrightarrow \begin{cases} p_i = \dfrac{\partial F_1}{\partial q_i} & \text{(a)} \quad n\text{-relation} \\ P_i = -\dfrac{\partial F_1}{\partial Q_i} & \text{(b)} \quad \text{"} \\ K = H + \dfrac{\partial F_1}{\partial t} & \text{(c)} \end{cases}$$

a) $\rightarrow \quad P_i = g_i(q,Q,t) \rightarrow Q_i = g_i'(q,P,t)$

b) $\rightarrow \quad \begin{cases} \underline{P}_i = h_i(q,Q,t) \\ Q_i = g_i'(q,P,t) \end{cases} \longrightarrow \underline{P}_i = h_i'(q,P,t)$

Now, q and p in H are expressed in terms of $\underline{P}$ and Q through the inverse of $\begin{cases} Q_i = g_i' \\ \underline{P}_i = h_i' \end{cases}$

Then the q_i in $\frac{\partial F_1}{\partial t}$ are expressed in terms of Q and $\underline{P}$ in a similar manner.

$\longrightarrow K = K(Q,\underline{P},t)$

Thus with a given F_1 one may obtain the set of:

$$Q_i = g_i'(q,P,t) \equiv Q_i(q,P,t)$$
$$\underline{P}_i = h_i'(q,P,t) \equiv \underline{P}_i(q,P,t)$$
$$K = K(Q,\underline{P},t)$$

Inverse of the Problem:

Given the set $\begin{cases} Q_i = Q_i(q,P,t) \\ \underline{P}_i = P_i(q,P,t) \end{cases}$ one may find an appropriate generating func. F_1.

These equs. are inverted to find the following set;

$$p_i = p_i(q, Q, t)$$
$$\underline{P}_i = \underline{P}_i(q, Q, t)$$

into (a) $\rightarrow \begin{cases} p_i(q,Q,t) = \dfrac{\partial F_1}{\partial q_i} \\ \underline{P}_i(q,Q,t) = \dfrac{\partial F_1}{\partial Q_i} \end{cases}$
" (b) $\rightarrow$

These partial differential equs. can be integrated to find F_1, Providing the tr. is indeed canonical.

$$\longrightarrow F_1 = F_1(q, Q, t) + \underbrace{f(t)}_{\text{uncertain within}}$$

II- Sometimes it is not suitable to describe the canonical tr. by the generating func. $F_1(q, Q, t)$.

For example, the tr. may be such that p_i cannot be written as fucs of q, Q and t, but rather will be fucs. of $q, \underline{P}$ and t.

Then one may seek a generating func. of the form $F_2(q, \underline{P}, t)$

A generating func. of the form

$$F = F_2(q, \underline{P}, t) - Q_i \underline{P}_i$$

solves the problem.

$$p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{dF}{dt}$$

$$= \cancel{P_i\dot{Q}_i} - K + \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P_i}\dot{P}_i + \frac{\partial F_2}{\partial t} - \dot{P}_i Q_i - \cancel{P_i\dot{Q}_i}$$

$$\longrightarrow \begin{cases} p_i = \dfrac{\partial F_2}{\partial q_i} & (a) \\ Q_i = \dfrac{\partial F_2}{\partial P_i} & (b) \\ K = H + \dfrac{\partial F_2}{\partial t} & (c) \end{cases}$$

a) $\longrightarrow \quad p_i = g_i(q, P, t) \longrightarrow P_i = g_i'(q, p, t)$

b) $\longrightarrow \begin{cases} Q_i = h_i(q, P, t) \\ P_i = g_i'(q, p, t) \end{cases} \longrightarrow Q_i = h_i'(q, p, t)$

$K = K(Q, P, t)$ can be constructed in a similar manner in case I.

III- $\quad F = q_i P_i + F_3(p, Q, t)$

$$\cancel{p_i}\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{dF}{dt}$$

$$= P_i\dot{Q}_i - K + \cancel{\dot{q}_i P_i} + q_i\dot{P}_i + \frac{\partial F_3}{\partial p_i}\dot{p}_i + \frac{\partial F_3}{\partial Q_i}\dot{Q}_i + \frac{\partial F_3}{\partial t}$$

$$\longrightarrow \begin{cases} q_i = -\dfrac{\partial F_3}{\partial p_i} \\ P_i = -\dfrac{\partial F_3}{\partial Q_i} \\ K = H + \dfrac{\partial F_3}{\partial t} \end{cases}$$

IV- $F = q_i p_i - Q_i P_i + F_4(p, P, t)$

$$\cancel{p_i \dot{q}_i} - H = P_i \dot{Q}_i - K + \frac{dF}{dt}$$

$$= \cancel{P_i \dot{Q}_i} - K + \cancel{\dot{q}_i p_i} + q_i \dot{p}_i - \cancel{\dot{Q}_i P_i} - Q_i \dot{P}_i + \frac{\partial F_4}{\partial p_i} \dot{p}_i + \frac{\partial F_4}{\partial P_i} \dot{P}_i + \frac{\partial F_4}{\partial t}$$

$$\longrightarrow \begin{cases} q_i = -\frac{\partial F_4}{\partial p_i} \\ Q_i = \frac{\partial F_4}{\partial P_i} \\ K = H + \frac{\partial F_4}{\partial t} \end{cases}$$

The four types of generating funcs. can be related to each other through the Legendre tr..

For exaple, the transition from $F_1 \xrightarrow{\text{to}} F_2$ is equivalent to going from the variables $q, Q \xrightarrow{\text{to}} q, P$

$$f(q, Q) \longrightarrow g(q, P)$$

$$g = f \mp Q P$$

$$g = F_2, \quad f = F_1 \longrightarrow F_2(q, P, t) = F_1(q, Q, t) \mp P_i Q_i$$

This is equivalent to the second type of generating func..

As another example consider the 4^{th} type of generating func.:

$$F = q_i p_i - Q_i P_i + F_4(p, P, t)$$

If now we consider $F = F_1(q, Q, t)$

$$\longrightarrow \quad F_1 = q_i p_i - Q_i P_i + F_4(p, P, t)$$

This is a double Legendre tr..

Remark: For any given canonical tr. it <u>is not</u> always possible to find generating funcs. of all 4-types, by means of these Legendre trs.

Remark: It is possible and for some canonical trs. necessary, to use generating funcs., that is a mixture of the 4-types.

Ex.: It may be desirable for a particular canonical tr. with n=2 deg. of freedom to be defined by a generating func. of the form:

$$F' = (q_1, p_2, P_1, Q_2, t)$$

$$\rightarrow F = F'(q_1, p_2, P_1, Q_2, t) - Q_1 P_1 + q_2 p_2$$

together with $p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{\partial F}{\partial t}$

$$\Rightarrow \quad p_1 = \frac{\partial F'}{\partial q_1}, \quad Q_1 = \frac{\partial F'}{\partial P_1}$$

$$q_2 = -\frac{\partial F'}{\partial p_1}, \quad P_2 = -\frac{\partial F'}{\partial Q_2}$$

$$K = H + \frac{\partial F'}{\partial t}$$

Ex.: $F_2(q, P, t) = q_i P_i$

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i \quad , \quad Q_i = \frac{\partial F_2}{\partial P_i} = q_i \quad , \quad K = H$$

Hence → F_2 generates <u>indentity</u> tr. .

Ex.: $F_2(q, P, t) = f_i(q_1 \ldots q_n, t) P_i$

$$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q, t)$$

This is a <u>point</u> tr. as we faced before

$$Q_i = Q_i(q, t)$$

Here f_i must be <u>independent</u> and invertable.
Since f_i are otherwise completely arbitrary, then all <u>point trs.</u> are <u>canonical</u> .

Ex.: $F_2 = f_i(q_1 \ldots q_n, t) P_i + g(q_1 \ldots q_n, t)$
where g: differentiable func of the old coords. at t.

is another <u>point tr.</u> .

Ex.: $F_1 = (q, Q, t) = q_i Q$;

$$p_i = \frac{\partial F_1}{\partial q_i} = Q; \qquad P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i;$$

This tr. is called <u>exchange tr.</u> This is a canonical tr.

This can be seen from Hamilton's equs.,

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$p_i \longrightarrow q_i$$
$$q_i \longrightarrow -p_i$$

Remark: Exchange tr. cannot be derived from $F_2(q, P, t)$.

Because: $$p_i = \frac{\partial F_2(q, P, t)}{\partial q_i} = f(q, P, t)$$

But in the exchange tr. $p_i \neq f(P)$

It should be $p_i = f(Q)$!

Remark: Identity tr. can not be derived from $F_1(q, Q, t)$.

Because: $$p_i = \frac{\partial F_1(q, Q, t)}{\partial q_i} = f(q, Q, t)$$

In identity tr. $p_i \neq f(Q)$

It should be $p_i = f(P)$

Remark: Exchange tr. can not also be derived by defing an F_1 func. through a Legendre tr.

$$F_1(q,Q,t) = F_2(q,P,t) - P_i Q_i$$

Since $F_2 = q_i P_i = Q_i P_i$ (identity tr.)

$$F_1 = Q_i P_i - Q_i P_i = 0$$

Remark: A similar dead end is obtained in attempting to construct an F_2 func. to generate the exchange tr..

Remark: However an F_3 func. can generate the identity tr., and that the exchange tr. can be derived from an F_4 func..

Ex.: Define a tr. which leaves some of the (q,p) pairs unchanged and interchanges the rest (with a sign change). This is canonical tr.. The generating func. can not be derived from one of the 4-pure forms discussed. It will have a mixed form.

The tr.
$$Q_1 = q_1 \quad P_1 = p_1$$
$$Q_2 = p_2 \quad P_2 = -q_2$$

is generated by
$$F = \underbrace{q_1 P_1}_{F_2} + \underbrace{q_2 Q_2}_{F_1}$$

Ex. Harmonic Oscillator in one-dim.

$$H = \frac{p^2}{2m} + \frac{k}{2} q^2$$

Since $\frac{k}{m} = \omega^2 \longrightarrow H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$

We are interested for a tr. in which, H is cyclic in the new coords..

$$\text{Tr.:} \begin{cases} (1)\ p = f(P)\cos Q \\ (2)\ q = \frac{f(P)}{m\omega}\sin Q \end{cases} \qquad \begin{aligned} p &= p(P,Q) \\ q &= q(P,Q) \end{aligned}$$

$$K = H = \frac{f^2(P)}{2m}(\cos^2 Q + \sin^2 Q) = \frac{f^2(P)}{2m} = K(P)$$

So Q is cyclic. $\longrightarrow$ (P is conserved)

$f(P)$ must be found in such a way the tr. to be canonical.

(1), (2) $\rightarrow$ $p = m\omega q \cot Q$ indep of $f(P)$

This equ is of the form of $p_i = \frac{\partial F_1(q,Q)}{\partial q_i} = f(q,Q)$

$$\rightarrow p = \frac{\partial F_1(q,Q)}{\partial q} = m\omega q \cot Q$$

$$F_1 = m\omega \frac{q^2}{2} \cot Q + \underbrace{\varphi(P)}_{\text{verify, it must be zero}}$$

using the other half of the eqns. of tr.

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2 S^2 Q} \rightarrow q = \sqrt{\frac{2P}{m\omega}}\, S_i Q$$

but we had $q = \frac{f(P)}{m\omega} S Q$

$\rightarrow f(P) = \sqrt{2m\omega P}$ (to be canonical)

$$p = m\omega q \operatorname{cotg} Q = \sqrt{2m\omega P}\, C Q$$

$$\rightarrow H = \frac{p^2}{2m} + \frac{k}{2} q^2 = \omega P \rightarrow K = \omega P$$

K is cyclic in Q $\quad -\dot{P} = \frac{\partial H}{\partial Q} = 0 \rightarrow P = \text{const}$

$$P = \frac{E}{\omega}$$

Also $\dot{Q} = \frac{\partial H}{\partial P} = \omega \rightarrow Q = \omega t + \alpha$

$$q = \sqrt{\frac{2P}{m\omega}}\, S Q = \sqrt{\frac{2E}{m\omega^2}}\, S_i (\omega t + \alpha)$$

9-3 The Symplectic approach to canonical tr.

Consider a restricted tr. (t doesn't appear)

$$\begin{cases} Q_i = Q_i(q,p) \\ P_i = P_i(q,p) \end{cases} \longrightarrow \begin{cases} q_i = q_i(Q,P) \\ p_i = p_i(Q,P) \end{cases}$$

$$\longrightarrow H(q,p) \longrightarrow H(Q,P)$$

$$K = H$$

(1) $\dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j = \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j}$

where we have used equs. of motion for $\dot{q}_j$ and $\dot{p}_j$

On the other hand:

(2) $\dot{Q}_i = \frac{\partial H}{\partial P_i} = \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i} + \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i}$

Comparing (1) and (2):

$$\begin{cases} \left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \xrightarrow{\text{means}} p_j = p_j(Q,P) \\ \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P} \end{cases}$$

A similar comparison for $\dot{P}_i$ leads:

$$\begin{cases} \left(\dfrac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\dfrac{\partial p_j}{\partial Q_i}\right)_{Q,P} \\ \left(\dfrac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\dfrac{\partial q_j}{\partial Q_i}\right)_{Q,P} \end{cases}$$

These equs. are called direct conditions for restricted canonical tr.

In symplectic notation: $\dot{\eta} = J \dfrac{\partial H}{\partial \eta}$ (Hamilton's equs.)

$$\eta = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \begin{cases} \left(\dfrac{\partial H}{\partial \eta}\right)_i = \dfrac{\partial H}{\partial q_i} & i \leq n \\ \left(\dfrac{\partial H}{\partial \eta}\right)_{i+n} = \dfrac{\partial H}{\partial p_i} \end{cases}$$

Ex.: $n=2$

$$\dot{\eta} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \frac{\partial H}{\partial q_2} \\ \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ -\frac{\partial H}{\partial q_1} \\ -\frac{\partial H}{\partial q_2} \end{pmatrix} \qquad \begin{cases} \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \\ \dot{q}_i = \dfrac{\partial H}{\partial p_i} \end{cases}$$

Similarly:

$$\xi = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \\ P_1 \\ \vdots \\ P_n \end{pmatrix} \qquad \xi = \xi(\eta) \longrightarrow \eta = \eta(\xi)$$

(1) $\rightarrow$ $\dot{\xi}_i = \dfrac{\partial \xi_i}{\partial \eta_j} \dot{\eta}_j$ $\qquad \dot{\xi} = M \dot{\eta}$

$M \rightarrow M_{ij} = \dfrac{\partial \xi_i}{\partial \eta_j}$ Jacobian matrix of tr.

Ex.: $n = 2$ $\qquad M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$

Now;

$$\dot{\xi}_i = M J \frac{\partial H}{\partial \eta}$$

By the inverse of tr.: $\dfrac{\partial H}{\partial \eta_i} = \dfrac{\partial H}{\partial \xi_j} \dfrac{\partial \xi_j}{\partial \eta_i}$

$$\frac{\partial H}{\partial \eta} = \tilde{M} \frac{\partial H}{\partial \xi} \rightarrow \dot{\xi} = [M J \tilde{M}] \frac{\partial H}{\partial \xi}$$

We obtained for the restricted canonical tr.:

$\dot{\xi} = J \dfrac{\partial H}{\partial \xi}$ $\qquad$ (as the form $\dot{\eta} = J \dfrac{\partial H}{\partial \eta}$)

Therefore $\xi = \xi(\eta)$ will be canonical if:

$$M J \tilde{M} = J$$

This cond. is also a necessary cond. for restricted tr.. (It can be seen by reversing the steps of proof)

Note: For extended time-indep. canonical tr.:

$$M J \tilde{M} = \lambda J$$

Remark: $M J \tilde{M} = J \rightarrow M J \tilde{M} \tilde{M}^{-1} = J \tilde{M}^{-1}$

$\rightarrow M J = J \tilde{M}^{-1}$

Since $(\tilde{A})^{-1} = \widetilde{(A^{-1})}$

$M J = J \widetilde{M^{-1}}$

$J M J = J^2 \widetilde{M^{-1}} \longrightarrow J M J (-J) = J^2 \widetilde{M^{-1}} (-J)$

$\longrightarrow J M [J(-J)] = J^2 \widetilde{M^{-1}} (-J)$

$\rightarrow J M [J \tilde{J}] = -I \widetilde{M^{-1}} (-J) \rightarrow J M = -\widetilde{M^{-1}} (-J)$

$\rightarrow J M = \widetilde{M^{-1}} J \rightarrow \tilde{M} J M = \tilde{M} \widetilde{M^{-1}} J$

$\longrightarrow \tilde{M} J M = J$

$M J \tilde{M} = J$ and $\tilde{M} J M = J$ are symplectic conds. for canonical tr.. (M: sympelectic matrix)

Remark: For a canonical tr. that contains t as a parameter, the simple derivation given for the symplectic cond. <u>no longer</u> holds.

Time-dep. canonical tr. also satisfies $M J \tilde{M} = J$

Proof:

Consider a canonical tr. of the form:

$\xi = \xi(\eta, t)$ (we consider t as a parameter)

which evolves continuously as time increases from t_0 -

If the tr. $\eta \longrightarrow \xi(t)$ is canonical (1)

$\xrightarrow{\text{obviously}}$ $\eta \longrightarrow \xi(t_0)$ " (2)

From def. of canonical tr.

$\xi(t_0) \longrightarrow \xi(t)$ " (3)

Since t_0 in (2) is a fixed const, this canonical tr. satisfies $M J \tilde{M} = J$.

If now (3) obeys $M J \tilde{M} = J$
$\longrightarrow$ (1) will obey also "

Proof: Consider an infinitesimal canonical tr. (I.C.T.)

The tr. eqns.: $Q_i = q_i + \delta q_i$
$P_i = p_i + \delta p_i$

or $\xi = \eta + \delta\eta$

$\delta\eta$: infinitesimal actual displacement

In generating formalism; the tr. is in such a way that it differs slightly from the identity tr.

Suitable generating func.: F_2

$$F_2 = q_i P_i + \epsilon G(q, P, t)$$

$\downarrow$ identity tr.

G: differentiable func of 2n+1 parameter

$$\begin{cases} p_j = \dfrac{\partial F_2}{\partial q_j} = P_j + \epsilon \dfrac{\partial G}{\partial q_j} \\ Q_j = \dfrac{\partial F_2}{\partial P_j} = q_j + \epsilon \dfrac{\partial G}{\partial P_j} \end{cases}$$

$$\begin{cases} \delta p_j \equiv P_j - p_j = -\epsilon \dfrac{\partial G}{\partial q_j} \\ \delta q_j \equiv Q_j - q_j = \epsilon \dfrac{\partial G}{\partial P_j} \end{cases} \longrightarrow \delta \eta = \epsilon J \frac{\partial G}{\partial \eta}$$

The tr. $\zeta(t_0) \longrightarrow \zeta(t)$ when t differs from t_0 by an infinitesimal is canonical.

$\zeta(t_0) \longrightarrow \zeta(t_0 + dt)$ with $\epsilon = dt$

Continuous evaluation of the tr. $\zeta(\eta, t)$ from $\zeta(\eta, t_0)$, means that the tr. $\zeta(t_0) \longrightarrow \zeta(t)$ can be build up as a succesion of such I.C.T. in dt steps.
It is therefore suffice to show $\zeta(t_0) \longrightarrow \zeta(t_0 + dt)$ satisfies the symplectic cond.

$$\zeta = \eta + \delta\eta \longrightarrow \frac{\partial \zeta}{\partial \eta} = I + \frac{\partial(\delta\eta)}{\partial \eta} = I + \epsilon J \frac{\partial^2 G}{\partial\eta\partial\eta}$$

where $\left(\frac{\partial^2 G}{\partial\eta\partial\eta}\right)_{ij} = \frac{\partial^2 G}{\partial\eta_i\partial\eta_j}$

Since J is antisym. $\rightarrow \tilde{J} = -J$

also $\widetilde{\frac{\partial^2 G}{\partial\eta_i\partial\eta_j}} = \frac{\partial^2 G}{\partial\eta_j\partial\eta_i} = \frac{\partial^2 G}{\partial\eta_i\partial\eta_j}$

$$\rightarrow \tilde{M} = 1 - \epsilon \frac{\partial^2 G}{\partial\eta\partial\eta} J$$

$$\rightarrow \quad M J \tilde{M} = \left(1 + \epsilon J \frac{\partial^2 G}{\partial\eta\partial\eta}\right) J \left(1 - \epsilon \frac{\partial^2 G}{\partial\eta\partial\eta} J\right)$$

$$M J \tilde{M} = J + \epsilon J \frac{\partial^2 G}{\partial\eta\partial\eta} J - \epsilon J \frac{\partial^2 G}{\partial\eta\partial\eta} J + O(\epsilon^2)$$

Keeping the first order terms:

$M J \tilde{M} \approx J$ symplectic cond. holds.

Remark: Despite the symplectic approach has been treated independently from the generating func. method, these are related.

Symplectic cond. $\longrightarrow$ existance of generating func.

Both are valid ways of looking at canonical tr.
Both of these can be used to show that the canonical trs. have the group properties:

1- The identity tr. is canonical
2- If a tr. is canonical so is its inverse.
3- Two successive canonical trs define a tr. that is also canonical
4- The product operation is associative.

9-4 Poisson Brackets and other Canonical Invariants

The poisson bracket of two funcs. u, v with respect to the canonical variables (q, p) is defined as:

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

In matrix form:

$$[u, v]_\eta = \frac{\widetilde{\partial u}}{\partial \eta} J \frac{\partial v}{\partial \eta}$$

i.e. $$\left(\frac{\partial u}{\partial \eta_1}, \cdots \frac{\partial u}{\partial \eta_n}\right)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial v}{\partial \eta_1} \\ \vdots \\ \frac{\partial v}{\partial \eta_n} \end{pmatrix}$$

If we let $u = q_j \quad v = q_k$:

$$\rightarrow [q_j, q_k]_{q,p} = \underbrace{\frac{\partial q_j}{\partial q_i}}_{\delta_{ji}} \underbrace{\frac{\partial q_k}{\partial p_i}}_{0} - \underbrace{\frac{\partial q_j}{\partial p_i}}_{0} \underbrace{\frac{\partial q_k}{\partial q_i}}_{\delta_{ki}} = 0$$

Similarly:

$$[p_j, p_k]_{q,p} = 0$$

and

$$[q_j, p_k]_{q,p} = \underbrace{\frac{\partial q_j}{\partial q_i}}_{\delta_{ji}} \underbrace{\frac{\partial p_k}{\partial p_i}}_{\delta_{ki}} - \underbrace{\frac{\partial q_j}{\partial p_i}}_{0} \underbrace{\frac{\partial p_k}{\partial q_i}}_{0} = \delta_{jk} = -[p_j, q_k]_{q,p}$$

Summarizing all of them in a square matrix Poisson bracket:

(1) $$[\eta, \eta]_\eta = J \qquad (\eta : q, p)$$

Now we consider u,v the members of transfered variables (Q,P), or ξ:

$$[\xi,\xi]_\eta = \frac{\widetilde{\partial \xi}}{\partial \eta} J \frac{\partial \xi}{\partial \eta}$$

But since $M = \frac{\partial \xi}{\partial \eta} \rightarrow [\xi,\xi]_\eta = \tilde{M} J M$

If $\eta \longrightarrow \xi$ is canonical the $\tilde{M} J M = J$

(2) $\longrightarrow [\xi,\xi]_\eta = J$

Conversely if this cond. is valid then the tr. is canonical.

(3) From (1) $\longrightarrow [\xi,\xi]_\xi = J$ fundamental Poissons braket

Then the fundamental Poissons brackets of the ξ-variables have the same value when evaluated with respect to any canonical coord. set.

In other words:

Fundamental Poissons brackets are invariant under canonical tr. -

From $[\xi,\xi]_\eta = \tilde{M} J M$ we see the invariance is a necessary and sufficient cond. for the tr. matrix to be symplectic.

All poissons brakets are invariant under canonical tr..

Proof:

$$[u,v]_\eta = \frac{\widetilde{\partial u}}{\partial \eta} J \frac{\partial v}{\partial \eta}$$

$$\frac{\partial v}{\partial \eta_i} = \frac{\partial v}{\partial \xi_j}\frac{\partial \xi_j}{\partial \eta_i} \longrightarrow \frac{\partial v}{\partial \eta} = \widetilde{M}\frac{\partial v}{\partial \xi}$$

$$\frac{\widetilde{\partial u}}{\partial \eta} = \widetilde{\widetilde{M}\frac{\partial u}{\partial \xi}} = \frac{\widetilde{\partial u}}{\partial \xi} M$$

$$[u,v]_\eta = \frac{\widetilde{\partial u}}{\partial \xi} MJ\widetilde{M}\frac{\partial v}{\partial \xi}$$

For canonical tr.

$$[u,v]_\eta = \frac{\widetilde{\partial u}}{\partial \xi} J \frac{\partial v}{\partial \xi} \equiv [u,v]_\xi$$

Thus the poissons bracket has the same value when evaluated with respect to any canonical set of variables.

Remark:

In canonical tr. $\xrightarrow{\text{implies}}$ Hamilton's equs. of motion are <u>invariant</u> in form under tr.

Canonical invariance of Poisson brackets $\xrightarrow{\text{implies}}$ The equs. expressed in terms of Poisson brackets are <u>invariant</u> in form under canonical tr.

We will use these properties in developping structure of classical mechanics, paralleling Hamiltonian formulation expressed in terms of Poisson brackets.

The Poisson bracket formulation, which has the same form in all canonical coords., is useful for carrying out the transition from classical to quantum mechanics.

The properties of Poisson brackets:

$$[u,u]=0$$
$$[u,v]=-[v,u]$$
$$[au+bv\,,w]=a[u,w]+b[v,w]$$
$$[uv,w]=[u,w]v+u[v,w]$$
$$[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0 \quad \text{Jacobi's identity}$$

Proof of Jacobi's identity:

let $u_i \equiv \dfrac{\partial u}{\partial \eta_i}$ $\quad v_{ij} \equiv \dfrac{\partial v}{\partial \eta_i \partial \eta_j}$

$\rightarrow [u,v]=u_i J_{ij} v_j \qquad [u,[v,w]]=u_i J_{ij}[v,w]_j = u_i J_{ij}(v_k J_{k\ell} w_\ell)_j$

$[u,[v,w]]=u_i J_{ij}(v_k J_{k\ell} w_{\ell j}+v_{kj}J_{k\ell}w_\ell)$ (because the elements of J are consts.)

The other Poisson brackets can be obtained by cyclic permutation of u,v,w. Consider the term involving the second derivative of w, $J_{ij}J_{k\ell}u_i v_k w_{\ell j}$

The only other second derivative of w appears in $[v,[w,u]]$. That is $J_{ji}J_{k\ell}u_i v_k w_{j\ell}$, Since $w_{j\ell}=w_{\ell j}$

$\rightarrow \underbrace{(J_{ij}+J_{ji})}_{\text{antisym.}} J_{k\ell}u_i u_k w_{\ell j}=0$ and so the other terms

Lagrange brackets:

Lagrange bracket of u and v with respect to (q,p) variables is defined as:

$$\{u,v\}_{q,p} = \frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u}\frac{\partial q_i}{\partial v}$$

In matrix notation:

$$\{u,v\}_\eta = \frac{\widetilde{\partial \eta}}{\partial u} J \frac{\partial \eta}{\partial v}$$

Lagrange bracket are canonical invariant:

consider Lagrange bracket with respect to ξ:

$$\{u,v\}_\xi = \frac{\widetilde{\partial \xi}}{\partial u} J \frac{\partial \xi}{\partial v}$$

but $\frac{\partial \xi_i}{\partial v} = \frac{\partial \xi_i}{\partial \eta_j}\frac{\partial \eta_j}{\partial v} \longrightarrow \frac{\partial \xi}{\partial v} = M\frac{\partial \eta}{\partial v}$

$$\{u,v\}_\xi = \frac{\widetilde{\partial \eta}}{\partial u}\tilde{M} J M \frac{\partial \eta}{\partial v}$$

By virtue of symplectic cond. $\tilde{M}JM = J$

$$\rightarrow \quad \{u,v\}_\xi = \frac{\widetilde{\partial \eta}}{\partial u} J \frac{\partial \eta}{\partial v} = \{u,v\}_\eta$$

which proofs the canonical invariance of Lagrange bracket.

One may verify:

$$\{q_i, q_j\} = 0 \quad \{p_i, p_j\} = 0 \quad \{q_i, p_j\} = \delta_{ij}$$

or

$$\{\eta, \eta\} = J$$

which are called Fundamental Lagrange brackets.

Remark: For $n=1$ $(2n=2)$ $[u,u] = \begin{pmatrix} [u_1,u_1] & [u_1,u_2] \\ [u_2,u_1] & [u_2,u_2] \end{pmatrix}$

The reciprocal character of the two brackets manifests itself in the relation:

$$\{u,u\}[u,u] = -I$$

This can be seen for special case $u=\eta$

$$\{\eta,\eta\} = J\ ,\ [\eta,\eta] = J \quad \text{and} \quad J^2 = -I$$

But it is valid for any u.

Another invariant:

The magnitude of the volume element in phase space is canonical invariant.

$$(d\eta) = dq_1 \cdots dq_n dp_1 \cdots dp_n$$
$$(d\xi) = dQ_1 \cdots dQ_n dP_1 \cdots dP_n$$

The size of two volume element is related by the absolute value of Jacobian determinant.

$$(d\xi) = \|M\|(d\eta)$$

$$\tilde{M} J M = J \longrightarrow |\tilde{M} J M| = |J|$$

$$\longrightarrow |M|^2 |J| = |J| \longrightarrow |M|^2 = 1$$

$$|M| = \pm 1 \longrightarrow ||M|| = 1$$ absolute value of det.

This proves the invariance of volume element in phase space.

It follows, also, that the volume of any arbitrary region in phase space

$$J_n = \int \dots \int (d\eta)$$

is a canonical invariant.

<u>Theorem</u>: The symplectic cond. implies the existance of a generating func.

We consider only one-deg. of freedom, then the method can be extended for many degs. of freedom.

consider:

$$\begin{cases} Q = Q(q,p) & (1) \\ P = P(q,p) & (2) \end{cases} \rightarrow p = \varphi(q,Q) \xrightarrow{\text{sub. into (2)}} P = \psi(q,Q)$$

In such case we would expect the tr. to be generated by F_1; i.e.

$$p = \frac{\partial F_1(q,Q)}{\partial q} \qquad \underline{P} = \frac{-\partial F_1(q,Q)}{\partial Q} \qquad (1),(2)$$

Remark: If the Q tr. equ is not invertable, as in the identity tr., then we would invert the P equ. and be led to a generating func of the form F_2.

If (1), (2) hold, then:

$$\begin{cases} \dfrac{\partial p}{\partial Q} = \dfrac{\partial^2 F_1}{\partial q \partial Q} \\ \dfrac{\partial P}{\partial q} = -\dfrac{\partial^2 F_1}{\partial q \partial Q} \end{cases} \longrightarrow \frac{\partial p}{\partial Q} = -\frac{\partial \underline{P}}{\partial q} \longrightarrow \frac{\partial \varphi}{\partial Q} = -\frac{\partial \psi}{\partial q} \qquad (3)$$

Conversely, if we can show that equ (3) is valid, then there must exist a func. F_1 such that p and $\underline{P}$ are given by (1) and (2).

To demonstrate the validity of (3) we try to look on all quantities as funcs of q and Q.

$$\begin{cases} Q = Q(q,p) \\ p = \varphi(q,Q) \end{cases} \longrightarrow Q = Q(q, \varphi(q,Q))$$

$$\frac{\partial Q}{\partial Q} = \frac{\partial Q}{\partial p}\frac{\partial p}{\partial Q} = \frac{\partial Q}{\partial p}\frac{\partial \varphi}{\partial Q} = 1$$

But $\frac{\partial Q}{\partial Q} = 1 \longrightarrow \frac{\partial Q}{\partial p}\frac{\partial \varphi}{\partial Q} = 1$ (1)

Now

Since $\tilde{M} J M = J \longrightarrow [Q, P] = 1$

$$[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1 \qquad (P = \psi(q,Q))$$

$$[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial \psi}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q}$$

$$= \frac{\partial Q}{\partial q}\frac{\partial \psi}{\partial Q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\left(\frac{\partial \psi}{\partial q} + \frac{\partial \psi}{\partial Q}\frac{\partial Q}{\partial q}\right)$$

$$[Q,P] = \frac{\partial \psi}{\partial Q}\underbrace{\cancel{\left(\frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q}\right)}}_{0} - \frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q}$$

$$= -\frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q} \longrightarrow -\frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q} = 1 \qquad (2)$$

$$(1),(2) \longrightarrow \frac{\partial Q}{\partial p}\frac{\partial \varphi}{\partial Q} = -\frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q} \longrightarrow \frac{\partial \varphi}{\partial Q} = -\frac{\partial \psi}{\partial q}$$

Note $\frac{\partial Q}{\partial p} \neq 0$ else the Q-eqn could not be inverted.

Remark: $P = P(q,p) \quad \psi = \psi(q,Q)$

$$\frac{\partial P}{\partial q} = \frac{\partial \psi}{\partial q} + \frac{\partial \psi}{\partial Q}\frac{\partial Q}{\partial q} = \frac{d\psi}{dq}$$

Ex. – $\begin{cases} Q = q^2 + p & \rightarrow p = Q - q^2 \\ P = q + p & \end{cases} \longrightarrow P = \psi = q + Q - q^2$

$$\frac{\partial P}{\partial q} = 1 \qquad \frac{d\psi}{dq} = 1 - 2q + (1)2q = 1$$

9-5 Equs. of motion, Infinitesimal canonical trs., and Conservation Theorems in the Poisson Bracket Formulation –

Almost the entire framework of Hamiltonian mechanics can be restarted in terms of Poisson brackets.

Canonical invariance of Poisson brackets $\xrightarrow{\text{implies}}$ All relations so obtained be invariant in Form under a canonical tr.

Let $u(q,p,t)$: Canonical variable

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t}$$

$$\rightarrow \quad \frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$ Generalized equ. of motion of u

In terms of the symplectic notation:

$$\frac{du}{dt} = \frac{\partial u}{\partial \eta}\dot{\eta} + \frac{\partial u}{\partial t} = \widetilde{\frac{\partial u}{\partial \eta}} J \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial t}$$

For special case:

$$\dot{q}_i = [q_i, H] \quad , \quad \dot{p}_i = [p_i, H] \quad \left(\text{since } \frac{\partial q_i}{\partial t} = 0 \quad \frac{\partial p_i}{\partial t} = 0\right)$$

In symplectic notation:

(1) $$\dot{\eta} = [\eta, H]$$ identical to Hamilton's equs. of motion.

Note: $\frac{dq}{dt} = \dot{q} = \frac{\partial q}{\partial q}\frac{\partial H}{\partial p} - \cancel{\frac{\partial q}{\partial p}}\frac{\partial H}{\partial q} = \frac{\partial H}{\partial p} \longrightarrow \dot{q} = [q, H]$

$$\frac{dp}{dt} = \dot{p} = \cancel{\frac{\partial p}{\partial q}}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial q} = -\frac{\partial H}{\partial q} \qquad \dot{p} = [p, H]$$

Now;

$$[u,v]_\eta = \frac{\partial \tilde{u}}{\partial \eta} J \frac{\partial v}{\partial \eta} \longrightarrow [\eta, H] = J\frac{\partial H}{\partial \eta} \qquad (2)$$

$(1)(2) \longrightarrow \dot{\eta} = J\frac{\partial H}{\partial \eta}$ (we had it before)

Also

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} \xrightarrow{\text{if } u = H} \frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Remark: If the canonical tr. is t-dep. then H must be changed to K.

Now, if u is const. of motion, i.e. $\frac{du}{dt} = 0$

then $[H, u] = \frac{\partial u}{\partial t}$

For those const. of motions of u, that $\frac{\partial u}{\partial t} = 0$

$$\longrightarrow [H, u] = 0$$

If u and v are consts. of motion, with $\frac{\partial u}{\partial t} = 0$, $\frac{\partial v}{\partial t} = 0$

From the Jacobi identity, with $w = H$,

$$[u,\underbrace{[v,H]}_{0}] + [v,\underbrace{[H,u]}_{0}] + [H,[u,v]] = 0$$

$\longrightarrow [H,[u,v]] = 0 \longrightarrow [u,v]$ const. of motion

Even if $\frac{\partial u}{\partial t} \neq 0$, $\frac{\partial v}{\partial t} \neq 0$, it can be shown that $[u,v]$ is const. of motion. (Poisson's Theorem)

Infinitesimal canonical tr. in terms of Poissons brackets.

let η be a canonical variable

$$\xi = \eta + d\eta$$

we obtained before $d\eta = \epsilon J \frac{\partial G(\eta,t)}{\partial \eta}$ (1)

from $F_2 = \underset{\text{identity tr.}}{\underset{\uparrow}{q_i P_i}} + \epsilon G(q,P,t)$

Now since:

$[u,v]_\eta = \widetilde{\frac{\partial u}{\partial \eta}} J \frac{\partial v}{\partial \eta} \longrightarrow [\eta, u] = J \frac{\partial u}{\partial \eta}$ (2)

if $u \equiv G \xrightarrow{(1)(2)} \delta\eta = \epsilon[\eta, G]$

Now consider an infinitesimal canonical tr. in which the Continuous parameter is t, so that $\epsilon = dt$

and let $G \equiv H$

$\longrightarrow d\eta = dt\,[\eta, H] = \dot{\eta}\, dt$ (Since $\dot{\eta} = J \frac{\partial H}{\partial \eta}$)

This tr. changes all $\left\{\begin{matrix}\text{coords.}\\ \text{momenta}\end{matrix}\right.$ at t $\xrightarrow{\text{to}}$ $\left\{\begin{matrix}\text{coords.}\\ \text{momenta}\end{matrix}\right.$ at $t+dt$

Thus the motion of a system in a time interval dt can be described by an Infinitesimal contact (canonical) tr. generated by H.

The system motion in a finite time interval $t_0 \to t$ is represented by a succession infinitesimal contact trs., which is equivalent to a single finite canonical tr.

Result:

Hamiltonian is the generator of the system motion with Time.

q_0, p_0 at t_0 $\xrightarrow[\text{Tr}(t)]{\text{Canonical tr which is}}$ q, p at t

continuous

Conversely, there must exist a canonical tr. in the following way:

$$q, p \text{ at } t \xrightarrow{\text{tr.}} q_0, p_0 \text{ at } t_0$$

This is equivalent to solving the prob. of the system of motion.

Passive view, Active view -

At the begining of this chap. it was pointed out that

$$\text{A mechanical Prob.} \xrightarrow[\text{a canonical tr.}]{\text{can be reduced to finding}} \text{in which all } P_i = \alpha_i = \text{const.}$$

Then we could solve the Prob. easily.

Now we indicate the possibility of an alternative sol. by means of the canonical tr. for which both the momenta and coords. are consts. of motion.

Assume a canonical tr. (Passive view):

Passive view of canonical tr.

The state of the system is described by point A as well as A' at a given time. (For example A or A' can be a point of trajectory) -

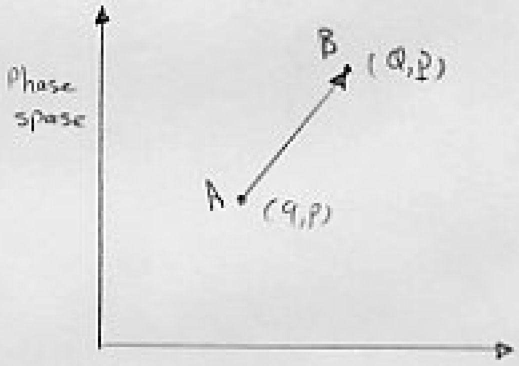
Any func. of the system variables would have the same value for a given system configuration whether it was described by the (q,p) set or by the (Q,P) set.

$\longrightarrow$ the func. would have the same value at A' as at A.

$$f(A') = f(A) \qquad f(q,p) = f(Q,P)$$

Now, consider a canonical tr. in active view:

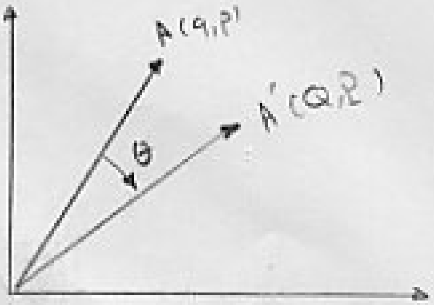
In contrast to this view, in active view the canonical tr. generated by the Hamiltonian relates the coords of one point in phase space to another point in the same phase space.



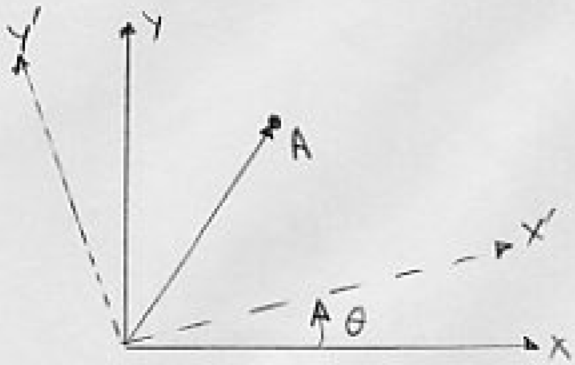
Active view of canonical tr.

Ex.

Rotation



Active view



Passive view

In some cases the active view is not helpful.

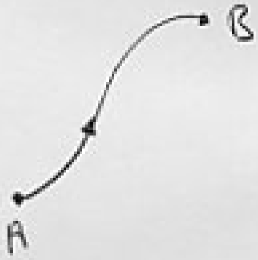
Ex:

$$\text{Cartezian coord} \xrightarrow{\text{Point tr.}} \text{spherical Polar coord.}$$

An active interpretation of this is not helpful.

The active viewpoint is useful for trs. depending continuously on a single parameter.

The effect of tr. (in active view) is to move the system point <u>continuously</u> on a curve in phase space as the parameter changes continuously.



When the generator associated I.C.T is H $\longrightarrow$ the curve is the trajectory in phase space.

In passive view:

$$u(A') = u(A)$$

Ex. $\begin{cases} \begin{cases} Q = q^2 \\ P = q + p \end{cases} \rightarrow \begin{cases} q = \sqrt{Q} \\ p = P - \sqrt{Q} \end{cases} \\ u = q - p \\ u(A) = q - p \quad u(A') = \sqrt{Q} - P + \sqrt{Q} \end{cases}$

but functional dependence of u on Q, P are <u>different</u> than " " " " " q, p.

In active view:

$$u(A) \neq u(B) \qquad \text{(For example; } f(q_1,p_1) \neq f(q_2,p_2))$$

But, functional dep. of u on $Q, \underline{P}$ at point B is the

same as " " " " " q, p " " A.

Ex. $\begin{cases} q = at \\ p = B \end{cases}$ $\begin{cases} u(A) = q + p \rightarrow u(A) = a t_1 + B \\ u(B) = q + p \rightarrow u(B) = a t_2 + B \end{cases}$

Under I.C.T. $\quad \partial u = u(B) - u(A)$ (in active view)

where A and B are infinitesimal close.

In matrix notation: $\partial u = u(\eta + \delta\eta) - u(\eta)$

Expanding in a Taylor series: $\left(u(\eta+\delta\eta) = u(\eta) + \frac{\partial u}{\partial \eta}\delta\eta + \dots \right)$

$$\partial u = \frac{\partial u}{\partial \eta}\delta\eta = \frac{\partial u}{\partial \eta}\left(\epsilon J \frac{\partial G}{\partial \eta}\right) = \epsilon \frac{\partial u}{\partial \eta} J \frac{\partial G}{\partial \eta}$$

Recalling the def. of the Poisson bracket:

$$\partial u = \epsilon[u, G] \xrightarrow[\text{application}]{\text{immadiate}} \underbrace{\partial\eta = \epsilon[\eta, G] = \delta\eta}_{\text{we had it before}}$$

This is obvious result, the change in the coords. from A to B is just the infinitesimal difference between the old and new coord.

Generalization to Hamiltonian:

Remember: Hamiltonian does not mean a specific func. (not the same in all coord. systems).

Rather $\longrightarrow$ H refers to a func. which defines the canonical equs. of motion in phase space.

If (canonical) $T_r = T_r(t) \longrightarrow$

$$H(A) \not\longrightarrow H(A')$$

But $H(A) \longrightarrow K(A')$ (and $H(A) \neq K(A')$ in general)

$\longrightarrow \delta H = H(B) - K(A')$

(Under the two interpretaion, active, passive. Because: $\eta \longrightarrow \xi$ (Passive) and we have t-dependence also)

This new def is identical to $\partial U = U(B) - U(A)$ when the func. itself does not change under canonical tr. (Since $U(A') = U(A)$)

i.e. $\partial H = H(B) - K(A') \longrightarrow \partial H = H(B) - H(A') \equiv \partial H = H(B) - H(A)$

We have: $K = H + \frac{\partial F}{\partial t}$

where $F_2 = q_i P_i + \epsilon G(q, P, t)$

$$K(A') = H(A') + \epsilon \frac{\partial G}{\partial t} = H(A) + \epsilon \frac{\partial G}{\partial t}$$

$$\partial H = H(B) - H(A) - \epsilon \frac{\partial G}{\partial t}$$

As before, (Page 149);

$$\delta H = \epsilon [H, G] - \epsilon \frac{\partial G}{\partial t}$$

using the generalized equ. of motion:

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} \qquad \text{with} \quad u \equiv G$$

$$\longrightarrow \delta H = -\epsilon \frac{dG}{dt} \qquad (1)$$

If G = const. of motion equ (1) says that:

It generates an I.C.T. that does not change the value of the Hamiltonian.

In other words:

The consts. of motion are the generating funcs. of those I.C.T. that leave the Hamiltonian invariant.

Symmetry properties of the system $\xleftrightarrow[\text{a connection}]{\text{There is}}$ Conserved quantities

If the conserved quantity G (i.e. $\frac{dG}{dt}=0$) does not explicitly depend on time (i.e. $\frac{\partial G}{\partial t}=0$) then:

$$\partial H = H(B) - H(A) = 0$$

and $\partial H = H(B) - H(A)$ (if $\frac{dG}{dt} \neq 0$ but $\frac{\partial G}{\partial t}=0$)

If the system is symmetric under a tr. (G does not change), then the H will obviously remain unafected under this tr.

Thus if the system is symmetrical about a given direction, the the Hamiltonian will not change in value if the system as a whole is rotated about that direction.

It follows then that the quantity that generates (through an I.C.T.) such a rotation of the system must be conserved.

Ex. — Momentum Cons.

If q_i is cyclic $\longrightarrow \frac{\partial H}{\partial q_i} = 0$

$\longrightarrow$ under an infinitesimal tr. that involves q_i, H is invariant.

Now consider a tr. generated by the generalized momentum conjugate to q_i

$$G(q,P) = P_i$$

Since $F_2 = q_i P_i + \epsilon G(q,P,t)$ $\begin{cases} \delta p_j = -\epsilon \frac{\partial G}{\partial q_j} \\ \delta q_j = \epsilon \frac{\partial G}{\partial p_j} \end{cases}$

$$\longrightarrow \begin{cases} \delta q_j = \epsilon \delta_{ij} \\ \delta p_j = 0 \end{cases}$$

$\longrightarrow$ if a coord. is cyclic, its conjugate momentum is conserved.

In general form:

$$G = (J\eta)_\ell = J_{\ell r}\eta_r$$

then the eqns. of tr. are $\delta\eta = \epsilon J \frac{\partial G}{\partial \eta}$

$$\longrightarrow \delta\eta_k = \epsilon J_{ks}\frac{\partial G}{\partial \eta_s} = \epsilon J_{ks} J_{\ell r}\delta_{rs} = \epsilon J_{ks} J_{\ell s} \underset{\text{orthogonality}}{=} \epsilon\delta_{k\ell}$$

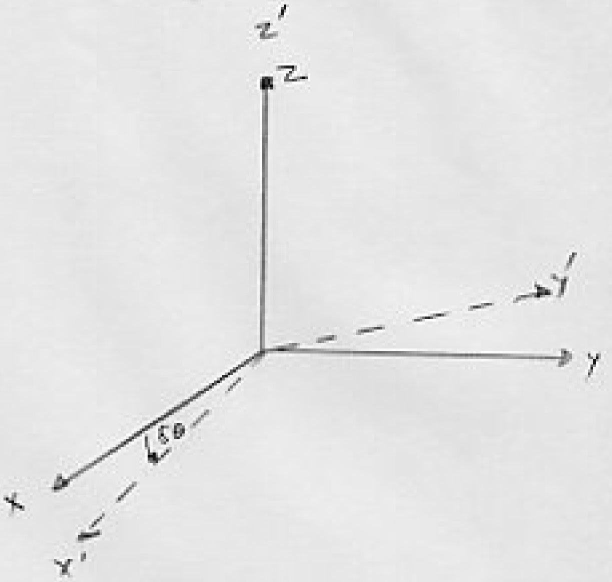
Ex. —

If $\eta_\ell = q_i \longrightarrow G = P_i$

If $\eta_\ell = P_i \longrightarrow G = -q_i$

Ex. — Infinitesimal Rotation about z-axis.

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\longrightarrow \quad A = \begin{pmatrix} 1 & \delta\theta & 0 \\ -\delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X' = AX \longrightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & \delta\theta & 0 \\ -\delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y\,\delta\theta \\ y - x\,\delta\theta \\ z \end{pmatrix}$$

$$\begin{cases} \delta x_i = -y_i\,\delta\theta \\ \delta y_i = x_i\,\delta\theta \\ \delta z_i = 0 \end{cases}$$

Similarly:
$$\begin{cases} \delta P_{ix} = -P_{iy}\,\delta\theta \\ \delta P_{iy} = P_{ix}\,\delta\theta \\ \delta P_{iz} = 0 \end{cases}$$

Comparing this tr. equs. with
$$\begin{cases} \delta P_j = -\epsilon \dfrac{\partial G}{\partial q_j} \\ \delta q_j = \epsilon \dfrac{\partial G}{\partial P_j} \end{cases}$$

$$\longrightarrow \quad G = x_i P_{iy} - y_i P_{ix} \equiv L_z$$

$$G = L_z \equiv (r_i \times P_i)_z$$ canonical angular momentum

In general: $G = L \cdot \hat{n}$ n: rotation axis

Canonical angular momentum may differ from the mechanical angular momentum.

If $V = V(\dot{q}_i, \ldots) \longrightarrow$ Canonical momentum vectors P_i are not necessarily the same as linear momentum vector and so for the L

Note: H : is the generator of displacement of the system in time

L : " " " " spacial rotation

P : " " " " spacial tr.

The finite tr. of a system can be obtained by sum of an infinite succession of I.C.T. (integrating the expression for infinitesimal displacements along the curve)

Let the continuous parameter to be α with initial value of $\alpha = 0$. Also u is some func. of the system configration, then u will be a continuous func. of α along the trajectory, $u(\alpha)$.

For simplicity we assume $\frac{\partial u}{\partial t} = 0$

We obtained $\partial u = \epsilon [u, G]$

$\epsilon \to d\alpha \longrightarrow \frac{du}{d\alpha} = [u, G]$

From the Taylor series

$$u(\alpha) = u_0 + \alpha \frac{du}{d\alpha}\Big|_0 + \frac{\alpha^2}{2!}\frac{d^2u}{d\alpha^2} + \frac{\alpha^3}{3!}\frac{d^3u}{d\alpha^3} + \cdots$$

But $\frac{du}{d\alpha}\Big|_0 = [u, G]_0$

also $\frac{d^2u}{d\alpha^2} = [[u,G],G]$

$$\rightarrow \quad u(\alpha) = u_0 + \alpha[u,G]_0 + \frac{\alpha^2}{2!}[[u,G],G]_0 + \frac{\alpha^3}{3!}[[[u,G],G],G]_0 + \cdots$$

Ex. — Let $G = L_z$, then the final canonical tr. should correspond to a finite rotation about z-axis.

$\alpha = \theta$

$[x_i, L_z] = ?$

$$[x_i, x_i p_{iy} - y_i p_{ix}] = \frac{\partial x_i}{\partial x_i}\frac{\partial\, x_i p_{iy} - y_i p_{ix}}{\partial p_{ix}} + \frac{\partial x_i}{\partial y_i}\frac{\partial\, x_i p_{iy} - y_i p_{ix}}{\partial p_{iy}}$$

$$+ \frac{\partial x_i}{\partial z_i}\frac{\partial\, x_i p_{iy} - y_i p_{ix}}{\partial p_{iz}} - \frac{\partial x_i}{\partial p_{ix}}\frac{\partial\, x_i p_{ix} - y_i p_{ix}}{\partial x_i} - \frac{\partial x_i}{\partial p_{iy}}\frac{\partial \cdots}{\partial y_i}$$

$$- \frac{\partial x_i}{\partial p_{iz}}\frac{\partial \cdots}{\partial z_i} = -y_i$$

$[x_i, L_z]_\alpha = -y_i$

$[[x_i, L_z], L_z]_\alpha = -x_i \qquad [y_i, L_z]_\alpha = x_i$

$[[[x_i, L_z], L_z], L_z]_\alpha = y_i$

But $[x_i, L_z]_0 = -y_{i0}$

$$X_i = x_{i0} - y_{i0} - x_{i0}\frac{\theta^2}{2} + y_{i0}\frac{\theta^3}{3!} + x_{i0}\frac{\theta^4}{4!} + \cdots$$

$$X_i = X_{i0}\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) - y_{i0}\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$X_i = X_{i0}\, C\theta - y_{i0}\, S\theta$$

Similarly:

$$Y_i = X_{i0}(-S\theta) + Y_{i0}\, C\theta$$

Ex. – Consider the situation when $G = H$, $\alpha = t$

Then $\frac{du}{d\alpha} = [u, G] \longrightarrow \frac{du}{dt} = [u, H]$

$$\rightarrow u(t) = u_0 + t\,[u,H]_0 + \frac{t^2}{2!}[[u,H],H]_0 + \frac{t^3}{3!}[[[u,H],H],H]_0 + \cdots \quad (1)$$

Now consider one-dim motion, with a const. acceleration a,

$$H = \frac{p^2}{2m} - max$$

$$\rightarrow [x, H] = \frac{p}{m}, \quad [[x,H],H] = \frac{1}{m}[p, H] = a = \text{const}$$

Since $[[\ ],\ ] = \text{const} \longrightarrow$ all higher order brackets vanish.

$$X = x_0 + \underbrace{\frac{p_0}{m}}_{v_0} t + \frac{a}{2}t^2$$

Relation between cl. and Q. mechanics:

Eq.(1) can simbolically written as

$$u(t) = u\, e^{\hat{H}t}\Big|_0$$

This is similar to what we had in Heisenberg picture in Q.M., where

$u(t)$: time-varying OP., whose time-dependence is given in terms of $e^{iHt/\hbar}$ in such manner as to lead to the same equ. of motion:

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

$i/\hbar$ arises out of the correspondence between the cl. Poisson brackets and the Q. commutator.

9-6 The angular momentum Poisson bracket relations:

we obtained:

$$\partial u = \epsilon [u, G]$$

This eqn (on the "active" view) is also valid if u is taken as the vector-component along a fixed axis in ordinary space.

Thus if F is a vector func. of the system configration, then

$$\partial F_i = [F_i, G]$$

Remark: The direction along which the component is taken must be fixed, (not affected by the canonical tr.)

If the direction itself is determined in terms of the system variables $\xrightarrow{\text{then}}$ the tr. changes not only the value of the func., but the nature of the func. (just as with the Hamiltonian).

Therefore, the change in a vector F under a rotation of the system about a fixed axis $\bar{n}$, generated by L.n can be written:

$$\partial F = d\theta\,[F, L.n] \qquad (1)$$

The unit vectors $\hat{i}, \hat{j}, \hat{k}$ that form the basic set for F are not themselves rotated by L.n.

Remark: Rotation of the system under the I.C.T., is not necessarily the rotation of the vector F.

The generator L.n induces a spacial rotation of the system variables (q, p..) not for external vectors (for examples magnetic field, acceleration gravity).

If $F = F(q, p..)$
system variables

and $F \neq F$(external quantities or vectors)

Only under these conds. does a spacial rotation imply a corresponding rotation of F.

Such vectors are designated a System Vectors.

The change in a vector under infinitesimal rotation about an axis $\bar{n}$ is given by:

$$dF = \bar{n}\, d\theta \times \bar{F}$$

For a system vector $\bar{F}$ the change by I.C.T., generated by $L.n$ we have;

$$\partial F = d\theta\, [F, L.n] = \bar{n}\, d\theta \times \bar{F}$$

For all system vectors:

$$[F, L.n] = n \times F \qquad \text{Poisson bracket identity}$$

Note: Here there is no reference to a canonical tr. or even to a spacial rotation.
It is simply a statement about the value of certain Poisson brackets for a specific class of vectors.

Verification;
For system vectors, like $\bar{P}$ $(\bar{F} = \bar{P})$
Take L along z-dir.
$[P, L_z] = ?$

$$[P_x\ ,\ xP_y - yP_x] = -P_y$$
$$[P_y\ ,\ \quad " \quad] = P_x$$
$$[P_z\ ,\ \quad " \quad] = 0$$

The right hand side to the components of $\hat{k}\times\bar{P}$

Now consider $F = A$, where $A = \frac{1}{2}(\bar{r}\times\bar{B})$ and $\bar{B} = B\hat{i}$ (external to the system)

A is the vector potential corresponding to the uniform magnetic field $\bar{B}$ in x-dir.

$$[0\ ,\ xP_y - yP_x] = 0$$
$$[\tfrac{1}{2}zB\ ,\ \quad " \quad] = 0$$
$$[-\tfrac{1}{2}yB\ ,\ \quad " \quad] = -\tfrac{1}{2}Bx$$

whereas the vector $\hat{k}\times\bar{A}$ has instead the components:

$(-\frac{1}{2}Bz, 0, 0)$

Thus the Poisson bracket identity does not satisfy.

Poisson bracket identity can also be written as:

$$[F_i, L_j n_j] = \epsilon_{ijk} n_j F_k$$

$$\rightarrow \quad [F_i, L_j] = \epsilon_{ijk} F_k$$

For dot product of two system vectors $F.G$ we have;

$$[F.G, L.n] = F.[G, L.n] + G.[F, L.n]$$

$$= F.(n \times G) + G.(n \times G)$$

$$= F.n \times G + F.G \times n = 0$$

Now we apply the Poisson bracket identity for L;

$F = L$

$$\rightarrow [L, L.n] = n \times L$$

or $\quad [L_i, L_j] = \epsilon_{ijk} L_k$

and $\quad [L^2, L.n] = 0$

Using the Poisson's theorem, which states that "the Poisson bracket of any two consts of motion is also a const. of the motion (even if the conserved quantities depend on time explicitly)" we have;

If $\begin{cases} L_x \\ L_y \end{cases}$ are consts. of motion $\longrightarrow$ $[L_x, L_y] = L_z$ is also const of motion

$\longrightarrow$ the total angular momentum is conserved.

If in addition to L_x and L_y being conserved, there is a Cartezian vector of canonical momentum P with P_z const. of motion;

Then $\longrightarrow$ $[P_z, L_x] = P_y$ (conserved)

$[P_z, L_y] = -P_x$ "

$\longrightarrow$ both $\bar{L}$ and $\bar{P}$ are conserved.

But instead if P_x, P_y and L_z were the consts. of motion then their Poisson brackets are

$[P_x, P_y] = 0$
$[P_x, L_z] = P_y$
$[P_y, L_z] = P_x$

No new const. can be obtained from Poisson's theorem.

We showed before

$$[P_i, P_k]_{q,p} = 0$$

but from the Poisson bracket identity:

$$[L_i, L_j] \neq 0$$

Thus while $L = \sum_i \bar{r}_i \times \bar{p}_i$ canonical angular momentum.

(1) No two components of L can simultaneously be canonical variables

(2) However any components of $\bar{L}$ and its magnitude L, can be chosen to be canonical variables at the same time. (due to $[L^2, L \cdot n] = 0$)

Remark: Correspondence between quantum and classical mechanics:

$$\underset{\text{Poisson brackets}}{\overset{\text{Cl. M.}}{[\quad]}} \xrightarrow[h \to 0]{} \underset{\text{commutators}}{\overset{\text{Q. M.}}{[\quad]}}$$

Fact (1) $\longrightarrow$ L_i and L_j $(i \neq j)$ can not have simultaneous eigen-values.

Fact (2) $\longrightarrow$ L_i and L^2 can be quantized together