Chapter 8

The Hamilton Equs. of motion

8-1 Legendre transformation and the Hamilton equs. of motion -

We consider holonomic systems and monogenic forces (i.e. forces drived from V(r only))

In nonrelativistic limit: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad (i=1,\ldots n)$

There are n- <u>Second order</u> equs. $\rightarrow$ $2n$ initial values

initial values $\begin{cases} n \; q_i\text{'s} \\ n \; \dot{q}_i\text{'s} \end{cases}$ at t_0 or $\{ n \; q_i\text{'s}$ at t_1 and t_2

There are n- independent variable q_i and $\dot{q}_i$ (totally $2n$)

$q_i = q_i(t, C_{i1}, C_{i2})$

q_i and $\dot{q}_i$ are independent (inspite $\dot{q}_i = \frac{dq}{dt}$)

Because the coord. q doesn't give any information about the $\dot{q}$.

The Hamilton formulation (different picture) -

There are $2n$- <u>first order</u> equs.

$\rightarrow$ $2n$-independent variables

we choose n-q_i's and n-P_i's as the unknown-variables.

$$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i} \quad \text{canonical momentum}$$

q, p: canonical variables

If the forces are velocity dependent $\rightarrow P_c \neq P_M$

canonical Momentum   Mechanical Momentum

Lagrangian formalism $\longrightarrow$ Hamiltonian formalism

$$(q_i, \dot{q}_i, t) \longrightarrow (q_i, P_i, t)$$

The procedure for switching variables in this manner is provided by the Legendre Transformation.

Legendre tranformation:

Consider a func. of only two-variables, $f(x,y)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$u = \frac{\partial f}{\partial x}, \ V = \frac{\partial f}{\partial y} \longrightarrow df = u\,dx + V\,dy$$

We wish to change the basis of description in the following way: $x, y \longrightarrow u, y$

$\longrightarrow$ differential quantities $\xrightarrow{\text{in terms of}}$ du, dy

Let $g = f - ux$ (Legendre tr.) $g = g(u,y)$

$$dg = df - u\,dx - x\,du$$

$$dg = u\cancel{dx} + V\,dy - u\cancel{dx} - x\,du = V\,dy - x\,du$$

This has the desired form $dg = (\ \ )dy + (\ \ )du$

But $dg = \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial u} du$

$$\rightarrow \quad V = \frac{\partial g}{\partial y} \quad , \quad x = -\frac{\partial g}{\partial u}$$

Ex. - compare

$$\begin{array}{l} f \longrightarrow g \\ L \longrightarrow H \end{array}$$

$$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot q} d\dot q + \frac{\partial L}{\partial t} dt$$

$dL = V\,dq + u\,d\dot q + \frac{\partial L}{\partial t} dt$ where $u = \frac{\partial L}{\partial \dot q}$, $V = \frac{\partial L}{\partial q}$

$-H = L - u\dot q \quad \rightarrow -dH = dL - u\,d\dot q - \dot q\,du$

$$\rightarrow -dH = \frac{\partial L}{\partial q} dq + \cancel{\frac{\partial L}{\partial \dot q} d\dot q} + \frac{\partial L}{\partial t} dt - \cancel{\frac{\partial L}{\partial \dot q} d\dot q} - \dot q\, d\left(\frac{\partial L}{\partial \dot q}\right)$$

$$-dH = \frac{\partial L}{\partial q} dq - \dot q\, d\left(\frac{\partial L}{\partial \dot q}\right) + \frac{\partial L}{\partial t} dt$$

But $\frac{\partial L}{\partial \dot q} = p \qquad \frac{\partial L}{\partial q} = \dot p \quad \rightarrow dH = -\dot p\,dq + \dot q\,dp - \frac{\partial L}{\partial t} dt$

This is the desired form.

also $\longrightarrow \quad \dot p = \frac{\partial H}{q} \quad , \quad \dot q = -\frac{\partial H}{\partial p}$

Because; $dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial t} dt$

Transformation from $(q,\dot{q},t) \xrightarrow{\text{to}} (q,P,t)$

$$H(q,P,t) = \underset{\substack{\uparrow \\ \text{by convention}}}{-}\left[L(q,\dot{q},t) - \sum_{i=1}^{n} \dot{q}_i \underbrace{\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i}}_{P_i}\right]$$

Here we have more than one variable (n variable) to be transformed.

(1) $$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial P_i} dP_i + \frac{\partial H}{\partial t} dt$$

on the other hand;

(2) $$dH = -\frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt + \dot{q}_i dP_i + P_i d\dot{q}_i$$

Since def. of the generalized momentum $P_i = \frac{\partial L}{\partial \dot{q}_i}$ then the term $d\dot{q}_i$ is canceled.

From the lagrange's equ.; $\frac{\partial L}{\partial q_i} = \dot{P}_i$

(3) $$\rightarrow dH = \dot{q}_i dP - \dot{P}_i dq - \frac{\partial L}{\partial t} dt$$

(1),(3) $\rightarrow$ $$\dot{q}_i = \frac{\partial H}{\partial P_i}, \quad -\dot{P}_i = \frac{\partial H}{\partial q_i}, \quad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \qquad (2n+1 \text{ equs.})$$

These $2n+1$ equs. are Known as Cannonical equs. of Hamilton.

The constitute $2n$- first order equs. of motion

The hamiltonian H and the energy func. h have the same numerical value, but their functional dependence are different.

$$H \equiv H(q,P,t) \quad , \quad h \equiv h(q,\dot{q},t)$$

Remember: $h = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$

How to obtain the $H(q,p,t)$

I – Write down the Lagrangian $L(q,\dot{q},t)$

II – Use the def. $p_i = \frac{\partial L(q_i,\dot{q}_i,t)}{\partial \dot{q}_i}$ to obtain $\dot{q}_i \equiv \dot{q}_i(q,p,t)$

III – Construct $H = \sum_i p_i \dot{q}_i - \mathcal{L}(q,\dot{q},t)$

IV – Eliminate $\dot{q}_i$ from H using the results of II –

In many problems: $L = L_2 + L_1 + L_0$

If the forces are derivable from the potentials not involving the velocities, the Lagrangian has the above form.

Even for some of the velocity-dependent potentials the Lagrangian has the above form.

Now, $H(q,p,t) = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \sum_i \dot{q}_i p_i - L(q,\dot{q},t)$

using the Euler's theorem;

$$\sum_i x_i \frac{\partial f}{\partial x_i} = n f \qquad f\text{: homogeneous func. of deg. } n \text{ in } x_i$$

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 2L_2 + L_1$$

$$H = 2L_2 + L_1 - L = L_2 - L_0$$

and since $T = M_0 + \sum_j M_j \dot{q}_j + \frac{1}{2}\sum_{j,k} M_{jk} \dot{q}_j \dot{q}_k$

$$T = T_0 + T_1 + T_2$$

where $M_0 = \sum_i \frac{1}{2} m_i \left(\frac{\partial \bar{r}_i}{\partial t}\right)^2 \qquad M_j = \sum_i m_i \frac{\partial \bar{r}_i}{\partial t} \cdot \frac{\partial \bar{r}_i}{\partial q_j}$

$$M_{jk} = \sum_i m_i \frac{\partial \bar{r}_i}{\partial q_j} \cdot \frac{\partial \bar{r}_i}{\partial q_k}$$

If the eqns. defining the generalized coords.,

$$\bar{r}_i = \bar{r}_i(q_1, \ldots q_{3N-k}, t) \qquad i=1, \ldots n$$

do not depend on time explicitly, then

$$T = T_2$$

If further, the potential does not depend on the generalized velocities, then

$$L = T - V \qquad L_0 + L_1 + L_2 = T_0 + T_1 + T_2 - V$$

$$L_0 + L_1 + L_2 = T_2 - V(r)$$

quadratic terms of L.H.S. = quadratic terms of R.H.S.

$$L_2 = T_2 = T$$

Coord.-dep terms of L.H.S. = Coord.-dep terms of R.H.S

$$L_0 = -V$$

$$H = 2L_2 - L_0 = T + V = E \quad \text{total energy}$$

In this case, much of the algebra in step III is eliminated.

Remember if the potential V does not depend on expilicitly on t $\xrightarrow{\text{then}}$ L does not depend on t explicitly

then, since $\frac{dh}{dt} = -\frac{\partial L}{\partial t} = 0$

h is conserved.

Hamilton's equs. of motion do not treat the coords. and momenta in a completely symmetric fashion.

$$\dot{q}_i = +\frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

To bring this treatment in a symmetric form;

For a system of n-deg of freedom we construct column matrix η with 2n elements such that:

$$\begin{cases} \eta_i = q_i & i = 1 \cdots n \\ \eta_i = p_i & i = n+1 \cdots 2n \end{cases}$$

$$\left(\frac{\partial H}{\partial \eta}\right)_i = \frac{\partial H}{\partial q_i} \qquad \left(\frac{\partial H}{\partial \eta}\right)_{i+n} = \frac{\partial H}{\partial p_i} \qquad i \leq n$$

We also introduce: $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \cdots & & - \\ 0 & \cdots & & 0 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \\ \vdots & & & & \\ 0 & \cdots & & - & 1 \end{pmatrix} \qquad n \times n$$

$$J^2 = -I \qquad \tilde{J} J = I \qquad \tilde{J} = -J = J^{-1} \qquad |J| = +1$$

$$\longrightarrow \quad \dot{\eta} = J \frac{\partial H}{\partial \eta} \quad \text{Hamilton's equs of motion;}$$

Ex. $n = 2$

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \\ \dot{\eta}_4 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \frac{\partial H}{\partial q_2} \\ \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix}$$

Conservation Theorems:

Since $$\dot{p}_j = \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j}$$

a coord. that is syclic will thus also be absent from the Hamiltonian.

If $\frac{\partial H}{\partial q_j} = 0 \longrightarrow p_j = \text{const}$

All conservation theorems discussed in lagrangian formulation are valid also in Hamiltonian formulation. (only change $L \to H$)

$$\frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i}_{0} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

But $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

Thus if $\frac{\partial L}{\partial t} = 0 \longrightarrow \frac{\partial H}{\partial t} = 0 \to H = \text{const}.$

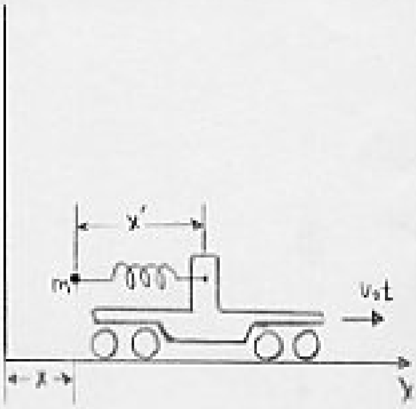
$$\text{If } \begin{cases} \frac{\partial r_n}{\partial t} = \frac{\partial r_n(q_1 \cdots q_n, t)}{\partial t} = 0 \\ \frac{\partial V}{\partial \dot{q}_i} = 0 \end{cases} \longrightarrow H = T + V \quad \text{total energy}$$

$$\text{If } \begin{cases} \frac{\partial r_m}{\partial t} \neq 0 \\ \frac{\partial H}{\partial t} = 0 \end{cases} \longrightarrow H = \text{const} \quad \text{not total energy}$$

H is dependent in $\begin{cases} \text{1- magnitude} \\ \text{2- functional form} \end{cases}$ on the initial choice of generalized coords.

While for the L with the prescription $L = T - V$, a change of $q_i \to q_i'$ may change the functional appearance of L, but not its magnitude.

Ex. - A point mass is attached to a spring fixed to a uniformally moving cart. (Cart is massless).



There exists external force to keep the motion of cart uniform.

$$L(x,\dot{x},t) = T - V = \frac{m\dot{x}^2}{2} - \frac{k}{2}(x - v_0 t)^2$$

At $t=0$ the cart passes through the origin.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \longrightarrow m\ddot{x} + k(x - v_0 t) = 0$$

let; $\quad x' = x'(t) = x - v_0 t \quad \rightarrow m\ddot{x}' - kx' = 0$

x': displacement of the particle relative to the cart

thus, in a coord. system fixed on cart the motion of the particle is simple harmonic.

Since $\begin{cases} 1\text{- } x \text{ is cartezian coord.} \\ 2\text{- } \dfrac{\partial V}{\partial \dot{x}} = 0 \end{cases} \longrightarrow H(q,p,t) = T + V$ total energy

$$H(q,p,t) = \frac{p^2}{2m} + \frac{k}{2}(x - v_0 t)^2$$

We see that $\dfrac{\partial H}{\partial t} \neq 0$ not conserved.

Physically this is understandable; energy must flow into and out of the external physical device to keep the cart moving uniformally against the reaction of the oscillating particle.

Indeed the moving cart constitutes a time-dep constraint on the particle and the force of constraint does do work in actual (not virtual) displacement of the system.

Now we consider the system in x'-coord.:

$$L(x',\dot{x}',t) = \frac{1}{2}m(\dot{x}'+v_0)^2 - \frac{k}{2}x'^2 = \frac{m\dot{x}'^2}{2} + m\dot{x}'v_0 + \frac{mv_0^2}{2} - \frac{k}{2}x'^2$$

$$p' = \frac{\partial L}{\partial \dot{x}'} \qquad p' = m\dot{x}' + mv_0 \longrightarrow \dot{x}' = \frac{p'-mv_0}{m}$$

$$H(x',p',t) = \dot{x}'p' - L = \left(\frac{p'-mv_0}{m}\right)p' - \left[\frac{m}{2}\left(\frac{p'-mv_0}{m}\right)^2 + m\left(\frac{p'-mv_0}{m}\right)v_0 + \frac{mv_0^2}{2} - \frac{k}{2}x'^2\right]$$

$$H(x',p',t) = \underbrace{\frac{(p'-mv_0)^2}{2m} + \frac{kx'^2}{2}}_{\text{total energy relative to the cart}} - \underbrace{\frac{mv_0^2}{2}}_{=\text{const, can be dropped}}$$

$$\begin{cases} \dfrac{\partial H}{\partial t} = 0 & \text{conserved} \\ H: \text{not the total energy} \end{cases}$$

$H(x,p,t)$ and $H(x',p',t)$ are different in $\begin{cases} \text{1- magnitude} \\ \text{2- t-dep} \\ \text{3- functional behavior} \end{cases}$

But both $\xrightarrow{\text{lead}}$ to the same motion

8-5 Derivation of Hamilton's Equs. from a Variational Principle

Remember the Hamilton's principle:

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0 \qquad \text{(in configration space)}$$

$$\delta I = \delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q,p,t))\,dt = 0 \qquad \text{(in phase space)}$$

$$\delta t = 0 \quad , \quad \delta q\big|_1^2 = 0 \quad , \quad \delta p\big|_1^2 = 0$$

Thus any varied path must be in the neighborhood of this phase-space trajectory.

P and q must be treated as indep.-coord. of phase-space.

Note: Variational principle in phase space, is sometimes refered to the <u>Modified Hamilton's principle</u>.

The problem is in the form of:

$$\delta I = \delta \int_{t_1}^{t_1} f(q,\dot{q},p,\dot{p},t)\,dt = 0 \qquad 2n\text{-dim.}$$

The 2n-Euler Lagrange equs. are:

(1) $$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_j}\right) - \frac{\partial f}{\partial q_j} = 0 \qquad j=1,\dots n$$

(2) $$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_j}\right) - \frac{\partial f}{\partial p_j} = 0 \qquad j=1,\dots n$$

f contains $\dot{q}_j$ through the $\underbrace{p_i \dot{q}_i}$ and q_j only in H.

let $f = \sum_i p_i \dot{q}_i - H$

(1) $\longrightarrow \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$, (2) $\longrightarrow \quad \dot{q}_j = \frac{\partial H}{\partial p_j}$

1-5 Velocity-dep. Potentials:

In the case the potentials are velocity-dep.;

$$V \longrightarrow U(q_i, \dot{q}_i)$$

If the generalized forces Q_i are obtained from $U(q_i\ \dot{q}_i)$ by the prescription;

$$Q_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right)$$

Substituting in;

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i$$

$$\rightarrow \quad \frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_i}\right) - \frac{\partial (T-U)}{\partial q_i} = 0$$

$\longrightarrow \quad L = T - U$ $\qquad$ U: generalized velocity-dep. potential.

Remark:

Using D'Alembert principle, we obtained;

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i$$

$$\begin{cases} \text{If } \ F_j = -\nabla_j V \longrightarrow Q_i = \sum_j \bar{F}_j \cdot \frac{\partial \bar{r}_j}{\partial q_i} = -\sum_j \nabla_j V \cdot \frac{\partial r_j}{\partial q_i} \\ \text{where } \ V = V(\bar{r}_1, \bar{r}_2 \cdots \bar{r}_n, t) \end{cases}$$

$$Q_i = -\frac{\partial V}{\partial q_i} \longrightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial (T-V)}{\partial q_i} = 0$$

$$\text{Since } V \neq V(\dot{q}_i) \rightarrow \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_i}\right) - \frac{\partial (T-V)}{\partial q_i} = 0$$

$$\rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad L = T - V$$

Electromagnetic forces on a charged particle are derivable from velocity-dep. potentials.

In Gaussian units the Maxwell equs.;

$$\begin{cases} (1)\quad \nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 & (3)\quad \nabla \cdot D = 4\pi\rho \\ (2)\quad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c} J & (4)\quad \nabla \cdot B = 0 \end{cases}$$

Lorentz force;

$$F = q\left\{E + \frac{1}{c}(V \times B)\right\}$$

B can be represented by $\quad B = \nabla \times A$

$$(1) \longrightarrow \nabla \times E + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times A) = 0 \rightarrow \nabla \times \left(E + \frac{1}{c}\frac{\partial A}{\partial t}\right) = 0$$

$$\rightarrow E + \frac{1}{c}\frac{\partial A}{\partial t} = -\nabla\varphi$$

$$\rightarrow E = -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}$$

$$\begin{cases} \varphi = \varphi(r,t) \\ B = B(r,t) \\ A = A(r,t) \end{cases}$$

$$F = q\left\{-\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t} + \frac{1}{c}(V \times (\nabla \times A))\right\}$$

Now we consider the x-component of F;

$$[V \times (\nabla \times A)]_i = \sum_{j=1}^{3} V_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)$$

$$= V_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - V_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

$$= V_y\frac{\partial A_y}{\partial x} + V_z\frac{\partial A_z}{\partial x} + \underbrace{V_x\frac{\partial A_x}{\partial x}}_{\text{We have added}} - V_y\frac{\partial A_x}{\partial y} - V_z\frac{\partial A_x}{\partial z} - \underbrace{V_x\frac{\partial A_x}{\partial x}}$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \left(V_x \frac{\partial A_x}{\partial x} + V_y \frac{\partial A_x}{\partial y} + V_z \frac{\partial V_z}{\partial z} \right)$$

$$\longrightarrow \left(V \times (\nabla \times A) \right)_x = \frac{\partial}{\partial x}(V \cdot A) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$$

Also $\quad (\nabla \varphi)_x = \dfrac{\partial \varphi}{\partial x}$

$$F_x = q \left\{ -\frac{\partial}{\partial x}\left(\varphi - \frac{1}{c} V \cdot A \right) - \frac{1}{c}\frac{d}{dt}\left(\frac{\partial}{\partial V_x}(A \cdot V) \right) \right\}$$

$$\rightarrow F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial V_x}$$

$$\longrightarrow U = q\varphi - \frac{q}{c} A \cdot V$$ generalized pot.

$$L = T - q\varphi + \frac{q}{c} A \cdot V$$

$$L = \frac{1}{2} m V^2 - q\varphi + \frac{q}{c} A \cdot V$$

Remark: $\dfrac{1}{c}\dfrac{d}{dt}\left(\dfrac{\partial}{\partial V_x}(A \cdot V) \right) = \dfrac{1}{c}\dfrac{dA_x}{dt}$

$$L(\bar{r},\dot{\bar{r}},t) = \frac{1}{2}m\dot{r}^2 - e\varphi(\bar{r},t) + \frac{e}{c}\dot{r}\cdot A(\bar{r},t)$$

where, in cgs units:

$$\bar{B}(\bar{r},t) = \bar{\nabla}\times A(\bar{r},t)$$
$$\bar{E}(\bar{r},t) = -\bar{\nabla}\varphi(\bar{r},t) - \frac{1}{c}\frac{\partial}{\partial t}A(\bar{r},t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \qquad i=1,2,3$$

$$\begin{cases} \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{e}{c}A_i(\bar{r},t) \\ \frac{\partial L}{\partial x_i} = -e\frac{\partial}{\partial x_i}\varphi(\bar{r},t) + \frac{e}{c}\sum_{j=1}^{3}\dot{x}_j\frac{\partial}{\partial x_i}A_j(\bar{r},t) \\ \frac{d}{dt}A_i(\bar{r},t) = \sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}A_i(r,t)\right)\dot{x}_j + \frac{\partial}{\partial t}A_i(\bar{r},t) \end{cases}$$

Lagrange's equs.:

$$m\ddot{x}_i = -e\left(\frac{\partial\varphi}{\partial x_i} + \frac{1}{c}\frac{\partial A_i}{\partial t}\right) + \frac{e}{c}\sum_{j=1}^{3}\dot{x}_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)$$

$$m\ddot{x}_i = e\left[(-\nabla\varphi)_i + \left(\frac{1}{c}\frac{\partial A}{\partial t}\right)_i\right] + \frac{e}{c}\left[\dot{r}\times(\nabla\times A)\right]_i$$

$$\rightarrow\ m\ddot{x}_i = e\left[E_i(r,t) + \frac{1}{c}\left(\dot{r}\times B(r,t)\right)_i\right]$$

$$\longrightarrow\ m\ddot{r} = e\left[E(r,t) + \frac{1}{c}r\times B(r,t)\right]$$

The problem in Hamiltonian formalism:

$$H = \sum_{i=1}^{3} p_i \dot{x}_i - L$$

$$p = \frac{\partial L}{\partial \dot{r}} = m\dot{r} + \frac{e}{c} A(r,t)$$ canonical momentum

$m\dot{r}$: mechanical momentum

$\frac{e}{c} A(r,t)$: electromagnetic contribution

If for example $\frac{\partial L}{\partial z} = 0 \longrightarrow P_z = \text{const.}$

$\longrightarrow P_z = m v_z + \frac{e}{c} A_z = \text{const}$ (const. of motion)

Here instead $m v_z$, the quantity $m v_z + \frac{e}{c} A_z$ is const., which reflects the presence of electromagnetic forces.

$$H = \frac{1}{2} m \dot{r}^2 + e\varphi(r,t) \longrightarrow H(r, \dot{r}, t)$$

H = Kinetic energy + Potential energy

However $H \neq$ const. of motion

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \neq 0$$ if E and B vary with time.

$$H(r,\dot{r},t) \longrightarrow H(r,P,t) = \frac{1}{2m} \left| P - \frac{e}{c} A(r,t) \right|^2 + e\varphi(r,t)$$

Hamilton's eqns:

$$\begin{cases} \dfrac{\partial H}{\partial p_i} = \dot{x}_i & \longrightarrow m\dot{x}_i = p_i - \frac{e}{c} A_i(r,t) \quad (1) \\ \dfrac{\partial H}{\partial x_i} = -\dot{p}_i & \longrightarrow \dot{p}_i = -e\frac{\partial}{\partial x_i}\varphi + \frac{e}{mc}\sum_{j=1}^{3}\left(p_j - \frac{e}{c}A_j\right)\left(\frac{\partial}{\partial x_i}A_j\right) \quad (2) \end{cases}$$

$$(1)\text{ in }(2) \longrightarrow \quad \dot{p}_i = -e\frac{\partial}{\partial x_i}\varphi + \frac{e}{c}\sum_{j=1}^{3}\dot{x}_j\left(\frac{\partial}{\partial x_i}A_j\right) \quad (3)$$

$$(1) \longrightarrow \quad m\ddot{x}_i = \dot{p}_i - \frac{e}{c}\frac{d}{dt}A_i(r,t) \quad (4)$$

Since; $$\frac{d}{dt}A_i(r,t) = \sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}A_i(r,t)\right)\dot{x}_j + \frac{\partial}{\partial t}A_i(r,t) \quad (5)$$

$$(3)(5)\text{ in }(4) \longrightarrow \quad m\ddot{r} = e\left[E(r,t) + \frac{1}{c}\dot{r}\times B(r,t)\right]$$