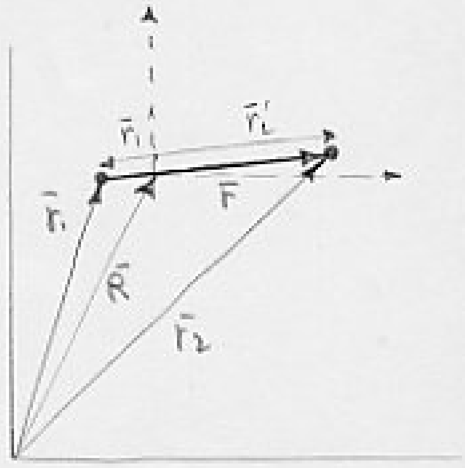
Chapter 3

The Two-body, Central Force Problem

3-1 Reduction of the problem to the equivalent one-body problem:



There exist 6-independent generalized coords. for two-body system

Let us choose:

$$\begin{cases} \bar{r} = \bar{r}_2 - \bar{r}_1 & \text{relative coords. (3-in number)} \\ \bar{R} & \text{center of mass " (" ")} \end{cases}$$

$$\begin{cases} \bar{r} = \bar{r}_2 - \bar{r}_1 \\ \bar{r} = \bar{r}_2' - \bar{r}_1' \\ \bar{R} = \dfrac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1+m_2} \end{cases} \qquad \text{If } \bar{R} = 0 \longrightarrow m_1\bar{r}_1' + m_2\bar{r}_2' = 0$$

$$\begin{cases} m_1\bar{r}_1' + m_2\bar{r}_2' = 0 \\ \bar{r} = \bar{r}_2' - \bar{r}_1' \end{cases} \longrightarrow \begin{cases} \bar{r}_1' = -\dfrac{m_2}{m_1+m_2}\bar{r} \\ \bar{r}_2' = \dfrac{m_1}{m_1+m_2}\bar{r} \end{cases}$$

The Lagrangian;

$$L = T(\dot{R}, \dot{r}) - U(r, \dot{r}, \ddot{r}, \cdots)$$

where

$$T = \underbrace{\tfrac{1}{2}(m_1+m_2)\dot{\bar{R}}^2}_{\text{K.E. of the center of mass}} + \underbrace{T'}_{\text{K.E. of motion about the center of mass}}$$

$$T' = \frac{1}{2} m_1 \dot{\bar{r}}_1'^2 + \frac{1}{2} m_2 \dot{\bar{r}}_2'^2$$

$$\rightarrow T' = \frac{1}{2} \frac{m_1 m_2}{m_1+m_2} \dot{\bar{r}}^2$$

$$T = \frac{1}{2}(m_1+m_2)\dot{\bar{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1+m_2} \dot{\bar{r}}^2$$

$$L = \frac{1}{2}(m_1+m_2)\dot{\bar{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1+m_2} \dot{\bar{r}}^2 - U(\bar{r}, \dot{\bar{r}}, \dots)$$

L is cyclic in $\bar{R} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{R}}}\right) = 0$

$\frac{d}{dt} \bar{P}_R = 0 \quad \bar{P}_R = \text{Const}$

Center of mass is either at rest or moving uniformally.

Therefore we drop the first term of the Lagrangian.

$$L' = \frac{1}{2} \mu \dot{\bar{r}}^2 - U$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ reduced mass

Thus the central force motion of two bodies about their center of mass can always be reduced to an equivalent One-body problem.

3-2 The equ. of motion

Since $\dot{\bar{r}}^2 = (\dot{r}^2 + r^2\dot{\theta}^2) \longrightarrow L' = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$ (Central force, $\bar{F}$ along $\bar{r}$)

$V(r)$: central force at the origin of coord. system

Since $V = V(r)$ (r-dep. only) the problem has spherical symmetry

$\longrightarrow \bar{L} = \bar{r} \times \bar{p}$ is conserved. $\rightarrow r \perp \bar{L}$ (motion on a plane)

If $\bar{L} = \bar{r} \times \bar{p} = \bar{r} \times m\dot{\bar{r}} = 0 \rightarrow \bar{r} \parallel \dot{\bar{r}} \rightarrow$ motion on a straight line.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_k} - \frac{\partial L}{\partial q_k} = 0$$

$$q_1 = \theta \qquad \begin{cases} P_\theta = \dfrac{\partial L}{\partial \dot\theta} = \mu r^2 \dot\theta \\ \dfrac{\partial L}{\partial \theta} = 0 \end{cases}$$

$$\frac{d}{dt} P_\theta = \dot P_\theta = \frac{d}{dt}(m r^2 \dot\theta) = 0 \quad \rightarrow m r^2 \dot\theta = \text{const} = \ell$$

Remark: $\bar V = \dot r \hat e_r + r\dot\theta \hat e_\theta$

$$\bar L = \bar r \times \bar p = \bar r \times m \bar v = r\hat e_r \times m(\dot r \hat e_r + r\dot\theta \hat e_\theta)$$

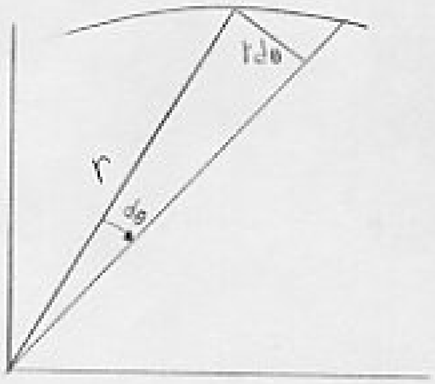
$$= 0 + m r^2 \dot\theta \underbrace{\hat e_r \times \hat e_\theta}_{1}$$

$$|L| = m r^2 \dot\theta = \ell$$

Also:

$$dA = \frac{1}{2} r (r\,d\theta)$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot\theta$$



But $\frac{d}{dt} P_\theta = \frac{d}{dt}(m r^2 \dot\theta) = 0$

$\rightarrow \frac{d}{dt}\left(\frac{1}{2} r^2 \dot\theta\right) = 0$

dA = The area swept out by the radius vector in a time dt.

$\rightarrow$ The areal velocity $= \frac{1}{2} r^2 \dot\theta$

(the area swept by the radius vector per unit time)

is const.

This is Kepler's second law.

However $\frac{dA}{dt} = 0$ is a general property for central force motion ($V(r)$) and is not restricted to an inverse square law of force.

The other equ of motion;

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \qquad \begin{cases} \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V(r)}{\partial r} \end{cases}$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

Designate, $f(r) = -\frac{\partial V}{\partial r}$ (along $\bar{r}$)

$$m\ddot{r} - mr\dot{\theta}^2 = f(r)$$

but $mr^2\dot{\theta} = l \qquad \dot{\theta} = \frac{l}{mr^2}$

$\rightarrow m\ddot{r} = \frac{l^2}{mr^3} + f(r) \qquad (m\ddot{r} : f_{eff})$

$\rightarrow m\ddot{r} = -\frac{d}{dr}\left(\underbrace{\frac{1}{2}\frac{l^2}{mr^2}}_{\text{centrifugal force}} + \underbrace{V}_{\text{central force}}\right) \qquad \left(V_{eff} = V + \frac{l^2}{2mr^2}\right)$

multiply both sides by $\dot{r}$:

$$m\ddot{r}\dot{r} = -\dot{r}\frac{d}{dr}\left(\frac{1}{2}\frac{l^2}{mr^2} + V\right)$$

L.H.S. $m\ddot{r}\dot{r} = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right)$

Since $\frac{d}{dt}g(r) = \frac{dg}{dr}\frac{dr}{dt}$

R.H.S. $\dot{r}\frac{d}{dr}\left(\frac{1}{2}\frac{\ell^2}{mr^2}+V\right)=\frac{dr}{dt}\frac{d}{dr}\left(\frac{1}{2}\frac{\ell^2}{mr^2}+V\right)$

$$=\frac{d}{dt}\left(V+\frac{\ell^2}{2mr^2}\right)$$

$$\longrightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right)=-\frac{d}{dt}\left(V+\frac{\ell^2}{2mr^2}\right)$$

$$\rightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2+\frac{1}{2}\frac{\ell^2}{mr^2}+V\right)=0$$

$$\rightarrow \frac{1}{2}m\dot{r}^2+\frac{1}{2}\frac{\ell^2}{mr^2}+V = \text{const} = E$$ energy conservation.

Proof. - $\frac{1}{2}\frac{\ell^2}{mr^2}=\frac{1}{2mr^2}m^2r^4\dot{\theta}^2=\frac{mr^2\dot{\theta}^2}{2}$

$$\rightarrow \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+V = E$$

$$T + V = E$$

Since there exist 2 variables r and θ, therefor 4 integrations are needed.

The first two integrations have left the Lagrange's equs as two first order equs.:

$$\begin{cases} mr^2\dot{\theta}=\ell & (1) \\ \frac{1}{2}m\dot{r}^2+\frac{\ell^2}{2mr^2}+V(r)=E & (2) \end{cases}$$

$$(2) \rightarrow \dot{r}=\sqrt{\frac{2}{m}\left(E-V-\frac{\ell^2}{2mr^2}\right)}$$

$$\rightarrow dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}}$$

let at $t=0$, $r = r_0$

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}}$$

Or inversely one can obtain $r = r(t)$

then if r is known as a fun of t, then

$$\ell = mr^2\dot{\theta} \longrightarrow d\theta = \frac{\ell\, dt}{mr^2}$$

if at $t=0$, $\theta = \theta_0$

$$\theta = \ell \int_0^t \frac{dt}{mr^2(t)} + \theta_0$$

Note: ℓ, E, r_0, θ_0 are the consts. of integrations.

We could consider another set of consts, like $r_0, \theta_0, \dot{r}_0, \dot{\theta}_0$, but of course E and ℓ can always be determined in terms of this set.

3-3 The equivalent one-dimensional Problem, and classification of orbits.

It is more convenient to perform the integration in some other fashion than we did for.

$$t = \int_{r_0}^{r_0} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}}$$

and $\quad \theta = \ell \int_0^t \frac{dt}{mr^2(t)} + \theta_0$

General information:

If E and ℓ (which are conserved) are known, then;

$E = \frac{1}{2}mv^2 + V(r) \longrightarrow v = \sqrt{\frac{2}{m}(E - V(r))}$ magnitude of velocity

we had also $\dot{r} = \sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}$ radial component of velocity

$\Longrightarrow$ we can get information about the direction of velocity

Alternatively conservation of $\ell = mr^2\dot{\theta}$, furnishes $\dot{\theta}$, this together with $\dot{r}$ gives both the magnitude and direction of $\dot{\mathbf{r}}$.

$$v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

The radial equ. $m\ddot{r} - \frac{\ell^2}{mr^3} = f(r)$ for the canteral motion is similar to the sol. for one-dimensional Poobl, in which a particle of mass m is subjected to a force

$$f' = f + \frac{\ell^2}{mr^3} = f + \frac{m^2 r^4 \dot{\theta}^2}{mr^3} = f + \underbrace{mr\dot{\theta}^2}_{\text{cantrifugal force}}$$

Alternatively, using the conseration of energy

$$\overbrace{\frac{1}{2} m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2}}^{T} + V = \text{const.} = E$$

$$\underbrace{\frac{1}{2}\frac{\ell^2}{mr^2} + V}_{V'}$$

$$\frac{1}{2} m\dot{r}^2 + V' = E \qquad \text{when } V' = V + \frac{1}{2}\frac{\ell^2}{mr^2}$$

The problem changes to a one-body problem with a fictitious potential V'

A check $\quad f' = -\frac{\partial V'}{\partial r} = f(r) + \frac{\ell^2}{mr^3}$

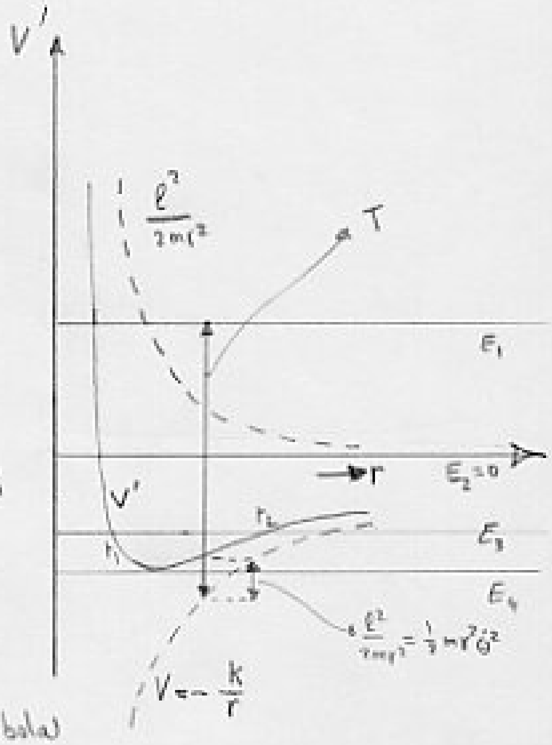
which agrees with the previous results.

Now consider an attractive inverse square law of force;

$$f = -\frac{k}{r^2} \qquad k > 0$$

$$\longrightarrow V = -\frac{k}{r}$$

$$V' = -\frac{k}{r} + \frac{\ell^2}{2mr^2}$$

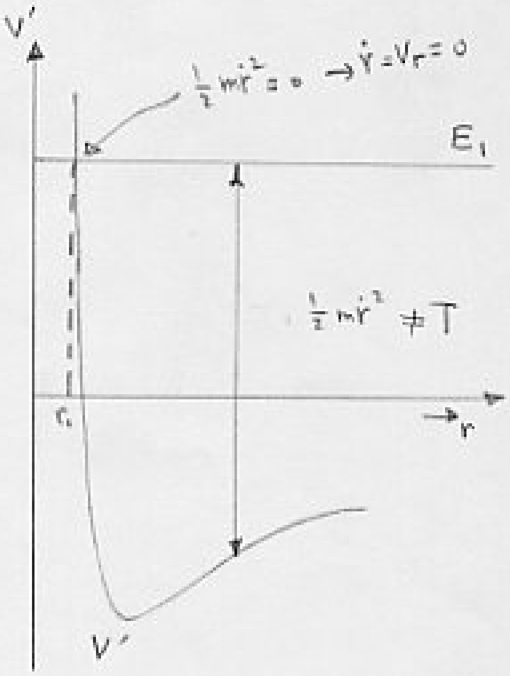


Now we consider a particle comming from infinity to the origin of force. We consider different total energies:

E_1: The particle face to a repulsive centrifugal barrier, and is repelled and travel back out to infinity. (hyperbola)

E_2: Similar to E_1 (Parabola)

E_3: The system is bounded between r_1 and r_2 (apsidal distances). The orbits are not necessarily closed. (ellipse)



E_4: $r_1 = r_2$ and $\dot r = 0$, and the orbit is a circle

$$f' = -\frac{\partial V'}{\partial r} = f(r) + \frac{\ell^2}{mr^3} = 0$$

$$f(r) = -\frac{\ell^2}{mr^3} = -mr\dot\theta^2$$

Remark: The min. point can be obtained by $\frac{\partial V'}{\partial r} = 0$

General qualitative division into open, bounded, and circular orbits will be true for any attractive potential that

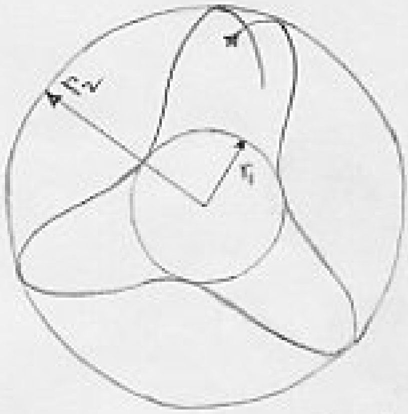
(1) falls off slower than $\frac{1}{r^2}$ as $r \longrightarrow \infty$

(2) becomes infinite slower than $\frac{1}{r^2}$ as $r \longrightarrow 0$

(1) $\longrightarrow$ ensures that the potential predominates over the centrifugal term for large r

(2) $\longrightarrow$ for small r, the centrifugal term is important

If the potential does not satisfy these requirements, the qualitative nature of the motion will be altered.



Bounded but not closed orbits

Ex.:

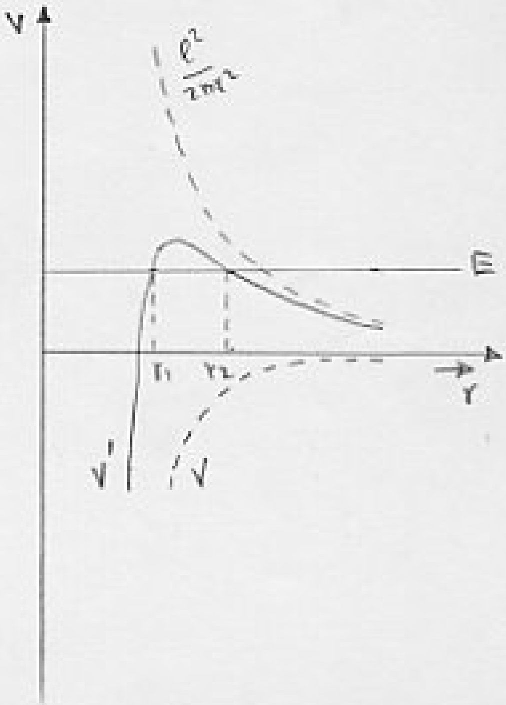
$$V = -\frac{a}{r^3} \longrightarrow f = -\frac{3a}{r^4}$$

For an energy E:

1) if $r_0 < r_1 \longrightarrow$ motion is bounded and always $r < r_1$

2) if initially $r_0 > r_2 \longrightarrow$ " " unbounded and always $r > r_2$

3) $r_1 < r_0 < r_2$ physically impossible because $V > E$ $(T_{min} = 0)$



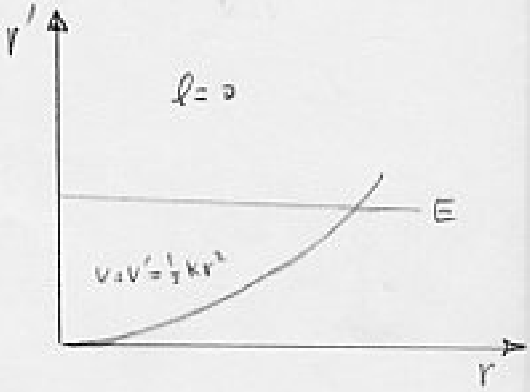
Ex.: Isotropic harmonic oscillator:

$$f = -kr \longrightarrow V = \frac{1}{2} k r^2$$

1) If $l = 0 \longrightarrow V' = V$

$\longrightarrow$ motion will be straight line.

For any $E > 0$ motion is bounded (simple harmonic).

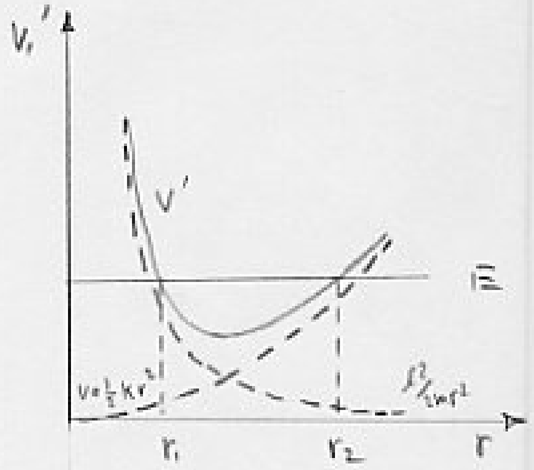


2) If $l \neq 0$

The motion is bounded for all physically possible energies, and does not pass through the center of force.

The orbit is ellipse.

$$\bar{f} = -k\bar{r} \longrightarrow \begin{cases} f_x = -kx \\ f_y = -ky \end{cases}$$



We found:

1) $mr^2\dot{\theta} = l \longrightarrow mr^2 d\theta = l\,dt \rightarrow \frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}$

2) also $m\ddot{r} - \frac{l^2}{mr^3} = f(r) \longrightarrow dt = \frac{dr}{\sqrt{\frac{2}{m}(E - V - \frac{l^2}{2mr^2})}}$

$$1) \rightarrow \frac{d^2}{dt^2} = \frac{l}{mr^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{d}{d\theta}\right)$$

$$2) \rightarrow m\frac{d^2r}{dt^2} - \frac{l^2}{mr^3} = f(r)$$

$$1), 2) \longrightarrow m\frac{l}{mr^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r)$$

$$\longrightarrow \frac{l^2}{mr^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r)$$

let $\frac{1}{r} = u \longrightarrow \frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$

$$-\frac{l^2}{mr^2}\frac{d}{d\theta}\frac{du}{d\theta} - \frac{l^2}{mr^3} = f(r) \longrightarrow -\frac{l^2}{m}u^2\frac{d^2u}{d\theta^2} - \frac{l^2}{m}u^3 = f(\tfrac{1}{u})$$

$$\longrightarrow \frac{l^2}{m}u^2\left(\frac{d^2u}{d\theta^2} + u\right) = -f(\tfrac{1}{u}) \longrightarrow \frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{1}{u^2}f(\tfrac{1}{u})$$

Also, $\frac{d}{du} = \frac{dr}{du}\frac{d}{dr} = -\frac{1}{u^2}\frac{d}{dr}$

$$\rightarrow \frac{dV(\frac{1}{u})}{du} = \frac{-1}{u^2}\frac{dV(\frac{1}{u})}{dr} \rightarrow u^2\frac{dV(\frac{1}{u})}{du} = -\frac{dV(\frac{1}{u})}{dr}$$

$$\rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2}\frac{1}{u^2}\left(-\frac{dV(\frac{1}{u})}{dr}\right)$$

$$\rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2}\frac{dV(\frac{1}{u})}{du}$$

Sol. for θ in terms of r:

(1) $\ell = mr^2\dot{\theta} \rightarrow \ell dt = mr^2 d\theta$

also we had $\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}$

$$\rightarrow dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E-V-\frac{\ell^2}{2mr^2}\right)}} \rightarrow \ell dt = \frac{\ell dr}{\sqrt{\frac{2}{m}\left(E-V-\frac{\ell^2}{2mr^2}\right)}}$$

using (1) $\rightarrow mr^2 d\theta = \dfrac{\ell dr}{\sqrt{\frac{2}{m}\left(E-V-\frac{\ell^2}{2mr^2}\right)}}$

$$\theta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}} + \theta_0$$

or

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - u^2}}$$

Chapter 6

Small Oscillations

6-1 Formulation of the problem

We consider conservative systems in which the potential energy is a func. of position only. $V = V(q_1, q_2 \cdots q_n)$

It is assumed that: $\frac{\partial q_i}{\partial t} = 0$ (not explicitly)

Thus $\longrightarrow$ time-dep. constraints are excluded.

The system is in equilibrium when:

$$Q_i = \left(\frac{\partial V(q_1, \cdots q_n)}{\partial q_i}\right)_0 = 0 \quad \text{generalized force}$$

$V(q_i)$ therefore has an extermum at the equilibrium configuration of the system $q_{10}, q_{20} \cdots q_{n0}$.

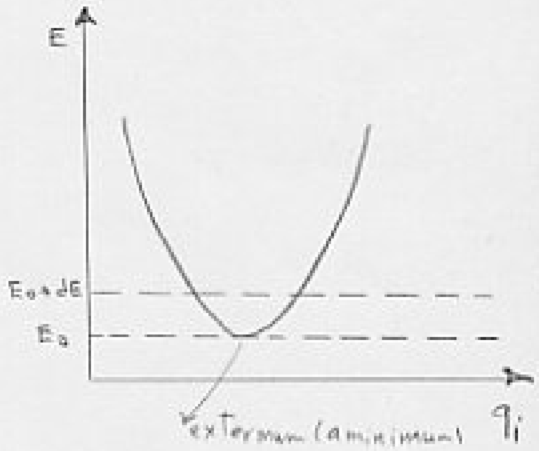
An Equilibrium Position is <u>stable</u> if a small disturbance of the system from equilibrium results only in <u>small bounded motion</u> about the rest Position.

The Equilibrium is <u>unstable</u> if an infinitesimal disturbance eventually produces unbounded motion.

Ex. Pendulum at rest (Stable)
Ex. the egg standing on end (Unstable)

If the extermum of the potential is a minimum, the equilibrium must be stable.

For a small departure from the equilibium, V can be expanded in a Taylor series

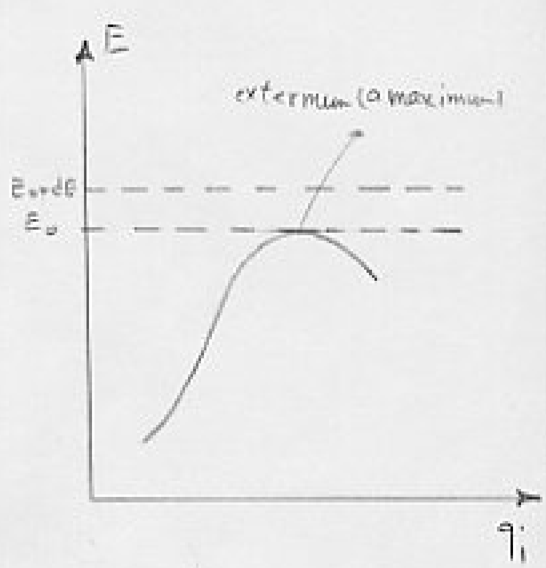


Suppose:

$$q_i = q_{i0} + \eta_i$$

η_i : infinitesimal departure

(new generalized coords.)



$$V(q_1, q_2 \cdots q_n) = V(q_{10}, q_{20} \cdots q_{n0}) + \sum_i \left(\frac{\partial V}{\partial q_i}\right)_{q_{i0}} \eta_i + \sum_{ij} \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j + \cdots$$

Due to $\left(\frac{\partial V}{\partial q_i}\right)_0 = 0$, (at equilibrium), the second term vanishes.

Remark: $V(q_{10}+\eta_1 \cdots q_{i0}+\eta_i) = V(\eta_{10} \cdots \eta_{n0}) + \left(\frac{\partial V}{\partial \eta_i}\right)_{\eta_{i0}} \eta_i + \left(\frac{\partial^2 V}{\partial \eta_i \partial \eta_j}\right)_0 \eta_i \eta_j + \cdots$

But $V(\eta_{10} \cdots \eta_{n0}) = V(q_{10} \cdots q_{n0})$ and $\left(\frac{\partial V}{\partial \eta_i}\right)_{\eta_{i0}} = \left(\frac{\partial V}{\partial q_i}\right)_{q_{i0}}$ and so forth.

Alternatively:

$$V(q_1 \cdots q_n) = V(q_{10} \cdots q_{n0}) + \left(\frac{\partial V}{\partial q_i}\right)_{q_{i0}} (q_i - q_{i0}) + \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 (q_i - q_{i0})(q_j - q_{j0}) + \cdots$$

$$= V(q_{10} \cdots q_{n0}) + \left(\frac{\partial V}{\partial q_i}\right)_{q_{i0}} \eta_i + \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right) \eta_i \eta_j + \cdots$$

By shifting the arbitrary zero of the potential the first term, may also made be vanish. Keeping the quadratic terms as the first approx.:

$$V(\eta_1 \cdots \eta_n) = \frac{1}{2} \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j = \frac{1}{2} \sum_{ij} V_{ij} \eta_i \eta_j$$

Obviously $V_{ij} = V_{ji}$ and real

Similarly:

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_j \frac{\partial \bar{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \bar{r}_i}{\partial t} \right)^2$$

$$T = M_0 + \sum_j M_j \dot{q}_j + \frac{1}{2} \sum_{jk} M_{jk} \dot{q}_j \dot{q}_k$$

$$M_0 = \sum_i \frac{1}{2} m_i \left(\frac{\partial \bar{r}_i}{\partial t} \right)^2$$

$$M_j = \sum_i m_i \frac{\partial \bar{r}_i}{\partial t} \cdot \frac{\partial \bar{r}_i}{\partial q_j}$$

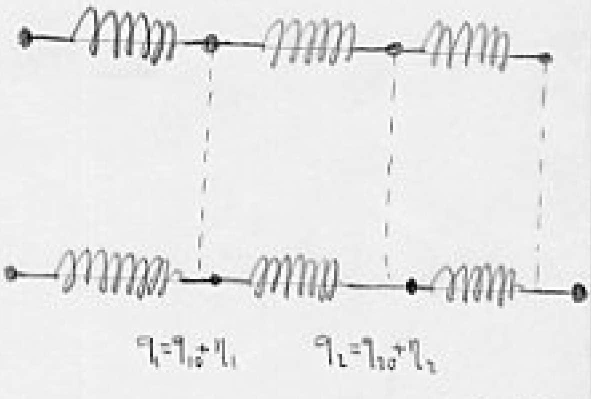
$$M_{jk} = \sum_i m_i \frac{\partial \bar{r}_i}{\partial q_j} \cdot \frac{\partial \bar{r}_i}{\partial q_k}$$

$$M = M(\bar{r}, t) = M(q_i, t)$$

If the transformation equs. do not contain the time explicitly, (as may occur when the constraints are time-independent), then only the last term remain.

$$T = \frac{1}{2} \sum_{ij} m_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} \sum_{ij} m_{ij} \dot{\eta}_i \dot{\eta}_j$$

$$m_{ij}(q_1 \cdots q_n) = m_{ij}(q_{10} \cdots q_{n0}) + \sum_k \left(\frac{\partial m_{ij}}{\partial q_k} \right)_0 \eta_k + \cdots$$



Since T is already quadratic in the $\dot{\eta}_i$'s, the lowest nonvanishing term is considered.

$$m_{ij}(q_1, \cdots q_n) \approx m_{ij}(q_{10}, \cdots q_{n0}) = T_{ij} \quad \text{consts.}$$

$$\rightarrow T = \frac{1}{2}\sum_{ij} T_{ij}\dot{\eta}_i\dot{\eta}_j$$

$$T_{ij} = T_{ji} \quad \text{(and real)}$$

$$L = T - V = \frac{1}{2}\sum_{ij}\left(T_{ij}\dot{\eta}_i\dot{\eta}_j - V_{ij}\eta_i\eta_j\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_i} - \frac{\partial L}{\partial \eta_i} = 0 \quad \text{(n equs.)}$$

$$\rightarrow \sum_j \left(T_{ij}\ddot{\eta}_j + V_{ij}\eta_j\right) = 0 \quad \left(\begin{matrix}\text{full set of coupled equs.}\\ \text{n-equ.}\end{matrix}\right)$$

6-2 The eigenvalue equ. and the principal axis transformation

For the n-linear differential equs., we try the oscillatory sols.:

$$\eta_i = C a_i e^{-i\omega t}$$

(first we assume all the coords. oscillate with the same frequency ω (normal mode).)

where Ca_i: complex amplitude for each η_i

C: scale factor, containing imaginary part (P. 67 Remark)

The real part of the sol. corresponds to the actual motion.

Linear Equs.-

$$\begin{cases} a_{11}x_1 + a_{12}x_2 \cdots a_{1n}x_n = y_1 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 \cdots a_{nn}x_n = y_n \end{cases}$$

n-linear inhomogeneous equs. ($y_i \neq 0$)

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

$$\xrightarrow{\text{sol.}} x_i = \frac{\begin{array}{c} \text{ith column} \\ \begin{vmatrix} a_{11} & \cdots & y_1 & \cdots & a_{1n} \\ \vdots & & & & \\ a_{n1} & \cdots & y_n & \cdots & a_{nn} \end{vmatrix} \end{array}}{\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}}$$

But if $y_i = 0 \quad \forall i$:

case I) $\det A \neq 0 \quad \xrightarrow{\text{Sol.}} x_i = 0 \quad \forall i$

case II) $\det A = 0$

In this case one of the equs., say the last one is linearly dependent. We discard this one. We assume at least one of the sols. say x_n is non-zero for the remainder $(n-1)$ - equs.

$$\longrightarrow a_{11}\frac{x_1}{x_n} + a_{12}\frac{x_2}{x_n} + \cdots a_{1,n-1}\frac{x_{n-1}}{x_n} = -a_{1n}$$

$$\cdots$$

$$a_{n-1,1}\frac{x_1}{x_n} + \cdots \cdots a_{n-1,n-1}\frac{x_{n-1}}{x_n} = -a_{n-1,n}$$

Now we have $n-1$ inhomogeneous equs. as before.

Remark: if one of the a_i's is chosen real the others will be real

Sub. of the trial sol. into the equs. of motion:

$$(1)\quad \sum_j (V_{ij}\, a_j - \omega^2 T_{ij}\, a_j) = 0 \qquad i = 1, \dots n$$

These are n- linear homogenous equs. for a_i's.

There exists sol. if:

$$\mathrm{Det}\, |V_{ij} - \omega^2 T_{ij}| = 0$$

$$\begin{vmatrix} V_{11} - \omega^2 T_{11} & V_{12} - \omega^2 T_{12} & \cdots \\ V_{21} - \omega^2 T_{21} & V_{22} - \omega^2 T_{22} & \cdots \\ & & \\ & & \end{vmatrix} = 0$$

This is an equ. of n^{th} degree for ω^2

$\longrightarrow$ n value of ω (i.e. ω_k $k = 1, n$)

For each ω, equs. (1) can be solved for n amplitude of a_i, or for $(n-1)$ amplitudes in terms of the remaining a_i.

In cartezian coord., let us choose the generalized coords. as following:

$$\eta_i = \frac{1}{\sqrt{m_i}} x_i$$

Since in Cartezian coords., only the squares of velocity components appear:

$$T = \frac{1}{2} \sum_i \dot{\eta}_i^2$$

so in this case $T_{ij} = \delta_{ij}$

let $\omega^2 = \lambda$

$$\sum_j V_{ij} a_j - \omega^2 T_{ij} a_j = 0 \longrightarrow \sum_j V_{ij} a_j - \lambda \delta_{ij} a_j = 0$$

Similar to eigenvalue problem (n-dim.)

$$\sum_j V_{ij} a_j = \lambda a_i$$

$$V \bar{a} = \lambda \bar{a}$$

V is symmetric and real $\Longrightarrow$ λ's are real (n-number)

n-eigenvector of $\bar{a}$ are orthogonal

(These are valid when T_{ij} is diagonal)
similar results can be proved for the general case

In general $V\bar{a} = \lambda T\bar{a}$

Hemitean property of V and T $\longrightarrow$ λ is real and it must be positive

Let $\bar{a}_k$ be the eigenvector corresponding to λ_k

(1) $V\bar{a}_k = \lambda_k T\bar{a}_k$ (no summation over k)

Meaning: $\left\{ \sum_j V_{ij} a_{jk} = \lambda_k \sum_j T_{ij} a_{jk} \quad \begin{pmatrix} V_{11} & \cdots & V_{1n} \\ \vdots & & \\ V_{n1} & \cdots & V_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \lambda_k \begin{pmatrix} T_{11} & \cdots & T_{n1} \\ \vdots & & \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right\}$

The adjoint equ (transposed complex conjugate) for λ_ℓ:

(2) $a_\ell^+ V = \lambda_\ell^* a_\ell^+ T$ (since $T^+ = T$, $V^+ = V$)

(1) $\longrightarrow$ $\bar{a}_\ell^+ V\bar{a}_k = \lambda_k \bar{a}_\ell^+ T\bar{a}_k$

(2) $\longrightarrow$ $\bar{a}_\ell^+ V\bar{a}_k = \lambda_\ell^* \bar{a}_\ell^+ T\bar{a}_k$

$\longrightarrow$ $(\lambda_k - \lambda_\ell^*)\, \bar{a}_\ell^+ T\bar{a}_k = 0$

For distinct roots $\ell \neq k$

$\bar{a}_\ell^+ T\bar{a}_k = 0$ (i.e. $\sum_{ij} T_{ij} a_{jk} a_{i\ell}^* = 0$)

If $\ell = k$

$(\lambda_k - \lambda_k^*)\, \bar{a}_k^+ T\bar{a}_k = 0$

Now let $\bar{a}_k = \bar{\alpha}_k + i\bar{\beta}_k$

$$\rightarrow \bar{a}_k^{+} T \bar{a}_k = \overbrace{\tilde{\alpha} T \bar{\alpha}_k}^{>0} + \overbrace{\tilde{\beta}_k T \bar{\beta}_k}^{>0} + i\left(\tilde{\bar{\alpha}}_k T \bar{\beta}_k - \tilde{\bar{\beta}}_k T \bar{\alpha}_k\right)$$

or

$$\sum_{ij} T_{ij} a_{jk} a^*_{ik} = \sum_{ij} T_{ij} \alpha_{jk} \alpha_{ik} + \sum_{ij} T_{ij} \beta_{jk} \beta_{ik}$$

$$\cancel{+ i\left[\sum_{ij} \underbrace{T_{ij}}_{\text{symmetric}} \underbrace{\left(\beta_{jk}\alpha_{ik} - \beta_{ik}\alpha_{jk}\right)}_{\text{antisymmetric}}\right]}$$

$$\rightarrow (\lambda_k - \lambda_k^*)\, \bar{a}_k^{+} T \bar{a}_k = 0 \quad \rightarrow \lambda_k = \lambda_k^*$$

real $\lambda_k \longrightarrow$ the ratio of a_{jk} real $(V\bar{a}_k = \lambda_k T \bar{a}_k)$

We choose a_i particular one to be real, then all the others will be real.

Any complex phase factor in the amplitude of the oscillation will be thrown into the factor C.

Now;

$$V\bar{a}_k = \lambda_k T \bar{a}_k \rightarrow \tilde{\bar{a}}_k V \bar{a}_k = \lambda_k \tilde{\bar{a}}_k T \bar{a}_k$$

$$\lambda_k = \frac{\tilde{\vec{a}}_k V \vec{a}_k}{\tilde{\vec{a}}_k T \vec{a}_k}$$

or $$\lambda_k = \frac{\sum_{ij} V_{ij} a_{jk} a_{ik}}{\sum_{ij} T_{ij} a_{jk} a_{ik}}$$

The denominator of this expression is equal to <u>twice</u> kinetic energy for velocities a_{ik} and since the eigenvectors are real, then

$$\sum_{ij} T_{ij} a_{jk} a_{ik} > 0 \quad \text{positive-definite}$$

Similarly, the numerator is the potential energy for coordinates a_{ik}, and the cond. that V be <u>a min.</u> <u>at equilibrium</u> requires that

$$\sum_{ij} V_{ij} a_{jk} a_{ik} \geq 0$$

When V is <u>not</u> a <u>local</u> Min., this summation might be negative $\longrightarrow$ imaginary ω's

$\longrightarrow$ unbounded exp. increase of the η_i's with t.

i.e. containing $e^{\pm \omega t}$ terms.

$$\rightarrow \begin{cases} \lambda > 0 & \text{for stable systems} \\ \lambda < 0 & \text{" unstable "} \end{cases}$$

The values of a_{jk}'s are not completely fixed by the eigenvalue equ.

$$V_{ij} a_j - \omega^2 T_{ij} a_j = 0$$

It is possible to remove this indeterminacy by requiring:

$$\tilde{a}_k T a_k = 1 \quad \left(\sum_{ij} T_{ij} a_{jk} a_{ik} = 1 \right)$$

(n-equs.)

$$\rightarrow \tilde{A} T A = I \qquad \left\{ a_k = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix}_{\text{for } \omega_k}, \; A = \begin{pmatrix} a_{11} \cdots & a_{1k} \cdots & a_{1n} \\ a_{21} & a_{2k} & a_{2n} \\ a_{n1} \cdots & a_{nk} & \cdots a_{nn} \end{pmatrix} \right\}$$

This argument is not true when there exists degeneracy

i.e. $\lambda_\ell = \lambda_k$

But we claim there exists a set of a_{jk} coeff's that satisfies

$$\begin{cases} V_{ij} a_j - \omega^2 T_{ij} a_j = 0 \\ \tilde{\bar{a}}_\ell T \bar{a}_k = 0 \quad \ell \neq k \end{cases}$$

hence $\rightarrow \tilde{A} T A = I$ (orthogonalized A is in this sense)

Note: $V\bar{a} = \lambda T\bar{a}$

$\rightarrow \tilde{A}VA = \lambda\tilde{A}TA$

The cond $\tilde{A}TA = I \longrightarrow \tilde{A}VA = \lambda$

where $\lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

$$A = \begin{pmatrix} [\ \ ] & [\ \ ] & \cdots & [\ \ ] \end{pmatrix} \qquad (n \times n)$$

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & \\ & & & \\ & & & T_{nn} \end{pmatrix}$$

$$V = \begin{pmatrix} V_{11} & V_{12} & \cdots & \\ & & & \\ & & & V_{nn} \end{pmatrix}$$

6-3 Frequencies of free vibration, and Normal coords. —

We found there exists a set of n-frequencies ω_k

Therefore the general sol.:

$$\eta_i = C a_i e^{-i\omega t} \longrightarrow \eta_i = \sum_k C_k a_{ik} e^{-i\omega_k t} \quad (1) \quad (C: \text{complex})$$

But for each $\lambda_k = \omega_k^2 \longrightarrow \begin{cases} -\omega_k \\ +\omega_k \end{cases}$

More general sol.:

$$\eta_i = \sum_k a_{ik} \left(C_k^+ e^{i\omega_k t} - C_k^- e^{-i\omega_k t} \right) \quad (2)$$

For the actual motion we take the real part:

$$\eta_i = \sum_k f_k a_{ik} \cos(\omega_k t + \delta_k) \qquad \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \cos(\omega_1 t + \delta_1) \\ \vdots \\ f_n \cos(\omega_n t + \delta_n) \end{pmatrix}$$

$\begin{cases} f_k \\ \delta_k \end{cases}$ are determined from the initial conds.

$\longrightarrow$ Either of sols. (1) and (2) will therefore represent the actual motion. But the former is more convenient.

In eqn (1), unless it happens that all of the ω_k's are commensurable (i.e. rational fractions of each other) η_i never repeats its initial value $\longrightarrow$ η_i is not aperiodic func. of t. (i.e. if at $t=t_0 \begin{cases} x=x_0 \\ v=v_0 \\ a=a_0 \end{cases}$ then at $t=t \begin{cases} \text{if } x=x_0 \\ v \neq v_0 \\ a \neq a_0 \end{cases}$)

However it is possible to make a transformation:

$$\eta_i \longrightarrow \xi_i$$

ξ_i : simple periodic funs of t (normal coords.)

$$\eta_i = \sum_j a_{ij} \xi_j \quad \text{or} \quad \bar{\eta} = A\bar{\xi}$$

The potential energy:

$$V = \frac{1}{2} \sum_{ij} \eta_i V_{ij} \eta_j \longrightarrow V = \frac{1}{2} \tilde{\bar{\eta}} V \bar{\eta}$$

Since $\tilde{\bar{\eta}} = \widetilde{A\bar{\xi}} = \tilde{\bar{\xi}} \tilde{A}$

$$V = \frac{1}{2} \tilde{\bar{\xi}} \tilde{A} V A \bar{\xi}$$

But $\tilde{A} V A = \lambda \longrightarrow V = \frac{1}{2} \tilde{\bar{\xi}} \lambda \bar{\xi} = \frac{1}{2} \sum_k \omega_k^2 \xi_k^2$

Also

$$T = \frac{1}{2} \tilde{\dot{\bar{\eta}}} T \dot{\bar{\eta}} = \frac{1}{2} \tilde{\dot{\bar{\xi}}} \tilde{A} T A \dot{\bar{\xi}} = \frac{1}{2} \tilde{\dot{\bar{\xi}}} I \dot{\bar{\xi}}$$

$$T = \frac{1}{2} \sum_k \dot{\xi}_k^2$$

The new Lagrangian:

$$L = T - V = \frac{1}{2} \sum_k \left(\dot{\xi}_k^2 - \omega_k^2 \xi_k^2 \right)$$

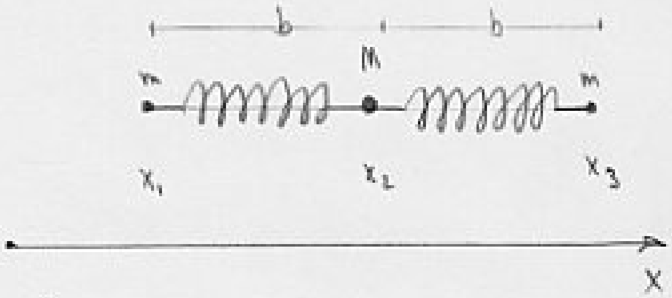
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\xi}_i} - \frac{\partial L}{\partial \xi_i} = 0$$

$$\rightarrow \ddot{\xi}_i + \omega_i^2 \xi_i = 0 \longrightarrow \xi_i = C_i e^{-i\omega_i t} \quad \text{(not summation)}$$

ξ_k: Periodic func with a single frequency ω_k

Ex.: Linear symmetrical triatomic molecule;

For linear harmonic oscillator:

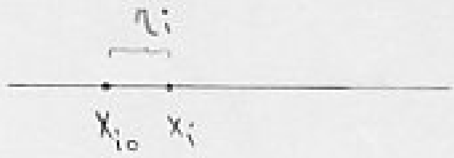


$$V = \frac{1}{2} k x^2$$

$$V = V_{spring_1} + V_{spring_2}$$

$$V = \frac{k}{2}(x_2 - x_1 - b)^2 + \frac{k}{2}(x_3 - x_2 - b)^2$$

Tr. to the new coord. (coords. relative to the equilibrium positions)



$$\eta_i = x_i - x_{i0}$$

where $\begin{cases} x_{20} - x_{10} = b \\ x_{30} - x_{20} = b \end{cases}$

The potential energy: $V = \frac{k}{2}(\eta_2 - \eta_1)^2 + \frac{k}{2}(\eta_3 - \eta_2)^2$

$$V = \frac{k}{2}(\eta_1^2 + 2\eta_2^2 + \eta_3^2 - 2\eta_1\eta_2 - 2\eta_2\eta_3)$$

$$V = \frac{1}{2}\sum_{ij} V_{ij}\,\eta_i\,\eta_j$$

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

$$T = \frac{m}{2}(\dot{\eta}_1^2 + \dot{\eta}_3^2) + \frac{M}{2}\dot{\eta}_2^2$$

$$T = \frac{1}{2} \sum_{ij} T_{ij} \dot{\eta}_i \dot{\eta}_j$$

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{12} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

The secular eqn.:

$$|V - \omega^2 T| = \begin{pmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 m \end{pmatrix} = 0$$

$$\omega^2 (k - \omega^2 m)[k(M+2m) - \omega^2 Mm] = 0$$

$$\rightarrow \omega_1 = 0 \quad , \quad \omega_2 = \sqrt{\frac{k}{m}} \quad , \quad \omega_3 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}$$

$\omega_1 = 0$ corresponds to the rigid-body motion (i.e. translation without oscillation). This frequency can be eliminated by imposing the constraint that the center of mass remain stationary at the origin:

$$m(x_1 + x_3) + Mx_2 = 0 \quad (3 \text{ deg.} \longrightarrow 2 \text{ deg})$$

Zero frequencies also arise when both, first and second derivatives of V vanish at equilibrium.

Small oscillations may still be possible in this case if the forth derivatives do not also vanish (the third derivatives must vanish for the stable equilibrium).

The vibrations will not be simple harmonic.

$$VA = \lambda TA \rightarrow (V - \lambda T)A = 0$$

$$\begin{pmatrix} k-\omega_j^2 m & -k & 0 \\ -k & 2k-\omega_j^2 M & -k \\ 0 & -k & k-\omega_j^2 m \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{pmatrix} = \begin{pmatrix} \\ 0 \\ \\ \end{pmatrix}$$

$$\begin{cases} (k-\omega_j^2 m)\, a_{1j} - k\, a_{2j} + 0 = 0 \\ -k\, a_{1j} + (2k-\omega_j^2 M)\, a_{2j} - k a_{3j} = 0 \\ 0 - k\, a_{2j} + (k-\omega_j^2 m)\, a_{3j} = 0 \end{cases}$$

Solving these equs. the ratio of the amplitudes can be obtained. A complete sol. can be achieved by imposing the normalization cond.:

$$\tilde{A} T A = I$$

or $\sum_{ij} T_{ij}\, a_{ik} a_{jk} = 1$

$$(a_{1j}\ a_{2j}\ a_{3j}) \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{pmatrix} = 1$$

$$(a_{1j}\ a_{2j}\ a_{3j}) \begin{pmatrix} m a_{1j} \\ M a_{2j} \\ m a_{3j} \end{pmatrix} = 1 \rightarrow m(a_{1j}^2 + a_{3j}^2) + M a_{2j}^2 = 1$$

For $\omega_1 = 0$ $\quad a_{11} = a_{21} = a_{31} = \dfrac{1}{\sqrt{2m+M}}$ (only translation)

For ω_2 $\quad a_{12} = \dfrac{1}{\sqrt{2m}}$, $a_{22} = 0$, $a_{32} = -\dfrac{1}{\sqrt{2m}}$

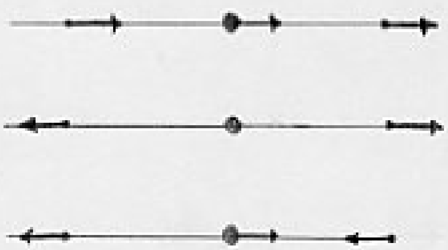
The center atom at rest
The other two exactly out of phase.

For ω_3 $\quad a_{13} = \dfrac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}}$, $a_{23} = \dfrac{-2}{\sqrt{2M\left(2+\frac{M}{m}\right)}}$, $a_{33} = \dfrac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}}$

(ω_1) (a)

(ω_2) (b)

(ω_3) (c)



Longitudinal normal modes of the linear symmetric triatomic molecule.

$$A = \begin{pmatrix} \frac{1}{\sqrt{2m+M}} & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m(1+\frac{2m}{M})}} \\ \frac{1}{\sqrt{2m+M}} & 0 & \frac{-2}{\sqrt{2M(2+\frac{M}{m})}} \\ \frac{1}{\sqrt{2m+M}} & -\frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m(1+\frac{2m}{M})}} \end{pmatrix}$$

$$\eta_i = \sum_k f_k a_{ik} \cos(\omega_k t + \delta_k)$$

$$\begin{cases} \eta_1 = f_1 a_{11} \cos(\omega_1 t + \delta_1) + f_2 a_{12} \cos(\omega_2 t + \delta_2) + f_3 a_{13} \cos(\omega_3 t + \delta_3) \\ \eta_2 = f_1 a_{21} \cos(\omega_1 t + \delta_1) + f_2 a_{22} \cos(\omega_2 t + \delta_2) + f_3 a_{23} \cos(\omega_3 t + \delta_3) \\ \eta_3 = f_1 a_{31} \cos(\omega_1 t + \delta_1) + f_2 a_{32} \cos(\omega_2 t + \delta_2) + f_3 a_{33} \cos(\omega_3 t + \delta_3) \end{cases}$$

f_1, f_2, f_3 and $\delta_1, \delta_2, \delta_3$ will be determined by the initial conds.

Normal coords.:

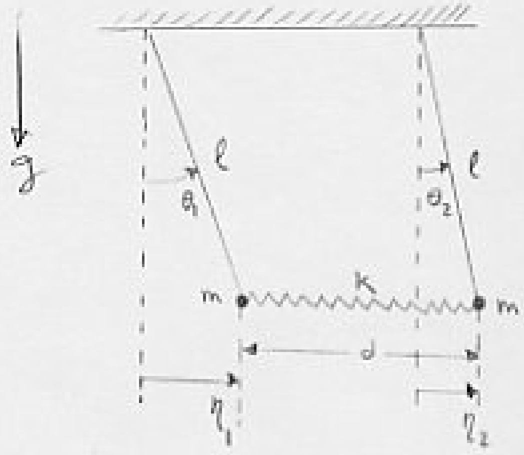
$$\eta = A\xi \;\rightarrow\; \xi = A^{-1}\eta \qquad \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = A^{-1}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

$$A^{-1} = \frac{(\text{cofac Matrix})^T}{\det A}$$

In normal coord.: $\ddot{\xi}_i + \omega_i^2 \xi_i = 0$

$$\rightarrow \xi_i = C_i\, e^{-i\omega_i t}$$

Ex.: Coupled identical Pendulum:



$$\frac{\eta}{l} = \sin\theta = \left(\theta - \frac{\theta^3}{3!} + \cdots\right) \simeq \theta \quad \text{For small } \theta$$

There are two kinds of potential, gravitational potential and the potential of the spring.

$$\text{Height} = l(1-\cos\theta) = l\left(1 - \left(1 - \frac{\theta^2}{2!} + \cdots\right)\right) \simeq \frac{1}{2} l\theta^2 = \frac{\eta^2}{2l}$$

$$V_g = \frac{mg}{2l}(\eta_1^2 + \eta_2^2)$$

$$d - d_0 = \eta_2 - \eta_1 \qquad d_0 : \text{the length of spring at rest}$$

$$V_s = \frac{k}{2}(\eta_2 - \eta_1)^2$$

$$V = V_g + V_s = \frac{mg}{2l}(\eta_1^2 + \eta_2^2) + \frac{k}{2}(\eta_2 - \eta_1)^2$$

$$= \left(\frac{mg}{2l} + \frac{k}{2}\right)(\eta_1^2 + \eta_2^2) - k\eta_1\eta_2$$

$$V_{ij} = \begin{pmatrix} \frac{mg}{l} + k & -k \\ -k & \frac{mg}{l} + k \end{pmatrix}$$

$$T = \frac{m}{2}\left[(\ell\dot\theta_1)^2 + (\ell\dot\theta_2)^2\right] \simeq \frac{m}{2}\left(\dot\eta_1^2 + \dot\eta_2^2\right)$$

$$T_{ij} = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|V - \omega^2 T| = 0 \qquad \begin{pmatrix} \frac{mg}{\ell} + k - \omega^2 m & -k \\ -k & \frac{mg}{\ell} + k - \omega^2 m \end{pmatrix} = 0$$

$$\frac{mg}{\ell} + k - \omega^2 m = \pm k \longrightarrow \begin{cases} \omega_1 = \sqrt{\frac{g}{\ell}} \\ \omega_2 = \sqrt{\frac{g}{\ell} + 2\frac{k}{m}} \end{cases}$$

$$(V - \lambda T)A = 0$$

$$\begin{pmatrix} \frac{mg}{\ell} + k - \omega_j^2 m & -k \\ -k & \frac{mg}{\ell} + k - \omega_j^2 m \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix} = 0$$

$$\begin{array}{ll} (1) & \left(\frac{mg}{\ell} + k - \omega_j^2 m\right) a_{1j} - k a_{2j} = 0 \\ (2) & -k a_{1j} + \left(\frac{mg}{\ell} + k - \omega_j^2 m\right) a_{2j} = 0 \\ (3) & \tilde{A} T A = I \Rightarrow (a_{1j} \;\; a_{2j}) \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix} = 1 \quad \Rightarrow a_{1j}^2 + a_{2j}^2 = \frac{1}{m} \end{array}$$

$$\omega_1 = \sqrt{\frac{g}{\ell}} \xrightarrow{\text{sub. in (1)}} a_{11} = a_{21} \xrightarrow{\text{sub. in (3)}} a_{11} = a_{21} = \sqrt{\frac{1}{2m}}$$

$$\omega_2 = \sqrt{\frac{g}{\ell} + 2\frac{k}{m}} \xrightarrow{\;\;''\;\;} a_{12} = -a_{22} \xrightarrow{\;\;''\;\;} a_{12} = -a_{22} \;\; \sqrt{\frac{1}{2m}}$$

$$\bar{a}_1 = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \bar{a}_2 = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad A = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Displacements of the bobs in the Normal modes of the coupled pendulums

$$\eta_i = \sum_k c_k a_{ik} e^{-i\omega_k t} \quad \text{or} \quad \eta_i = \sum_k f_k a_{ik} \cos(\omega_k t + \delta_k)$$

$$\xi = A^{-1}\eta \qquad A^{-1} = \frac{(\text{cofactor Matrix})^T}{\det A}$$

$$A^{-1} = \frac{\frac{1}{\sqrt{2m}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}}{-2\,\frac{1}{2m}} = \sqrt{\frac{m}{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(1) $$\xi = \sqrt{\frac{m}{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \sqrt{\frac{m}{2}}\begin{pmatrix} \eta_1+\eta_2 \\ \eta_1-\eta_2 \end{pmatrix} \rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \sqrt{\frac{m}{2}}\begin{pmatrix} \eta_1+\eta_2 \\ \eta_1-\eta_2 \end{pmatrix}$$

In η-coords.:

$$L = \frac{1}{2}m(\dot\eta_1^2 + \dot\eta_2^2) - \frac{mg}{2\ell}(\eta_1^2+\eta_2^2) - \frac{1}{2}k(\eta_1^2+\eta_2^2-2\eta_1\eta_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} = 0 \longrightarrow \begin{cases} m\ddot\eta_1 + \left(k+\frac{mg}{\ell}\right)\eta_1 - k\eta_2 = 0 \\ m\ddot\eta_2 + \left(k+\frac{mg}{\ell}\right)\eta_2 - k\eta_1 = 0 \end{cases}$$

The last term in each equ. reflects the presence of the spring coupling the motion of the two pendulums.

Substituting the anticipated sols. in the equs. of motions gives the oscillation frequencies. Again substituting the frequencies in the equs. of motion gives the amplitudes of oscillation.

$$\begin{cases} \eta_1 = f_1 a_{11}\cos(\omega_1 t+\delta_1) + f_2 a_{12}\cos(\omega_2 t+\delta_2) \\ \eta_2 = f_1 a_{21}\cos(\omega_1 t+\delta_1) + f_2 a_{22}\cos(\omega_2 t+\delta_2) \end{cases}$$

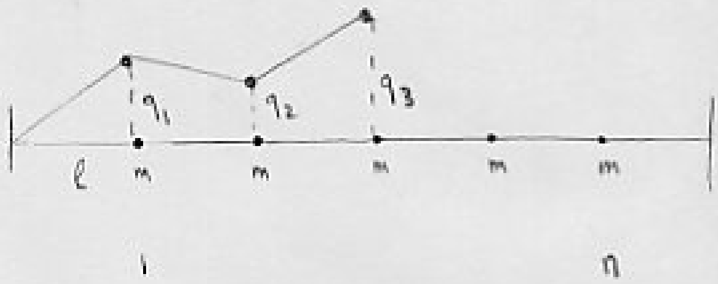
Using equ (1) the lagrangian reduces to:

$$L = \frac{1}{2}\sum_{k=1}^{2}\left(\dot\xi_k^2 - \omega_k^2\xi_k^2\right) \longrightarrow \ddot\xi_k + \omega_k^2\xi_k = 0$$

$$\longrightarrow \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} c_1\cos(\omega_1 t+\delta_1) \\ c_2\cos(\omega_2 t+\delta_2) \end{pmatrix}$$

Small Oscillation of Particles on String:
N-body problem:
Transverse Oscillations;

Consider a light string or spring stretched with force F, with n-equal masses m along it, with equal spacing of interval ℓ.



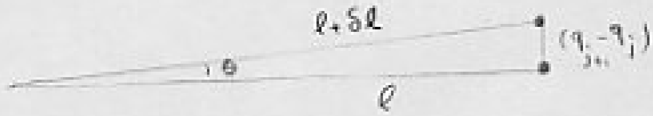
$L = (n+1)\ell$: total length

$\delta\ell$: the change in the length of the string between jth and $(j+1)$th particle,

$$\ell + \delta\ell = \sqrt{\ell^2 + (q_{j+1}-q_j)^2} \simeq \ell\left[1 + \frac{(q_{j+1}-q_j)^2}{2\ell^2}\right]$$

$$\delta\ell = \frac{(q_{j+1}-q_j)^2}{2\ell}$$

We assume the tension F in string in equilibrium position and in a stretched position is the same.



$$\begin{cases} \tan\theta = \dfrac{q_{j+1}-q_j}{\ell} \\ \tan\theta = \dfrac{\delta F}{F} \end{cases}$$

$$\rightarrow \frac{\delta F}{(q_{j+1}-q_j)} = \frac{F}{\ell}$$

$$\therefore \frac{\delta F}{q_{j+1}-q_j} = k \rightarrow \frac{F}{\ell} = k$$

$$V = \frac{1}{2}\sum_i k q_i^2$$

$$V = \frac{1}{2}\frac{F}{\ell}\left[q_1^2 + (q_2-q_1)^2 + (q_3-q_2)^2 + \dots (q_n-q_{n-1})^2 + q_n^2\right]$$

$$V = \frac{F}{2\ell}\sum_{j=1}^{n+1}(q_{j-1}-q_j)^2 \qquad q_0=0 \quad q_{n+1}=0$$

$$T = \frac{1}{2}m\sum_j \dot{q}_j^2$$

$$\mathcal{L} = T - V = \sum_{j=1}^{n+1}\left[\frac{1}{2}m\dot{q}_j^2 - \frac{F}{2\ell}(q_{j-1}-q_j)^2\right]$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

$$m\ddot{q}_i - \frac{F}{\ell}(q_{j-1} - 2q_j + q_{j+1}) = 0 \qquad (1)$$

$$\ddot{q}_i - \omega_0^2(q_{j-1} - 2q_j + q_{j+1}) = 0$$

where $\omega_0^2 = \frac{F}{m\ell}$

This is call nearest neighbour interaction.

The coupling is only between the neighbouring particles;

When n is small, we can solve the problem as before.

But when n is large, we proceed as below;

$$q_j(t) = a_j e^{i\omega t}$$

substitute in (1)

$$-\frac{F}{\ell}a_{j-1}+\left(2\frac{F}{\ell}-m\omega^2\right)a_j-\frac{F}{\ell}a_{j+1}=0 \qquad (1)$$

with $a_0=a_{n+1}=0$

we try $a_j=a\,e^{i(j\gamma-\delta)}$

a: real

If we find a, γ and δ to satisfy our problem conditions, the procedure is justified.

Introducing a_j in (1):

$$-\frac{F}{\ell}e^{-i\gamma}+\left(2\frac{F}{\ell}-m\omega^2\right)-\frac{F}{\ell}e^{i\gamma}=0$$

$$\omega^2=\frac{2F}{m\ell}-\frac{F}{m\ell}\left(e^{i\gamma}+e^{-i\gamma}\right)=\frac{2F}{m\ell}(1-\cos\gamma)$$

$$\omega^2=\frac{4F}{m\ell}\sin^2\frac{\gamma}{2}$$

Since we know secular determinant has n-value for ω:

$$\omega_r=2\sqrt{\frac{F}{m\ell}}\sin\frac{\gamma_r}{2} \qquad r=1,--n$$

Now applying the boundary conds. we determine γ_r and δ_r

$$a_{jr}=a_r\,e^{i(j\gamma_r-\delta_r)}$$

We take the real part that has physical meaning:

$$a_{jr} = a_r \cos(j\gamma_r - \delta_r)$$

Boundary conds.:

$$a_{0r} = a_{(n+1)r} = 0$$

$$a_{0r} = 0 \longrightarrow j = 0 \longrightarrow \delta_r = (2k+1)\frac{\pi}{2} \qquad k: \text{integer}$$

$$a_{jr} = a_r \cos(j\gamma_r - \tfrac{\pi}{2}) = a_r \sin j\gamma_r$$

and $a_{(n+1)r} = 0 \qquad a_r \sin[(n+1)\gamma_r] = 0$

$$\rightarrow (n+1)\gamma_r = k\pi \qquad k: \text{integer}$$

$$\gamma_r = \frac{k\pi}{n+1} \qquad k = 1, 2 \ldots \Longrightarrow \gamma_r = \frac{r\pi}{n+1} \qquad r = 1, 2 \ldots n$$

$$\rightarrow \quad a_{jr} = a_r \sin\left(j\frac{r\pi}{n+1}\right)$$

$$q_j(t) = \sum_r f_r a_{jr} e^{i\omega_r t} = \sum_r f_r a_r \sin\left(j\frac{r\pi}{n+1}\right) e^{i\omega_r t}$$

$$\omega_r = 2\sqrt{\frac{F}{m\ell}}\,\sin\left(\frac{r\pi}{2(n+1)}\right)$$

$$q_j(t) = \sum_r B_r \sin\left(j\frac{r\pi}{n+1}\right) e^{i\omega_r t}$$

Since in begining we did not consider a phase in $q_j(t) = a_j e^{i\omega t}$, we let B_r to be complex;

$$B_r = \mu_r + i\nu_r$$

$$\rightarrow q_j(t) = \sum_r \sin\left(j\frac{r\pi}{n+1}\right)\left(\mu_r \cos\omega_r t - \nu_r \sin\omega_r t\right) \quad \underline{\text{Real Part}}$$

$$\rightarrow q_j(0) = \sum_r \mu_r \sin\left(j\frac{r\pi}{n+1}\right) \quad (1)$$

$$\dot{q}(0) = -\sum_r \omega_r \nu_r \sin\left(j\frac{r\pi}{n+1}\right) \quad (2)$$

multiply (1) by $\sin\left(j\frac{s\pi}{n+1}\right)$ and sum over j.

$$\sum_j q_j(0) \sin\left(j\frac{s\pi}{n+1}\right) = \sum_{j,r} \mu_r \sin\left(j\frac{r\pi}{n+1}\right)\sin\left(j\frac{s\pi}{n+1}\right)$$

But
$$\sum_{j=1}^{n} \sin\left(j\frac{r\pi}{n+1}\right)\sin\left(j\frac{s\pi}{n+1}\right) = \frac{n+1}{2}\delta_{rs} \qquad r,s = 1 \dots n$$

$$\longrightarrow \sum_j q_j(0) \sin\left(j\frac{s\pi}{n+1}\right) = \sum_r \mu_r \frac{n+1}{2}\delta_{rs}$$

$$= \frac{n+1}{2}\mu_s$$

or
$$\mu_s = \frac{2}{n+1}\sum_j q_j(0)\sin\left(j\frac{s\pi}{n+1}\right)$$

using a similar procedure;

$$\nu_s = -\frac{2}{\omega_s(n+1)}\sum_j \dot{q}_j(0)\sin\left(j\frac{s\pi}{n+1}\right)$$

Ex. — Consider a loaded string consisting of 3-particles, regularly spaced on the string. At $t=0$ the center particle (only) is displaced a distance a and released from rest. Describe the subsequent motion.

Sol. $\quad n=3$

$$
\begin{cases}
(1) & q_1(0)=0 \quad q_2(0)=a \quad q_3(0)=0 \\
(2) & \dot{q}_1(0)=0 \quad \dot{q}_2(0)=0 \quad \dot{q}_3(0)=0
\end{cases}
$$

$$(2) \rightarrow \nu_r = 0 \qquad \nu_1=\nu_2=\nu_3=0$$

$$\mu_r = \frac{2}{n+1}\sum_j q_j(0)\sin\left(j\frac{r\pi}{n+1}\right) = \frac{1}{2}a\sin\left(\frac{r\pi}{2}\right)$$

$$\rightarrow \mu_1 = \frac{1}{2}a \qquad \mu_2=0 \qquad \mu_3=-\frac{1}{2}a$$

Now $\sin\left(j\frac{r\pi}{n+1}\right) =$

$j \backslash r$	1	2	3
1	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$
2	1	0	-1
3	$\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$

$$q_j(t) = \sum_r \sin\left(j\frac{r\pi}{n+1}\right)(\mu_r\cos\omega_r t - \nu_r\sin\omega_r t)$$

$$q_1(t) = \frac{\sqrt{2}}{4}a(\cos\omega_1 t - \cos\omega_3 t)$$

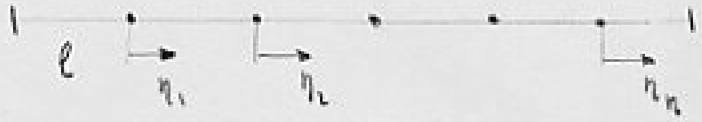
$$q_2(t) = \frac{1}{2}a(\cos\omega_1 t - \cos\omega_3 t)$$

$$q_3(t) = \frac{\sqrt{2}}{4}a(\cos\omega_1 t - \cos\omega_3 t)$$

where $\omega_r = 2\sqrt{\frac{F}{ml}}\sin\left(\frac{r\pi}{8}\right) \quad r=1,2,3$

Longitudinal Oscillation:

$$\mathcal{L} = T - V$$



$$\mathcal{L} = \frac{1}{2} m \sum_{j=1}^{n} \dot{\eta}_i^2 - \frac{1}{2} k \sum_{j=1}^{n+1} (\eta_{j-1} - \eta_j)^2$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\eta}_i} - \frac{\partial \mathcal{L}}{\partial \eta_i} = 0$$

$$m \ddot{\eta}_i - k(\eta_{j-1} - 2\eta_i + \eta_{j+1}) = 0 \qquad j = 1, \dots n$$