Chapter 2

Variational Principals and Lagrange's Equations

2-1 Hamilton's Principle:

Monogenic systems:

Those mechanical systems for which all forces (except the forces of constraint) are derivable from a generalized scalar potential that may be a function of the coordinates, velocities and time, are called monogenic systems.

Hamilton's principal for monogenic systems:

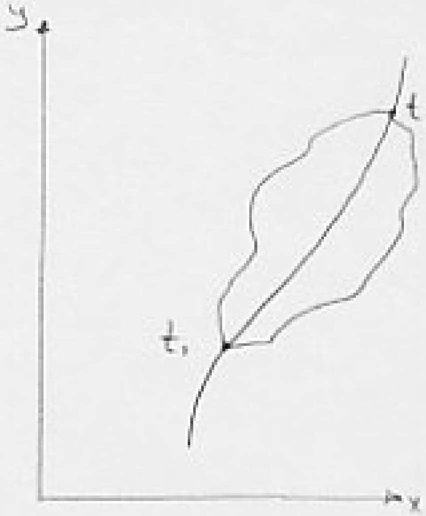
The motion of the system from time t_1 to time t_2 is such that the time integral

$$I = \int_{t_1}^{t_2} L\,dt \qquad \text{least action integral}$$

where $L = T - V$, has a stationary value for the correct path of motion.

By the term stationary value for a line integral we mean that the integral along the given path has the same value to within first order infinitesimals as that along all neighboring paths (i.e., those that differ from it by infinitesimal displacements.

The path is traced in configuration space.



The configuration space has no necessary connection with the physical three-dimensional space (position space), just as the generalized coordinates are not necessarily position coordinates.

Hamilton's principal by alternative statement:

The motion is such that the variation of the line integral I for a fixed t_1 and t_2 is zero:

$$\delta I = \delta \int_{t_1}^{t_2} \delta L(q_1, \ldots q_2, \dot{q}_1, \ldots \dot{q}_n, t)\, dt = 0$$

For holonomic constraints Hamilton's principale is both necessary and sufficient condition for Lagrang's equs.

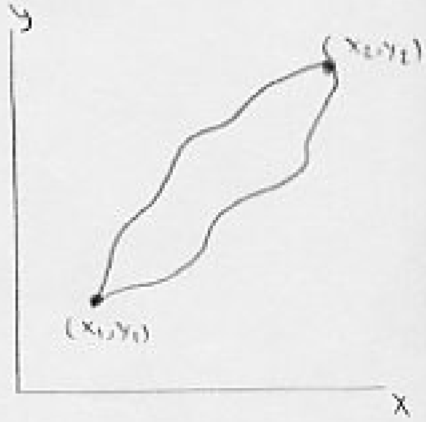
2-2 Some techniques of the calculus Variations:

One dimensional case:

We have a func. $f(y, \dot{y}, x)$ defined on a path $y = y(x)$ between two values x_1 and x_2

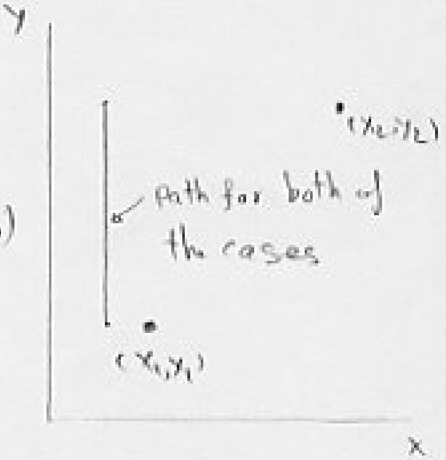
where $\dot{y} \equiv \frac{dy}{dx}$

We wish to find a particular path $y(x)$ such that the line integral J of the func. f between x_1 and x_2 has a stationary value.



(x plays the role of t)

let $y(x, 0)$ be the correct path
and $y(x, \alpha)$ neighboring path labeled
by an infinitesimal parameter α



We select any func. $\eta(x)$ (auxiliary func.) such that

$$\eta(x) \longrightarrow 0 \quad \text{at } x_1 \text{ and } x_2$$

The possible set of varried paths:

$$y(x, \alpha) = y(x, 0) + \alpha \eta(x)$$

Paths differ only in the functional relation between y and x

For simplicity, it is assumed that

$y(x)$ and $\eta(x)$ are well-behaved fncs.

i.e. $\begin{cases} \text{1- continuous} \\ \text{2- nonsingular between } x_1 \text{ and } x_2 \\ \text{3- with continuous first and second derivatives between } x_1 \text{ and } x_2 \end{cases}$

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), \dot{y}(x,\alpha), x)\, dx$$

The condition for being stationary is

$$\left(\frac{dJ}{d\alpha} \right)_{\alpha=0} = 0$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha} \right\} dx$$

Consider the second term

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha}\, dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \frac{\partial^2 y}{\partial x \partial \alpha}\, dx$$

Integrating by parts:

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \frac{\partial^2 y}{\partial x \partial \alpha}\, dx = \frac{\partial f}{\partial \dot{y}} \frac{\partial y}{\partial \alpha} \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) \frac{\partial y}{\partial \alpha}\, dx$$

All varied curves pass through the points (x_1, y_1) and (x_2, y_2),

hence $\left(\frac{\partial y}{\partial \alpha} \right)_{x_1} = 0 \qquad \left(\frac{\partial y}{\partial \alpha} \right)_{x_2} = 0$

Therefore:

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} \right) \frac{\partial y}{\partial \alpha}\, dx$$

$$\left(\frac{dJ}{d\alpha}\right)_{\alpha=0} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} \right) \left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=0} dx = 0$$

Lemma: If $\int_{x_1}^{x_2} M(x)\,\eta(x)\,dx = 0$

for all arbitrary funcs. $\eta(x)$ continuous through the second derivative then $M(x)$ must identically vanish in the interval (x_1, x_2).

Therefore J can have stationary value only if

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

The quantity $\left(\frac{\partial y}{\partial \alpha}\right)_0 d\alpha \equiv \delta y$

represents the infinitisimal departure of the varied path from the correct path $y(x)$ at the point x

Similarly
$\left(\frac{dJ}{d\alpha}\right)_0 d\alpha \equiv \delta J$
is the infinitesimal variation of J about the correct path.

The assertion that J is stationary for the correct path can thus be written:

$$\delta J = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} \right\} \delta y \, dx = 0$$

requiring $y(x)$ satisfy

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

Example: Shortest distance between two points in a plane.

$ds = \sqrt{dx^2 + dy^2}$ element of arc length in a <u>plane</u>

The total length of any curve between points 1 and 2.

$$I = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

The cond. that curve be the shortest path is that I be a minimum.

Comparing with $J = \int_{x_1}^{x_2} f(y, \dot{y}, x) \, dx$

$$\longrightarrow \quad f = \sqrt{1 + \dot{y}^2}$$

Substituting in

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

with
$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$$

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0$$

$$\longrightarrow \quad \frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = C \quad \text{(const.)}$$

This sol. can be valid only if

$\dot{y} = a$ (const.) (otherwise if $\dot{y}$ change, then $\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$ will change.)

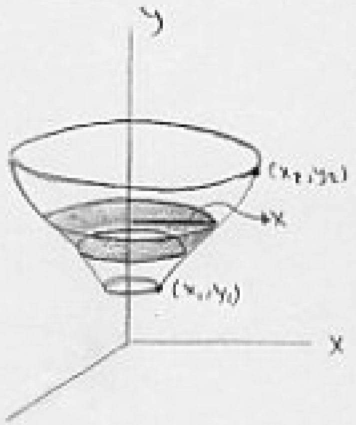
then: $a = \dfrac{c}{\sqrt{1-c^2}}$

$\dot{y} = a$ is clearly the equ. of a stright line.

$$y = ax + b$$

a and b can be determined by knowing the fact that the curve pass through the end points (x_1, y_1), (x_2, y_2).

Example: Minimum surface of revolution



The area of the strip:

$$2\pi x\, ds = 2\pi x \sqrt{1+\dot{y}^2}\, dx$$

Total area:

$$2\pi \int_1^2 x\sqrt{1+\dot{y}^2}\, dx$$

The extermum of this integral is given by

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

where $f = x\sqrt{1+\dot{y}^2}$

and $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}$

$$\frac{d}{dx}\left(\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0$$

or $\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = a$ (const.)

$$\rightarrow \dot{y}^2(x^2-a^2) = a^2$$

Solving $\frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}}$

The general sol.

$$y = a\int \frac{dx}{\sqrt{x^2-a^2}} + b = a \operatorname{Arc\,cosh}\frac{x}{a} + b$$

$$x = a\cosh\frac{y-b}{a} \quad \text{(catenary equ.)}$$

a and b are determined by the requirement that the curve pass through the given end points.

Note: 1- For some pairs of end points, unique constants of a and b can indeed be found.

2- For other end points two sols. result

3- Other regions no possible values for a and b

Equ $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$

represent a cond. for finding f(x), continuous through the second derivative, that render the integral stationary.

The catenary sols. therefore don't always present minimum values, but may give points of inflexion.

2-3 Derivation of Lagrange's Equs. from Hamilton's Principle:

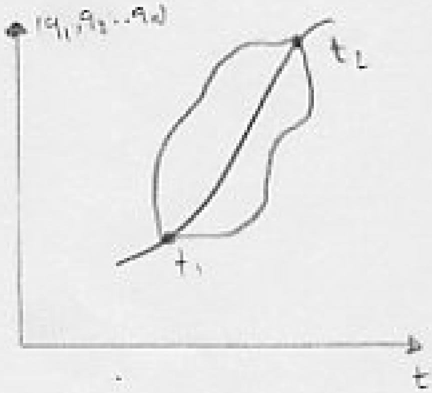
The least action integral is written:

$$I = \int_{t_1}^{t_2} L(q_1, q_2 \cdots q_n, \dot{q}_1, \dot{q}_2 \cdots \dot{q}_n, t)\, dt$$

$L = T - V$ for conservative system

Note that: $f \to L$, $y \to q$, $x \to t$

For a correct path:



$$\delta I = \delta \int_{t_1}^{t_2} L\, dt = 0$$

Since $\delta q_i \Big|_1^2 = 0$ $\quad \delta t = 0 \quad \cancel{\delta \dot{q} = 0}$

note: $\delta W = \left(\frac{\partial W}{\partial \alpha}\right)_0 d\alpha$

$$\delta I = \sum_i \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \cancel{\frac{\partial L}{\partial t} \delta t} \right) dt = 0$$

Integrating by parts: $\int v\,du = vu - \int u\,dv$

$$\delta I = \sum_i \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \right] \delta q_i\, dt + \cancel{\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_1}^{t_2}} = 0$$

Since q_i are independent coords. $\longrightarrow$ the variations δq_i are independent.

holonomic constraints

Hence the coeffs. of δq_i must seperately vanish.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$ Lagrange's equs. of motion $(i=1,\ldots n)$

Alternatively:

As before (Page 23):

$$\frac{\partial I}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial \alpha} d\alpha + \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \alpha} d\alpha \right) dt$$

Integration by parts for the second term

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \frac{\partial^2 q_i}{\partial \alpha \partial t} dt = \cancel{\frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \alpha} \Big|_{t_1}^{t_2}}^{\,0} - \int_{t_1}^{t_2} \frac{\partial q_i}{\partial \alpha} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) dt$$

$$\delta I = \left(\frac{\partial I}{\partial \alpha} \right)_0 d\alpha = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \, dt$$

$$\delta q_i = \left(\frac{\partial q_i}{\partial \alpha} \right)_0 d\alpha$$

Since q_i are independent $\longrightarrow$ δq_i are independent

e.g. $\longrightarrow$ $\eta_i(t)$ will be independent.

Hence the coeffs. of δq_i vanish seperately

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad (i=1,\ldots n)$$

(for monogenic systems with holonomic constraints)

Ex.: Free Particle:

For free particle $\quad V(x)=0$

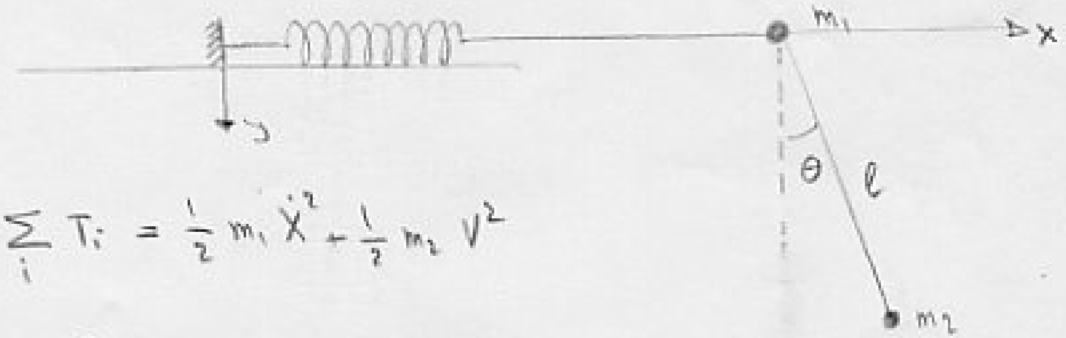
$$L = T - V = T = \frac{1}{2} m \dot{x}^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial(\frac{1}{2}m\dot{x}^2)}{\partial \dot{x}}\right) - \frac{\partial(\frac{1}{2}m\dot{x}^2)}{\partial x} = 0$$

$$\rightarrow m\ddot{x} = 0$$

Ex.:



$$T = \sum_i T_i = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 V^2$$

$$\begin{cases} V_x = \frac{d}{dt}(x + \ell \sin\theta) \rightarrow V_x = \dot{x} + \ell\dot{\theta}\cos\theta \\ V_y = \frac{d}{dt}(\ell\cos\theta) \rightarrow V_y = -\ell\dot{\theta}\sin\theta \end{cases}$$

$\{\theta, x$ are generalized coords.$\}$
6 dof. $\rightarrow$ 2 dof.

$$V^2 = V_x^2 + V_y^2 = \ell^2\cos^2\theta\,\dot{\theta}^2 + \ell^2\sin^2\theta\,\dot{\theta}^2 + \dot{x}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta$$
$$= \dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta$$

$$V = \sum V_i = \frac{1}{2}kx^2 - m_2 g\ell\cos\theta$$

$$L = T - V = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta) - \frac{1}{2}kx^2 + m_2 g\ell\cos\theta$$

2-4 Extension of Hamilton's principle to nonholonomic systems:

a) With nonholonomic constraints the generalized coords. are not all independent, and it is not possible to reduce them further by means of equs. of constraint of the form $f(q_1, q_2, \ldots q_n, t) = 0$

b) Another difference is that, it is important whether the varied path is or is not constructed by displacements consistent with the constraints.

However, a certain type of nonholonomic systems can be treated; that is when the equ. of constraint can be put in the form of:

$$\sum_{k}^{n} a_{\ell k}\, dq_k + a_{\ell t}\, dt = 0 \quad \begin{cases} \ell = 1, \ldots m \\ m \text{ equs.} \end{cases}$$

(a linear relation connecting the derivatives of q_i)

$$a_{\ell k} \equiv a_{\ell k}(q_1, \ldots q_n, t) \quad a_{\ell t} \equiv a_{\ell t}(q_1, \ldots q_n, t) \text{ in general}$$

It would be expected the varied paths or displacements constructing the varied path should satisfy the above constrainst (m equs.)

No such varied paths can be constructed, unless the above constrainst are integrable (they become holonomic).

The constraint equs. Valid for virtual displacements are then

$$\sum_k a_{lk}\,\delta q_k = 0 \qquad (\delta t = 0)$$

Note: { holonomic constraint $f(q_1, \ldots q_n, t) = 0 \rightarrow \delta f = \sum_{k=1}^{n} \frac{\partial f_l}{\partial q_k} \delta q_k = 0$

the varied path in general not satisfy

$$\sum_k a_{lk}\, dq_k + a_{lt}\, dt = 0$$

We can now use equs. $\sum_k a_{lk}\,\delta q_k = 0$, to reduce the number of virtual displacements to independent ones.

The procedure of this elemination is; the method of Lagrange undetermined multipliers.

$$\sum_k a_{lk}\,\delta q_k = 0 \longrightarrow \lambda_l \sum_k a_{lk}\,\delta q_k = 0$$

λ_l . $l = 1, 2, \ldots m$ undetermined multipliers

$$\lambda \equiv \lambda(q_i, t)$$

Hamilton's Principle is assumed to hold for nonholonomic systems:

$$\delta \int_{t_1}^{t_2} L\,dt = 0$$

$$\int_{t_1}^{t_2} \sum_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \, dt = 0$$

$$\lambda_\ell \sum_k a_{\ell k}\, \delta q_k = 0 \longrightarrow \int_{t_1}^{t_2} \sum_{k,\ell} \lambda_\ell a_{\ell k}\, \delta q_k \, dt = 0$$

Combining with Hamilton's principle:

$$\int_{t_1}^{t_2} dt \sum_{k=1}^{n} \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_\ell \lambda_\ell a_{\ell k} \right) \delta q_k = 0$$

δq_k's are still not independent, they are connected by m relation $\sum_k a_{\ell k}\, \delta q_k = 0$ (m number)

Now we choose λ_ℓ's to be such that:

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_\ell \lambda_\ell a_{\ell k} = 0 \quad \begin{cases} k = n-m+1 \dots n \\ m \text{ number} \end{cases}$$

These are in the nature of eqns. of motion for the last m of q_k variables.

With λ_ℓ determined by m equs. ;

$$\int_{t_1}^{t_2} dt \sum_{k=1}^{n-m} \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_\ell \lambda_\ell a_{\ell k} \right) \delta q_k = 0$$

Here the only δq_k's involved are the independent ones.

$$\longrightarrow \quad \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_\ell \lambda_\ell a_{\ell k} = 0 \qquad k=1, \ldots n-m$$

Combining these n-m equs. with the previous m equs. :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \sum_\ell \lambda_\ell a_{\ell k} \qquad k=1, \ldots n$$

(Lagrange's equs. for nonholonomic systems)

These equs. should be solved together with the

$$\sum_k a_{\ell k} \dot{q}_k + a_{\ell t} = 0 \qquad \ell = 1, \ldots m$$

for the m+n unknowns.

If we remove the constraints and apply external forces Q_k in such manner as to keep the motion of the system unchanged

$\longrightarrow$ the eqns. of motion would likewise remain the same.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k' \quad \text{generalized forces of constraint}$$

$$\rightarrow \quad Q_k' = \sum_{\ell} \lambda_\ell a_{\ell k}$$

Since $L = T - V$ with $V = V(q_1, \cdots q_n, t)$

$$\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_k} - \frac{\partial (T-V)}{\partial q_k} = Q_k'$$

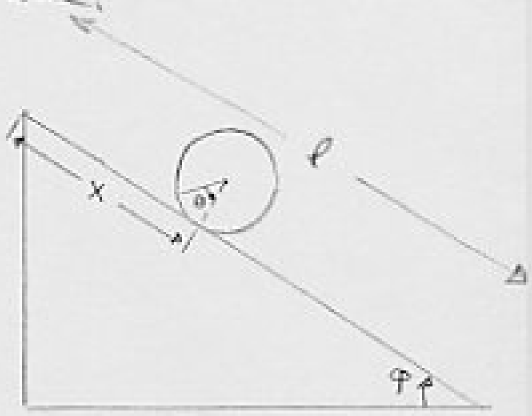
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k} + Q_k' \equiv Q_k \quad \text{Generalized forces}$$

$\sum_k a_{\ell k} dq_k + a_{\ell t} dt = 0$ does not include all the nonholonomic constraints, but it does include holonomic constraints:

$$f(q_1 \cdots q_n, t) = 0 \quad \longrightarrow \quad \sum_k \frac{\partial f_\ell}{\partial q_k} dq_k + \frac{\partial f_\ell}{\partial t} dt = 0$$

$$\longrightarrow \quad a_{\ell k} = \frac{\partial f_\ell}{\partial q_k} \qquad a_{\ell t} = \frac{\partial f_\ell}{\partial t}$$

Ex.: Rolling hoop on an inclined Plane:



The constraint: $\dot{x} = r\dot{\theta}$

$\longrightarrow r\,d\theta = dx$

Generalized Coord: x, θ

T = K.E. of motion of C.M. + K.E. about C.M.

$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} M\dot{x}^2 + \frac{1}{2}(Mr^2)\dot{\theta}^2$$

$$V = Mg(\ell - x)\sin\varphi$$

$$L = T - V = \frac{M\dot{x}^2}{2} + \frac{Mr^2\dot{\theta}^2}{2} - Mg(\ell - x)\sin\varphi$$

There is only one constraint equ. $m = 1$

$\longrightarrow$ there is only one λ

$n = 2$ number of generalized coords.

$$\sum_{k=1}^{n=2} a_{\ell k}\, dq_k = 0 \quad \rightarrow \quad \underset{r}{a_{11}}\,\underset{d\theta}{dq_1} + \underset{(-1)}{a_{12}}\,\underset{dx}{dq_2} = 0$$

$r\,d\theta - dx = 0$

$$\begin{cases} a_{11} = r \\ a_{12} = -1 \end{cases} \qquad \begin{cases} q_1 = \theta \\ q_2 = x \end{cases}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{\ell} a_{\ell k} \lambda_\ell$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \dot\theta} = Mr^2\dot\theta \\ \dfrac{\partial \mathcal{L}}{\partial \dot x} = M\dot x \end{cases} \qquad \begin{cases} \dfrac{\partial \mathcal{L}}{\partial \theta} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial x} = Mg\, S\varphi \end{cases}$$

$$\begin{cases} (1) \quad \dfrac{d}{dt}(Mr^2\dot\theta) - 0 = r\lambda \quad \rightarrow Mr^2\ddot\theta - r\lambda = 0 \\ (2) \quad \dfrac{d}{dt}(M\dot x) - Mg\,S\varphi = -\lambda \quad \rightarrow M\ddot x - Mg\,S\varphi + \lambda = 0 \end{cases}$$

$(3) \quad r\dot\theta = \dot x$ (constraint)

$\theta = ? \quad x = ? \quad \lambda = ?$

$$r\dot\theta = \dot x \rightarrow r\ddot\theta = \ddot x \xrightarrow{\text{sub. in (1)}} M\ddot x = \lambda$$

$$\xrightarrow{\text{sub. in (2)}} \lambda - Mg\,S\theta + \lambda = 0 \rightarrow \lambda = \frac{Mg\,S\varphi}{2}$$

$$M\ddot x = \lambda \rightarrow \ddot x = \frac{g\,S\varphi}{2}$$

$$r\ddot\theta = \ddot x \qquad \ddot\theta = \frac{g\,S\varphi}{2r}$$

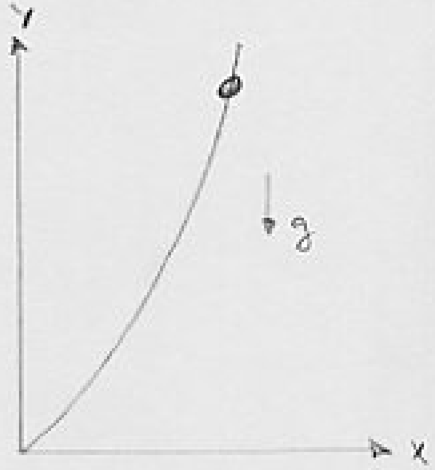
$$Q_k = \sum_\ell a_{\ell k}\lambda_\ell \longrightarrow \begin{cases} Q_1 = a_{11}\lambda = r\,\dfrac{Mg\,S\varphi}{2} \\ Q_2 = a_{12}\lambda = -\dfrac{Mg\,S\varphi}{2} \end{cases}$$

$$\ddot x = \frac{dv}{dt} = \frac{g\,S\varphi}{2} \longrightarrow v = \frac{g\,S\varphi}{2}\int dt \qquad (v_0 = 0)$$

$$\ddot x = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \qquad v\frac{dv}{dx} = \frac{g\,S\varphi}{2}$$

$$\int_0^v v\,dv = \frac{g\,S\varphi}{2}\int_0^\ell dx \qquad v = \sqrt{g\,\ell\,S\varphi}$$

Ex.: A bead sliding on a parabolic wire down under the gravitational potential



$Y = ax^2$ the equ. of constraint

$\rightarrow dy - 2ax\,dx = 0$

$$\sum_{k=1}^{n} a_{\ell k}\, dq_k + a_{\ell t}\, dt = 0$$

$\begin{cases} n = 2 \\ m = 1 \end{cases}$ $\begin{cases} k=1 & Y \\ k=2 & X \end{cases}$ generalized coord. (dependent) $\rightarrow$ 1 deg. of freedom

$$a_{11}\, dq_1 + a_{12}\, dq_2 = 0$$
$$dy - 2ax\, dx = 0$$
$$\Rightarrow \begin{cases} a_{11} = 1 \\ a_{12} = -2ax \end{cases} \quad \begin{cases} dq_1 = dy \\ dq_2 = dx \end{cases}$$

$$\mathcal{L} = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = \sum_{\ell} a_{\ell k} \lambda_\ell$$

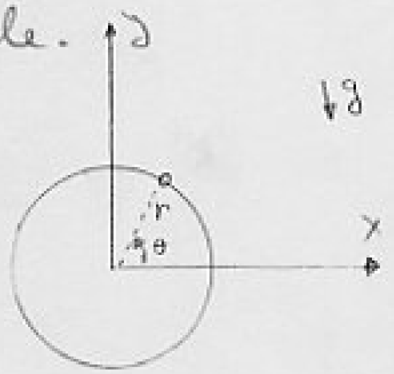
$$\begin{aligned} k=1 \rightarrow & \quad \frac{d}{dt}(m\dot{y}) + mg = (1)\,\lambda_1 \\ k=2 \rightarrow & \quad \frac{d}{dt}(m\dot{x}) - 0 = (-2ax)\,\lambda_1 \end{aligned}$$

$(\ell = 1$ only$)$

$$\begin{cases} m\ddot{y} + mg = \lambda_1 \\ m\ddot{x} = -2ax\lambda_1 \\ \dot{y} - 2ax\dot{x} = 0 \end{cases}$$

$$Q'_k = \sum_{\ell} a_{\ell k} \lambda_1 \Rightarrow \begin{cases} Q_y = \lambda_1 \\ Q_x = -2ax\lambda_1 \end{cases}$$ components of constraint force

Ex: A bead sliding on a vertical circle.



In polar coord.: $\begin{cases} r & (k=1) \\ \theta & (k=2) \end{cases}$ $n=2$

$r = \ell$ constraint $(m=1)$

$$\sum_{k=1}^{n=2} a_{\ell k}\, dq_k + a_{\ell t}\, dt = 0$$

$$\begin{aligned} a_{11} dq_1 + a_{12} dq_2 &= 0 \\ dr + 0 &= 0 \end{aligned} \Rightarrow \begin{cases} a_{11} = 1 \\ a_{12} = 0 \end{cases}$$

$\mathcal{L} = T - V$

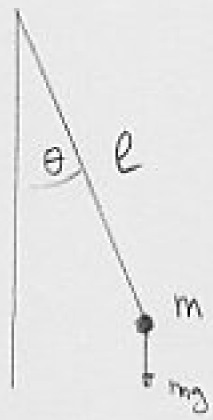
$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot r \hat e_r + r\dot\theta \hat e_\theta)^2$$
$$= \frac{1}{2} m (\dot r^2 + r^2 \dot\theta^2)$$

$$\mathcal{L} = \frac{1}{2} m (\dot r^2 + r^2 \dot\theta^2) - mg(r S\theta)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \sum_{\ell=1}^{m=1} a_{\ell k} \lambda_\ell$$

$$\begin{cases} \frac{d}{dt}(m\dot r) - mr\dot\theta^2 + mg\, S\theta = \lambda_1 \\ \frac{d}{dt}(mr^2\dot\theta) + mgr\, C\theta = 0 \end{cases} \Rightarrow \begin{cases} m\ddot r - mr\dot\theta^2 + mg S\theta = \lambda_1 \\ 2mr\dot r\dot\theta + mr^2\ddot\theta = -mgr C\theta \\ r = \ell \end{cases}$$

Ex.: Pendulum;



θ : generalized coord. (1 deg. of freedom)

$$\mathcal{L} = T - V = \frac{1}{2} m v^2 - (-mg\ell \cos\theta)$$

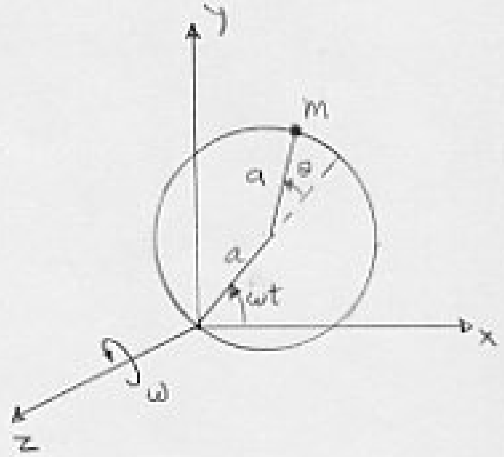
$$= \frac{1}{2} m \ell^2 \dot{\theta}^2 + mg\ell\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(m\ell^2\dot{\theta}\right) + mg\ell \sin\theta = 0$$

For small $\theta \longrightarrow \ddot{\theta} + \frac{g}{\ell}\theta = 0$

Ex.: A bead slides without friction on a hoop that rotates with const. angular velocity ω about an axis perpendicular to the plane of the hoop and passing through the edge of the hoop. The problem ignores both friction and gravity



θ: generalized coord.

The coords. of bead:

$$\begin{cases} x = a\cos\omega t + a\cos(\omega t+\theta) \\ y = a\sin\omega t + a\sin(\omega t+\theta) \end{cases}$$

$$\begin{cases} \dot{x} = -a\omega\sin\omega t - a(\omega+\dot{\theta})\sin(\omega t+\theta) \\ \dot{y} = a\omega\cos\omega t + a(\omega+\dot{\theta})\cos(\omega t+\theta) \end{cases} \quad \text{where } \omega = \text{const.}$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

Using the trigonometric identity:

$$\cos\omega t\cos(\omega t+\theta) + \sin\omega t\sin(\omega t+\theta) = \cos(\omega t+\theta-\omega t) = \cos\theta$$

$$L = T - V = T - 0 = T$$

$$L = \frac{1}{2}ma^2\left[\omega^2 + (\omega+\dot{\theta})^2 + 2\omega(\omega+\dot{\theta})\cos\theta\right]$$

Despite the explicit time-dependence in $x = x(t)$ and $y = y(t)$

L has no explicit time-dependence.

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{\partial L}{\partial\dot{\theta}} = \frac{1}{2}ma^2\left\{2(\omega+\dot{\theta}) + 2\omega\cos\theta\right\}$$

$$\frac{\partial L}{\partial\theta} = -\frac{1}{2}ma^2\left(2\omega(\omega+\dot{\theta})\sin\theta\right)$$

$$\frac{d}{dt}\left\{\frac{1}{2}ma^2\left[2(\omega+\dot\theta)+2\omega\cos\theta\right]\right\}-\frac{1}{2}ma^2\left[-2\omega(\omega+\dot\theta)\sin\theta\right]=0$$

$\rightarrow \quad \ddot\theta+\omega^2\sin\theta=0$ (like as pendulum equ.)

Remember $\ddot\theta+\frac{g}{l}\sin\theta=0$

This similarity illustrates the general equivalence between gravitational and centrifugal forces.

where $g \xrightarrow{\text{replaced by}} \omega^2 l$

Properties of Lagrange's Formulation:

I) Lagrange's eqns. are independent under coordinate transformation of the motion.

$$q_i = q_i(Q_1, \ldots Q_n, t) \qquad i = 1, \ldots n$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \longrightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Q}_i} - \frac{\partial \mathcal{L}}{\partial Q_i} = 0$$

II) Lagrangian is uncertain up to a total time derivative.

(The addition of an arbitrary time derivative to the Lagrangian does not affect the variational behavior of the integral)

$$\delta I = \delta \int_1^2 \left(\mathcal{L}(q_i, \dot{q}_i, t) + \frac{\partial F}{\partial t}\right) dt$$

$$\delta I = \delta \int_1^2 \mathcal{L}\, dt + \delta \int_1^2 \frac{\partial F}{\partial t} dt = 0$$

$$\delta \int_1^2 \mathcal{L}\, dt + \delta (F_2 - F_1) = 0$$

as the variation at the end points is zero.

III) Conservation laws:

Def.: $P_i = \frac{\partial L}{\partial \dot{q}_i}$ canonical momentum or conjugate "

$$\rightarrow \frac{d}{dt}(p_i) = \frac{\partial L}{\partial q_i}$$

If L does not depend on q_i explicitly (cyclic in q_i), then the corresponding generalized momentum is <u>conserved</u>.

Ex.: In a central potential

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$\frac{d}{dt} p_\theta = \frac{\partial L}{\partial \theta} = 0 \Rightarrow p_\theta = m r^2 \dot{\theta} = \text{const.}$$

Conservation of energy:

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i} \frac{dq_i}{dt} + \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} + \frac{\partial L}{\partial t}$$

$$= \sum_i \left(\frac{d}{dt} \frac{\partial L}{\dot{q}_i} \right) \frac{dq_i}{dt} + \quad '' \quad + ''$$

$$= \sum_i \frac{d}{dt} \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt} \left[L - \sum_i \dot{q}_i p_i \right] = \frac{\partial L}{\partial t}$$

$$h = \sum_{i=1}^{n} \dot{q}_i p_i - L \quad \rightarrow \quad \frac{dh}{dt} = -\frac{\partial L}{\partial t}$$

(Jacobi integral)

$h \equiv h(q_i, \dot{q}_i, t)$ energy func.

If L is cyclic in t (i.e. $\frac{\partial L}{\partial t} = 0$)

$\Longrightarrow h =$ const.

Note: h is the total energy when T is quadratic func. of velocities, and when constraints are time-indep. and holonomic.

(also $V \neq V(\dot{q})$)

($T = \cancel{T_0} + \cancel{T_1} + T_2$, i.e. tr. equs. don't contain t explicitly, as may occur when the constraints are t-indep. (a particle in moving system is subjected to a constraint))

Proof.:

Euler Theorem: If $f(x_1, x_2 \dots x_\alpha)$ is homogen of rank n with respect to all x's, then

$$n f = \sum_{k=1}^{\alpha} x_k \frac{\partial f}{\partial x_k}$$

$$h = \sum_{k=1}^{n} p_k \dot{q}_k - L = \sum_{k=1}^{n} \left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k - L$$

$$= \sum_{k=1}^{n} \left(\frac{\partial (T-V)}{\partial \dot{q}_k}\right) \dot{q}_k - L$$

$$= \sum_{k=1}^{n} \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k - L$$

(T: homogen of rank 2 with respect to $\dot{q}_i$)

V : $\dot{q}_i$ indep.

$$H = 2T - L = 2T - (T - V) = T + V = E$$

Ex.: A particle in one dimension

$$L = \frac{1}{2} m \dot{x}^2 - V(x)$$

$$p = m\dot{x} \qquad h = \dot{x}(m\dot{x}) - L = m\dot{x}^2 - V$$

$$h = \frac{1}{2} m \dot{x}^2 + V = T + V = E$$

Note: $H = H(P_i, q_i, t)$ allways

IV) Mechanical Similarity:

If $\mathcal{L}$ is multiplied by a const. value, it does not affect on the equs. of motion.

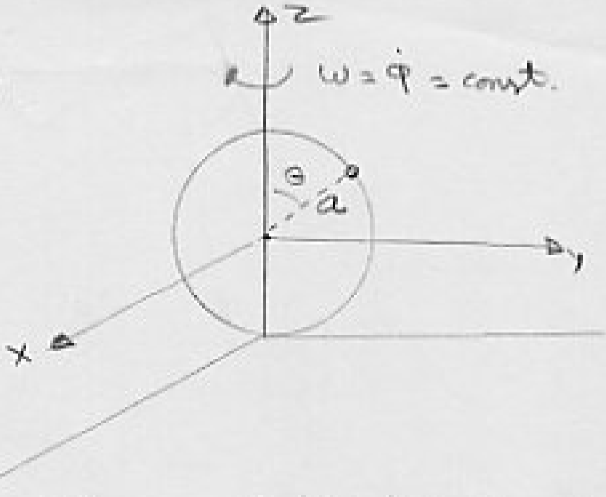
Remark: $T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_j \frac{\partial \bar{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \bar{r}_i}{\partial t} \right)^2$

$$T = M_0 + \sum_j M_j \dot{q}_j + \frac{1}{2} \sum_{j,k} M_{jk} \dot{q}_j \dot{q}_k$$

$$M_0 = \sum_i \frac{1}{2} m_i \left(\frac{\partial \bar{r}_i}{\partial t} \right)^2 \qquad M_j = \sum_i m_i \frac{\partial \bar{r}_i}{\partial t} \cdot \frac{\partial \bar{r}_i}{\partial q_j}$$

$$M_{jk} = \sum_i m_i \frac{\partial \bar{r}_i}{\partial q_j} \cdot \frac{\partial \bar{r}_i}{\partial q_k} \qquad T = T_0 + T_1 + T_2$$

Ex.: A bead on a vertical rotating circle wire, revolving on a circular motion around z.



$r = a$ holonomic constraint

$L = T - V$

$L = \frac{1}{2} m \left[(\cancel{\dot{r}^2} + r^2 \dot{\theta}^2) + r^2 \sin^2\theta \, \dot{\varphi}^2 \right] - mgr\cos\theta$

$L = \frac{1}{2} m (a^2 \dot{\theta}^2 + a^2 \sin^2\theta \, \dot{\varphi}^2) - mga\cos\theta$

$\left\{ \begin{array}{l} \text{deg of freedom} = 1 \\ \theta \end{array} \right.$

The equ of motion $\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$

$ma^2 \ddot{\theta} = ma^2 \sin\theta \cos\theta \, \dot{\varphi}^2 + mga\sin\theta$

For equilibrium circular orbits: $\left\{ \begin{array}{l} \theta = \theta_0 \\ \dot{\theta} = 0 \end{array} \right.$

$\rightarrow a\cos\theta_0 \, \dot{\varphi}^2 + g = 0$ $\quad \cos\theta_0 = -\frac{g}{a\omega^2}$