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Classical Mechanics

Useful Book : Classical Mechanics

Herbert Goldstein

Chapter 1

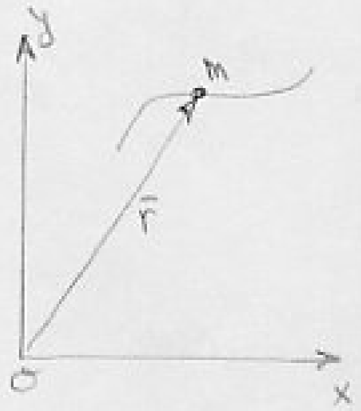
Survey of the Elementary principals

1-1 Mechanics of a Particle

$\bar{r} = \bar{r}(t)$ radius vector of a particle

$\bar{V} = \frac{d\bar{r}}{dt} = \dot{\bar{r}}$ vector velocity

$\bar{a} = \frac{d^2\bar{r}}{dt^2} = \ddot{\bar{r}} = \dot{\bar{V}}$



$\bar{P} = m\bar{V}$ linear momentum

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \qquad \begin{cases} V_x = \frac{dx}{dt} = \dot{x} \\ V_y = \frac{dy}{dt} = \dot{y} \\ V_z = \frac{dz}{dt} = \dot{z} \end{cases}$$

$$\begin{cases} a_x = \frac{d^2x}{dt^2} = \ddot{x} = \dot{V}_x \\ a_y = \frac{d^2y}{dt^2} = \ddot{y} = \dot{V}_y \\ a_z = \frac{d^2z}{dt^2} = \ddot{z} = \dot{V}_z \end{cases}$$

$$\begin{cases} P_x = mV_x \\ P_y = mV_y \\ P_z = mV_z \end{cases}$$

Newtons laws

I- The first law of motion

In the absence of a resultant force, a body at rest will remain at rest, and a body in motion will continue in motion in a straight line at constant speed.

II) The second law of motion

There exist frames of reference in which the motion of the particle is described by the differential equation:

$$\bar{F} = \frac{d\bar{p}}{dt}$$

or $$\bar{F} = \frac{d(m\bar{v})}{dt}$$

$$\bar{F} = m\frac{d\bar{v}}{dt} = m\bar{a} \quad \text{in nonrelativistic limit}$$

where

$$\bar{a} = \frac{d^2\bar{r}}{dt^2} \quad \text{vector acceleration}$$

A reference frame in which equ $\bar{F} = \frac{d\bar{P}}{dt}$ is valid, is called an inertial or Galilean system.

III) The third law of motion

When one body exerts a force on another body, the second body exerts a force on the first body of the same magnitude but in the opposite direction.

$$\bar{\bar{F}}_{ab} = -\bar{\bar{F}}_{ba}$$

Conservation Theorems

1- Conservation of the Linear Momentum

If $\bar{F} = 0$ total force

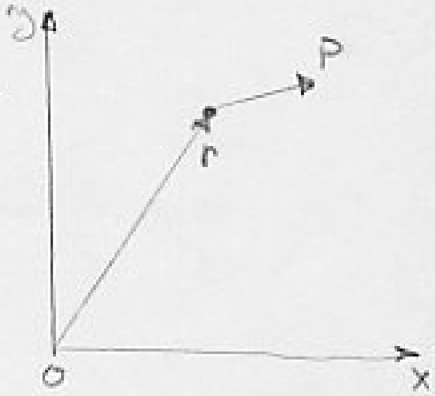
then $\bar{P} = \text{const.}$

$$\bar{F} = \frac{d\bar{P}}{dt} = 0 \qquad \dot{\bar{P}} = 0 \Rightarrow \bar{P} = \text{const.}$$

2- Conservation of the Angular Momentum

$\bar{L} = \bar{r} \times \bar{P}$ Angular Momentum

$\bar{N} = \bar{r} \times \bar{F}$ Moment of Force (Torque)



$$\bar{N} = \bar{r} \times \bar{F} = \bar{r} \times \frac{d\bar{P}}{dt} = \bar{r} \times \frac{d}{dt}(m\bar{v})$$

using

$$\frac{d}{dt}(\bar{r} \times m\bar{v}) = \frac{d\bar{r}}{dt} \times m\bar{v} + \bar{r} \times \frac{d}{dt}(m\bar{v})$$

$$= \cancelto{0}{\bar{v} \times m\bar{v}} + \bar{r} \times \frac{d}{dt}(m\bar{v}) = \bar{r} \times \frac{d}{dt}(m\bar{v})$$

$$\bar{N} = \bar{r} \times \frac{d}{dt}(m\bar{v}) = \frac{d}{dt}(\bar{r} \times m\bar{v}) = \frac{d}{dt}(\bar{r} \times \bar{P}) = \frac{d\bar{L}}{dt}$$

Note: $\bar{N}$ and $\bar{L}$ depend upon the point O

If $\bar{N} = 0 \longrightarrow \bar{N} = \frac{d\bar{L}}{dt} = 0$

$\dot{\bar{L}} = 0 \longrightarrow \bar{L} = \text{const.}$

3- Conservation of the Energy

By def.

$$W_{12} = \int_1^2 \bar{F} \cdot d\bar{s}$$

F: external force

$$\int_1^2 \bar{F} \cdot d\bar{s} = \int_1^2 \frac{d\bar{p}}{dt} \cdot d\bar{s} = \int_1^2 \frac{d(m\bar{v})}{dt} \cdot d\bar{s} = m \int_1^2 \frac{d\bar{v}}{dt} \cdot (\bar{v}\,dt)$$

for m = const.

$$W_{12} = \frac{m}{2} \int_1^2 \frac{d}{dt}(\bar{v}^2)\,dt = \frac{m}{2} \int_1^2 d(\bar{v}^2)$$

$$= \frac{m}{2}\left(\bar{V}_2^2 - \bar{V}_1^2\right) = \frac{1}{2} m V_2^2 - \frac{1}{2} m V_1^2$$

$$W_{12} = T_2 - T_1$$

If the force field is such that the work W_{12} is the same for any physically possible path between points 1 and 2, then the force (and the system) is said to be conservative.

i.e. $W_{12} = W_{12}' = W_{12}''$

If the integration is path independent, then:

$$\oint \bar{F} \cdot d\bar{s} = 0$$

$$\oint \bar{F} \cdot d\bar{s} = \int_a \operatorname{curl} \bar{F} \cdot \bar{n}\, da = 0$$

$\rightarrow \operatorname{curl} \bar{F} = 0 \rightarrow \bar{F} = -\bar{\nabla} V(r)$ (necessary and sufficient cond.)

$V(r)$ Potential

$$\bar{F} = -\bar{\nabla} V(r) = -\bar{\nabla}\left[V(r) + C\right]$$

Hence the zero level of $V(r)$ is arbitrary.

The existence of $V(r)$ can be infered intuitively by a simple argument:

If W_{12} is path independent between points 1 and 2

$$\int_1^2 \bar{F} \cdot d\bar{s} = V_2 - V_1 = \Delta V \rightarrow \bar{F} \cdot d\bar{s} = \pm dV$$

$$\bar{F} \cdot d\bar{s} = -dV \qquad F_s = \frac{-\partial V}{\partial s} \longrightarrow \bar{F} = -\bar{\nabla} V(\bar{r})$$

by def.

For a conservative system

$$W_{12} = \int_1^2 \bar{F}\cdot d\bar{s} = -\int_1^2 \bar{\nabla}V(r)\cdot d\bar{s} = -\int_1^2 dV = V_1 - V_2$$

Combining $W_{12} = T_2 - T_1$

$$\rightarrow \quad T_1 + V_1 = T_2 + V_2 = \mathcal{E} \quad \text{const. total energy}$$

Energy conservation Theorem:

If the forces acting on a particle are conservative, then the total energy of the particle, $T+V=\mathcal{E}$, is conserved.

1-2 Mechanics of a System of Particles

For a system of many particles there exist two kind of forces: I- internal, II-external

Consider N particle

Newton's Second law $\rightarrow \dot{\bar{P}}_k = m_k \ddot{\bar{r}}_k = \bar{F}^i_k + \bar{F}^e_k$

$\bar{F}^i_k$: internal force on k^{th} particle

$\bar{F}^e_k$: external " " "

$$\bar{F}^i_k = \sum_{\substack{j=1 \\ j\neq k}}^{N} \bar{F}^i_{kj} \qquad (\bar{F}^i_{kk} = 0)$$

(weak law of action and reaction)

We assume $\bar{F}^i_{kj}$ obey Newton's third law of motion (like $\bar{F}^e_k$)

Summed over all particales:

$$\frac{d^2}{dt^2} \sum_{k=1}^{N} m_k \bar{r}_k = \sum_k \bar{F}^e_k + \sum_k \sum_{\substack{j \\ k\neq j}} \bar{F}^i_{kj}$$

But $\sum_k \bar{F}^e_k = \bar{F}^e$ total external force

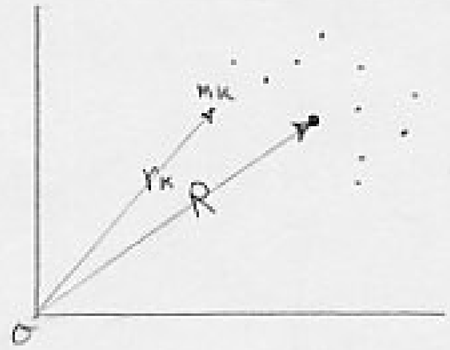
and by the Newton's third law $\bar{F}^i_{kj} + \bar{F}^i_{jk} = 0$

$$\rightarrow \sum_{\substack{k,j \\ k\neq j}} F^i_{kj} = 0$$

Now we define the vector

$$\bar{R} = \frac{\Sigma m_k \bar{r}_k}{\Sigma m_k} = \frac{\Sigma m_k \bar{r}_k}{M}$$

as the center of mass coordinate (or center of gravity).



With this definition

$$M \frac{d^2\bar{R}}{dt^2} = \bar{F}^e$$

This equ. states that the center of mass moves as if the total external force were acting on the entire mass of the system consentrated at the center of mass.

The total linear Momentum

$$\bar{P} = \Sigma m_k \frac{d\bar{r}_k}{dt} = M \frac{d\bar{R}}{dt}$$

(The analogous for angular momentum is more complicated)

$$\frac{d^2}{dt^2} \sum_{k=1}^{N} m_k \bar{r}_k = \frac{d}{dt} \sum_{k=1}^{N} m_k \bar{v}_k = \frac{d}{dt} \sum_{k=1}^{N} \bar{P}_k = \bar{F}^e$$

$$\frac{d\bar{P}}{dt} = \bar{F}^e \qquad \text{where} \quad \bar{P} = \sum_{k=1}^{N} \bar{P}_k$$

Conservation Theorem:

The total linear Momentum of a system of Particles is conserved, if the total external force is Zero.

Total angular momentum of a system of particales:

$$\bar{L}_{kQ} = (\bar{r}_k - \bar{r}_Q) \times \left[m_k (\dot{\bar{r}}_k - \dot{\bar{r}}_Q) \right]$$

the angular momentum of particle k about point Q

$$\frac{d}{dt}\bar{L}_{kQ} = (\bar{r}_k - \bar{r}_Q) \times \frac{d}{dt}\left[m_k (\dot{\bar{r}}_k - \dot{\bar{r}}_Q) \right] + \left[\frac{d}{dt}(\bar{r}_k - \bar{r}_Q) \right] \times \left[m_k (\dot{\bar{r}}_k - \dot{\bar{r}}_Q) \right]$$

$$\frac{d}{dt}\bar{L}_{kQ} = (\bar{r}_k - \bar{r}_Q) \times \frac{d\bar{P}_k}{dt} - m_k (\bar{r}_k - \bar{r}_Q) \times \ddot{\bar{r}}_Q$$

$$\cancelto{0}{- m_k (\dot{\bar{r}}_k - \dot{\bar{r}}_Q) \times (\dot{\bar{r}}_k - \dot{\bar{r}}_Q)}$$

But $\frac{dP_k}{dt} = \bar{F}^e_k + \bar{F}^i_k$

$$\frac{d}{dt}\bar{L}_{kQ} = (\bar{r}_k - \bar{r}_Q) \times \bar{F}^e_k + (\bar{r}_k - \bar{r}_Q) \times \bar{F}^i_k - m_k (\bar{r}_k - \bar{r}_Q) \times \ddot{\bar{r}}_Q$$

Define $\bar{L}_Q = \sum_k \bar{L}_{kQ}$ and $\bar{N}_Q = \sum_k (\bar{r}_k - \bar{r}_Q) \times \bar{F}^e_k$

$$\frac{d}{dt}\sum_k \bar{L}_{kQ} = \sum_k (\bar{r}_k - \bar{r}_Q) \times \bar{F}^e_k + \sum_k (\bar{r}_k - \bar{r}_Q) \times \bar{F}^i_k - \sum_k m_k (\bar{r}_k - \bar{r}_Q) \times \ddot{\bar{r}}_Q$$

$$\frac{d\bar{L}_Q}{dt} = \bar{N}_Q + \sum_k (\bar{r}_k - \bar{r}_Q) \times \bar{F}^i_k - M(\bar{R} - \bar{r}_Q) \times \ddot{\bar{r}}_Q$$

if $\begin{cases} \ddot{\bar{r}}_Q = 0 & \text{or} \\ \bar{R} = \bar{r}_Q & \text{or} \\ \bar{R} - \bar{r}_Q \parallel \ddot{\bar{r}}_Q \end{cases} \longrightarrow M(\bar{R} - \bar{r}_Q) \times \ddot{\bar{r}}_Q = 0$

Now consider

$$\sum_k (\bar{r}_k - \bar{r}_Q) \times \bar{F}_k^i = \sum_k \sum_{\substack{j \\ k \neq j}} (\bar{r}_k - \bar{r}_Q) \times \bar{F}_{kj}^i$$

This summation contains pairs of

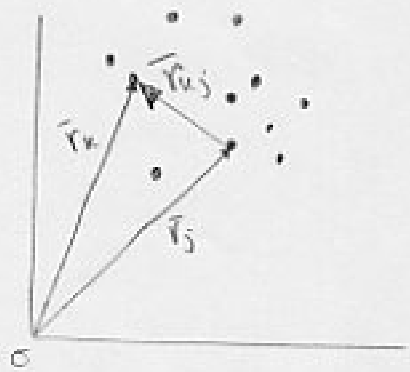
$$(\bar{r}_k - \bar{r}_Q) \times \bar{F}_{kj} + (\bar{r}_j - \bar{r}_Q) \times \bar{F}_{jk} \qquad \text{where } \bar{F}_{jk} = -\bar{F}_{kj}$$

$$[(\bar{r}_k - \bar{r}_Q) - (\bar{r}_j - \bar{r}_Q)] \times \bar{F}_{kj} = (\bar{r}_k - \bar{r}_j) \times \bar{F}_{kj} = \bar{r}_{kj} \times \bar{F}_{kj}$$

If the internal forces between two particles in addition to being equal and opposite also lie along the line jointing the particles (strong law of action and reaction), then:

$$\text{all} \quad \bar{r}_{kj} \times \bar{F}_{kj} = 0$$

$$\text{and} \quad \frac{d\bar{L}_Q}{dt} = \bar{N}_Q^e$$



Conservation Theorem:

The total angular momentum $\bar{L}$ is constant in Time if the applied external Torque is Zero.

This is a vector theorem, that is,

$$L_z = \text{const.} \quad \text{if} \quad N_z^e = 0$$

even N_x^e or/and $N_y^e \neq 0$

Note: The conservation of linear momentum in the absence of applied forces assumes that the <u>weak</u> law of action and reaction is valid for the internal forces.

The conservation of the total angular momentum of the system in the absence of applied torques requires the validity of the <u>strong</u> law of action and reaction (central forces).

Total angular momentum of the system of the particles:

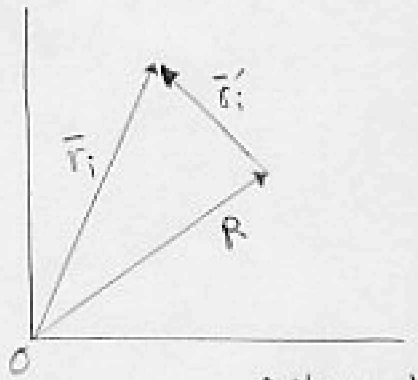
$\bar{L} = \sum_i \bar{r}_i \times \bar{P}_i$ about origin O

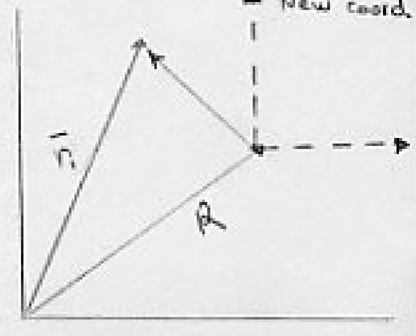
$\bar{R}$ radius vector from O to the center of mass

$\bar{r}_i = \bar{r}_i' + \bar{R}$ and $\bar{V}_i = \bar{V}_i' + V$

$\bar{V} = \frac{d\bar{R}}{dt}$ velocity of the center of mass relative to O

$\bar{V}_i' = \frac{d\bar{r}_i'}{dt}$ velocity of the i^{th} particle relative to the center of mass





$$\bar{L} = \sum_i (\bar{r}_i' + \bar{R}) \times m_i (\bar{V}_i' + \bar{V})$$

$$= \sum_i m_i (\bar{r}_i' \times \bar{V}_i' + \bar{r}_i' \times \bar{V} + \bar{R} \times \bar{V}_i' + \bar{R} \times \bar{V})$$

$$= \sum_i m_i \bar{r}_i' \times \bar{V}_i' + \underbrace{(\sum_i m_i \bar{r}_i')}_{0} \times \bar{V} + \bar{R} \times \underbrace{(\sum_i m_i \bar{V}_i')}_{0} + (\sum_i m_i) \bar{R} \times \bar{V}$$

Since $\bar{R} = \dfrac{\sum m_i \bar{r}_i}{M} \longrightarrow \sum_i m_i \bar{r}_i' = 0$

and $\sum_i m_i \bar{V}_i' = \dfrac{d}{dt} \sum_i m_i \bar{r}_i' = 0$

$$\bar{L} = R \times (M\bar{V}) + \sum_i \bar{r}_i' \times \bar{P}_i'$$

If the center of mass is at rest with respect to O, ($\bar{V} = 0$), the angular momentum will be independent of point of reference!

1-3 Constraints

The main equ. in all physical problems is:

$$m_k \ddot{\bar{r}}_k = \bar{F}^e_k + \sum_l \bar{F}^i_{lk}$$

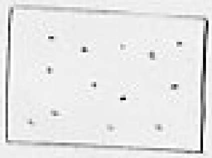
Constraints limit the motion of the system.

Examples:

1- rigid body all r_{ij} = const

2- beads on a wire 

3- Gas molecules within a container



4- a particle placed on the surface of solid sphere

I- Holonomic Constraints:

They have the following form:

$$f(\bar{r}_1, \bar{r}_2, \bar{r}_3 \dots t) = 0$$

II- Other constraints are called non-holonomic

Ex.: The constraint for rigid body which is

$$(\bar{r}_i - \bar{r}_j)^2 - c_{ij}^2 = 0$$

is a holonomic constraint.

Ex.: A particle constrained to move along any curve is another example for holonomic constraint

Ex.: A particle placed on a sphere is an example for nonholonomic constraint.

$$r^2 - a^2 \geqslant 0$$

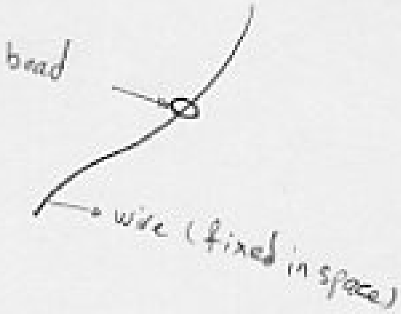
Rheonomous Constraints:

In these kind of constraints, the equations of constraint contain the time as an explicit variable.

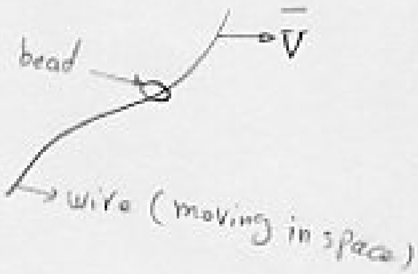
Scleronomous Constraints:

The equations of constraint in this case are not explicitly dependent on time.

Ex.: Scleronomous constraint



Ex.: Rheonomous constraint



Constraints introduce two types of difficulties:

1- $\bar{r}_k$ are no longer independent
hence the equs of motion are not all independent.

2- There exist forces of constraint, which must be obtained.
For example the force that the wire exerts on the bead (or the wall on the gas particle).

In the case of holonomic constraints, the first difficulty is solved by introduction of <u>generalized coordinates</u>.

A system of N particle, free from constraints, has 3N independent coordinates, or <u>degrees of freedom</u>.

If there exist K eqns. of the form $f(\bar{r}_1, \bar{r}_2, \bar{r}_3, \dots t) = 0$

$\longrightarrow$ we obtain 3N-K independent coordinates

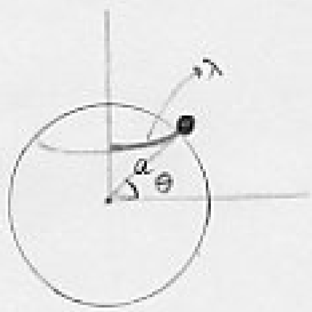
$\longrightarrow$ and the system has 3N-K degree of freedom.

Introduction 3N-K independent variables, $q_1, q_2, \dots q_{3N-K}$ the old coordinates are expressed as

$$\bar{r}_1 = \bar{r}_1(q_1, q_2, \dots q_{3N-K}, t)$$
$$\vdots$$
$$\bar{r}_N = \bar{r}_N(q_1, q_2, \dots q_{3N-K}, t)$$

Parametric representation of $(\bar{r}_i)$

and vice versa



Ex.: $r^2 - a^2 \geq 0$

2-deg. of freedom

Generalized Coordinates $\left\{ \begin{matrix} \theta \\ \lambda \end{matrix} \right.$

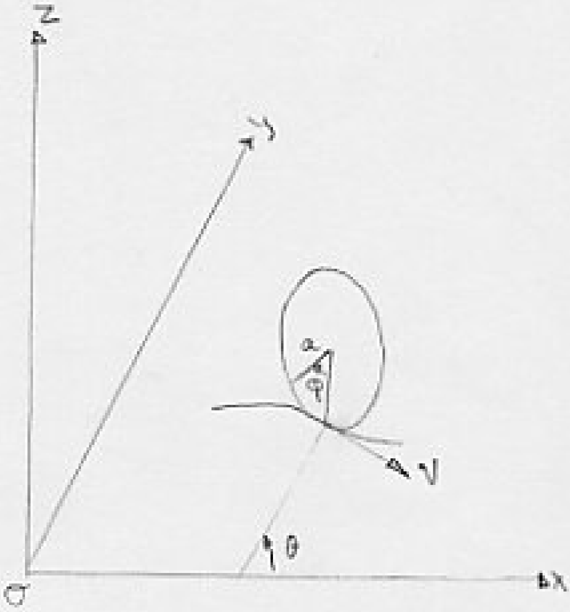
Ex.:

Double Pendulum

2-deg. of freedom

Generalized Coordinates $\left\{ \begin{matrix} \theta_1 \\ \theta_2 \end{matrix} \right.$

Ex.: Nonholonomic Case:



Generalized Coord. $\begin{cases} \begin{cases} x \\ y \end{cases} & \text{center of disk} \\ \begin{cases} \theta \\ \varphi \end{cases} & \text{orientation} \end{cases}$

As a result of constraint:

$$v = a\dot{\varphi}$$

also
$$\dot{x} = v \sin\theta$$
$$\dot{y} = v \cos\theta$$

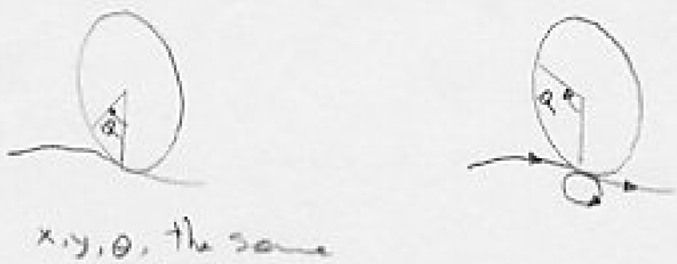
Combining $\Longrightarrow$ $\begin{cases} dx - a\sin\theta \, d\varphi = 0 \\ dy - a\cos\theta \, d\varphi = 0 \end{cases}$ nonholonomic constraints

It is not possible to turn these constraints in the form of

$f(\bar{r}_1, \bar{r}_2, \bar{r}_3 \ldots t) = 0$, without solving the problem.

Physically there is no direct functional relation between φ and the other coordinates x, y and θ.

i.e.:



x, y, θ, the same
but φ different

Nonholonomic constraints may involve higher order derivatives.