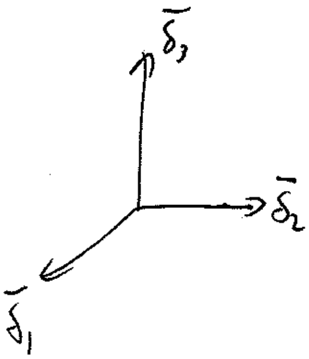
Tensor Review

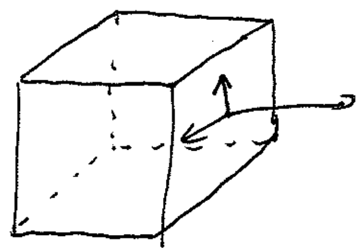
Scalar → specified by magnitude (0th rank tensor)

vector → magnitude and direction (1st rank tensor)

⇒ tensor specifies magnitude and a number of directions. (n → nth rank tensor)

Stress tensor = 2nd rank tensor





$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

Vector dot product

$$\bar{u} = u_1 \bar{\delta}_1 + u_2 \bar{\delta}_2 + u_3 \bar{\delta}_3$$

$$\bar{v} = v_1 \bar{\delta}_1 + v_2 \bar{\delta}_2 + v_3 \bar{\delta}_3$$

$$\bar{u} \cdot \bar{v} = \sum_i u_i \bar{\delta}_i \cdot \sum_j v_j \bar{\delta}_j = \sum_i \sum_j (u_i v_j)(\bar{\delta}_i \cdot \bar{\delta}_j)$$

Kronecker Delta $\equiv (\bar{\delta}_i \cdot \bar{\delta}_j) = \delta_{ij}$

$$\Rightarrow \quad \boxed{\bar{u} \cdot \bar{v} = u_i v_i}$$

Einstein summation convention: repeated indices imply summation.

<u>Cross product</u>:

$$\bar{u} \times \bar{v} = (u_2 v_3 - u_3 v_2)\bar{\delta}_1 + (u_3 v_1 - u_1 v_3)\bar{\delta}_2 + (u_1 v_2 - u_2 v_1)\bar{\delta}_3$$

introduce permutation symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & ijk \text{ cyclic order (eg } 123, 231, 312) \\ 0 & \\ -1 & ijk \text{ anticyclic order (eg } 321, 213, 132) \end{cases}$$

$$\Rightarrow \quad \boxed{\bar{u} \times \bar{v} = u_i v_j \varepsilon_{ijk} \bar{\delta}_k}$$

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} \qquad \boxed{\bar{\delta}_i \times \bar{\delta}_j = \varepsilon_{ijk} \bar{\delta}_k}$$

$$\varepsilon_{ijk} = -\varepsilon_{ikj}$$

<u>Second rank tensor</u>

$$\bar{\bar{\tau}} = \tau_{11} \bar{\delta}_1 \bar{\delta}_1 + \tau_{12} \bar{\delta}_1 \bar{\delta}_2 + \tau_{13} \bar{\delta}_1 \bar{\delta}_3 + \cdots$$

$$= \tau_{ij} \bar{\delta}_i \bar{\delta}_j$$

$$\boxed{\begin{aligned} \bar{u}\bar{v} &= u_i \bar{\delta}_i v_j \bar{\delta}_j \\ &= u_i v_j \bar{\delta}_i \bar{\delta}_j \end{aligned}}$$

$\bar{\delta}_i \bar{\delta}_j$ = "Dyadic tensor" unit tensor $\bar{\bar{I}} = \delta_{ij} \bar{\delta}_i \bar{\delta}_j$

Operations on tensors

1) $\bar{\delta}_i \bar{\delta}_j \cdot \bar{\delta}_k \bar{\delta}_l = \delta_{jk} \bar{\delta}_i \bar{\delta}_l \Rightarrow$ tensor

2) $\bar{\delta}_i \cdot \bar{\delta}_j \bar{\delta}_k = \delta_{ij} \bar{\delta}_k \Rightarrow$ vector

3) $\bar{\delta}_i \bar{\delta}_j : \bar{\delta}_k \bar{\delta}_l = \delta_{jk} \delta_{il} \Rightarrow$ Scalar

(double dot product)

4) $\bar{\bar{I}} : \bar{\bar{A}} = A_{ii} \Rightarrow$ trace of A

$\bar{\bar{I}} : \bar{\bar{I}} = 3$

Calculus of vectors and Tensors

Divergence: 1) $\nabla \cdot \bar{u} = \frac{\partial}{\partial x_i} \bar{\delta}_i \cdot u_j \bar{\delta}_j = \frac{\partial u_j}{\partial x_i} \delta_{ij}$

$= \frac{\partial u_i}{\partial x_i}$

2) $\left(\nabla \cdot \bar{\bar{\tau}}\right)_i = \frac{\partial \tau_{ij}}{\partial x_j}$

3) <u>curl</u>

$$\nabla \times \bar{u} = \frac{\partial}{\partial x_i}\bar{\delta}_i \times u_j \bar{\delta}_j = \frac{\partial u_j}{\partial x_i}(\bar{\delta}_i \times \bar{\delta}_j)$$

$$= \frac{\partial u_j}{\partial x_i}\varepsilon_{ijk}\bar{\delta}_k$$

If $\nabla \cdot \bar{u} = 0$ the field is called solenoidal

$\nabla \times \bar{u} = 0$ the field is called irrotational.

<u>Symmetric and Antisymmetric tensors</u>

1) A tensor is symmetric if $\boxed{B_{ij} = B_{ji}}$

2) antisymmetric if $\boxed{B_{ij} = -B_{ji}}$

3) A tensor can be represented as a sum of a symmetric and antisymmetric tensor

$$\boxed{B_{ij} = \underbrace{\tfrac{1}{2}(B_{ij} + B_{ji})}_{\text{symmetric}} + \underbrace{\tfrac{1}{2}(B_{ij} - B_{ji})}_{\text{antisymmetric}}}$$