Electromagnetic momentum density and the Poynting vector in static fields

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Many static configurations involving electrical currents and charges possess angular momentum in electromagnetic form; two examples are discussed here, an electric charge in the field of a magnetic dipole, and an electric charge in the vicinity of a long solenoid. These provide clear evidence of the physical significance of the circulating energy flux indicated by the Poynting vector, as the angular momentum of the circulating electromagnetic energy can be converted to mechanical angular momentum by turning off the magnetic field. Electromagnetic momentum is created whenever electric fields change in the presence of a magnetic field and whenever magnetic fields change in the presence of an electric field. When simple dielectrics are involved, the momentum density can be resolved into two components, a pure-field component $\epsilon_0\mathbf{E}\times\mathbf{B}$ and a component $\chi_e\epsilon_0\mathbf{E}\times\mathbf{B}$ associated with the polarization of the dielectric, the sum being $\epsilon_r\epsilon_0\mathbf{E}\times\mathbf{B}=\mathbf{D}\times\mathbf{B}$. It is argued that the latter component should be considered to be part of the electromagnetic momentum density, whose value then is $\mathbf{D}\times\mathbf{B}$.

I. INTRODUCTION

The concept of electromagnetic momentum in static electromagnetic configurations and the relationship between the vector potential of a current distribution and electromagnetic momentum were described at least as early as 1904 by Thomson,[1,2] but much of what he described has been overlooked or ignored. Many examples involving electromagnetic momentum are deceptively simple in their configurations, yet difficult to analyze, and many errors have appeared in the literature over the years in the analysis of simple problems. In this paper, we will discuss the angular momentum associated with the following systems: (1) an electric charge and a magnetic monopole; (2) an electric charge and a magnetic dipole; and (3) an electric charge and a long solenoid. All of these have been discussed in the past, in many cases either incompletely or erroneously, a notable exception being Furry's excellent analysis[3] of a variety of configurations. We will also discuss the linear momentum associated with a parallel-plate capacitor in a magnetic field. The latter example is of special value in making clear the role played by a dielectric in electromagnetic momentum whereby impulses are delivered to the dielectric whenever there is a change in the polarization of the dielectric in the presence of a magnetic field, or whenever there is a change in the magnetic field with constant polarization.

Thomson[1] identified the angular momentum $\mathbf{L}$ associated with a configuration consisting of magnetic monopole m and an electric charge q as

$$\mathbf{L}=\hat{\mathbf{r}}\mu_0 qm/4\pi, \tag{1}$$

where $\hat{\mathbf{r}}$ is the unit vector directed from q to m. Thomson evaluated the angular momentum in terms of the fields as

$$\mathbf{L}=\int \mathbf{r}\times(\epsilon_0\mathbf{E}\times\mathbf{B})d\tau, \tag{2}$$

where $\epsilon_0\mathbf{E}\times\mathbf{B}$ is the electromagnetic momentum density (emmd) in empty space and the integration is over all space. Thomson did not present the details of the integration; to reproduce his result, place q at $-b/2$ and $m+b/2$ on the z axis, as illustrated in Fig. 1. $\mathbf{E}\times\mathbf{B}$ is azimuthal about z, so all volume elements contribute to angular momentum in the $+z$ direction, and

$$L=\int \rho\epsilon_0 EB\sin\theta\, d\tau$$
$$=\epsilon_0\int \rho\frac{q}{4\pi\epsilon_0 r_1^2}\frac{\mu_0 m}{4\pi r_2^2}\sin(\theta_2-\theta_1)d\tau$$
$$=\frac{\mu_0 qm}{4\pi}\int\frac{\rho}{4\pi r_1^2 r_2^2}\sin(\theta_2-\theta_1)d\tau=\frac{\mu_0 qm}{4\pi}. \tag{3}$$

The details of the integration have been given by Adawi.[4]

Pugh and Pugh[5] have provided a very clear example in which use of the concept of emmd is essential in explaining the angular momentum of a system. Their system comprised a pair of concentric spheres with an electric field between them, the inner one magnetized; as the system is charged, it develops mechanical angular momentum without the application of any external mechanical torques. Romer[6] has also described two interesting examples in which emmd plays a role in understanding the angular momentum of the system—a solenoid with a coaxial cylindrical electrode inside it, and concentric spherical electrodes with the outer sphere wound so as to constitute a magnetic dipole. In both cases the inner electrode has a charged particle source on it from which particles are allowed to pass through an aperture in the outer electrode.

Thomson[2] was probably the first to provide a clear statement to the effect that the vector potential $\mathbf{A}(\mathbf{r})$ of a static current distribution $\mathbf{J}(\mathbf{r}')$ is equal to the electromagnetic momentum of a unit charge placed at $\mathbf{r}$ in the vicinity of that current distribution, where

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau'. \tag{4}$$

Nonetheless, the vector potential was long widely regarded as a purely mathematical convenience without physical meaning.[7] Calkin[8] and Konopinski[9] rediscovered this property of the vector potential that Thomson described. However, even before their publications, it was common practice in quantum mechanics to regard the product of the

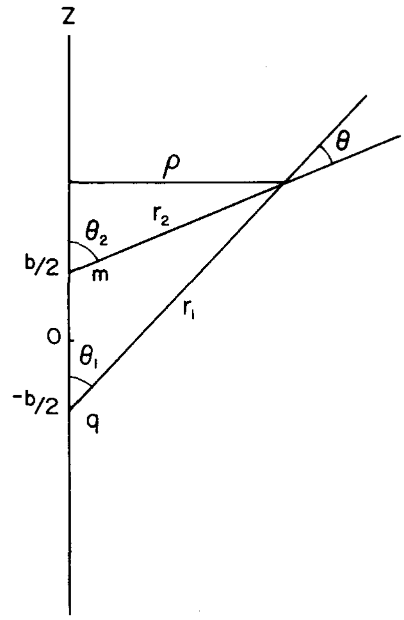

Fig. 1. Geometry associated with the fields of charge q at $-b/2$ and magnetic monopole m at $+b/2$ on the z axis. The field of $\epsilon_0 \mathbf{E}\times\mathbf{B}$ is azimuthal about the z axis, indicating the presence of angular momentum in the $+z$ direction. The angular momentum is $\mu_0 qm/4\pi$, directed from q to m.

vector potential and the charge q carried by a particle as part of the generalized momentum of the particle.[10]

The emmd in vacuum is simply the consequence of the flow of the mass equivalent of energy. This can be seen from the fact that $\epsilon_0 \mathbf{E}\times\mathbf{B}=\mathbf{S}/c^2$, where $\mathbf{S}$ is the Poynting vector. As $\mathbf{S}$ is the energy flux density, $\mathbf{S}/c^2$ is the flux density of the mass equivalent of the energy flow, and the rate of flow of mass per unit area is momentum density. In a material medium, the nature of the emmd is less obvious and sometimes complicated; it will be discussed later.

II. LONG SOLENOID AND AN ELECTRIC CHARGE

A useful example to consider in connection with electromagnetic momentum is a long solenoid with a charge q located a distance r from its axis, inside or outside the solenoid but otherwise near its midpoint. Let the radius of the solenoid be R and the magnetic induction within the solenoid be B. The magnetic field outside the solenoid is small and is considered at first to be zero. The charge is the source of the relevant electric field, and the electromagnetic momentum associated with the charge can be evaluated by integrating the field of emmd inside the solenoid even when the charge is outside the solenoid.[11] The momentum associated with q is

$$\int_\tau \epsilon_0 \mathbf{E}_q \times \mathbf{B}\, d\tau = q\mathbf{A}, \tag{5}$$

where the integration is over the volume within the solenoid. $\mathbf{A}$, given by Eq. (4), is the vector potential of the solenoid current at the point r where q is located. (In the approximation of an infinitely long solenoid, $A=rB/2$ for $r<R$, and $A=R^2B/2r$ for $r>R$.) The electric field $\mathbf{E}_q$ is the field of the charge q, disregarding any perturbation due to conducting properties of the solenoid. This is illustrated in Fig. 2(a) for $r>R$, along with the associated field of the Poynting vector $\mathbf{S}$. The field of the Poynting vector indicates an energy source on the left-hand side of the solenoid and an energy sink on the right-hand side, and the electromagnetic momentum describes the rate of flow of equivalent mass across the solenoid. Physically, the fields indi-

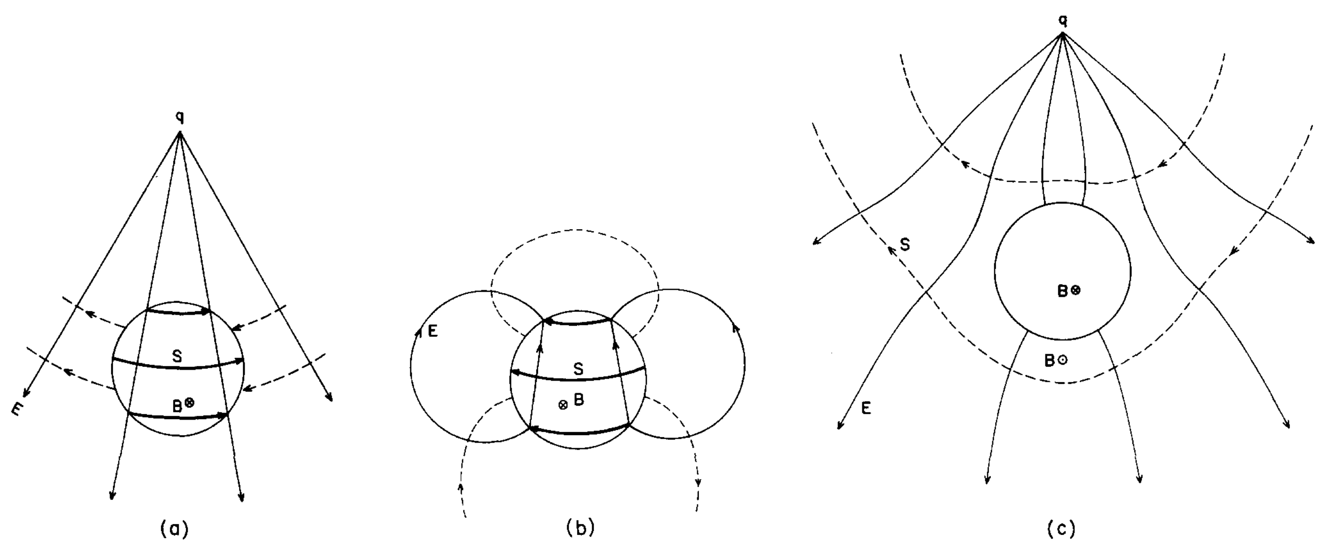

Fig. 2. Electric fields and fields of the Poynting vector in and around a long solenoid whose axis is perpendicular to the plane of the diagram. (a) Electric field of charge q, indicated by light solid lines. The field of the Poynting vector within the solenoid is shown by heavy solid lines, indicating a flow of energy across the solenoid from left to right. The dashed lines show the field of the Poynting vector outside the solenoid, indicating a weak flow of energy across the solenoid that supplements the much stronger flow inside the solenoid. (b) Field of the charge distribution induced on the surface of the solenoid by the presence of charge q and the field of the associated Poynting vector. The total flow of energy across the solenoid from right to left exactly cancels the total flow of energy across the solenoid shown in (a). (c) Combined fields of charge q and the induced charges on the solenoid. The Poynting vector field lines (the dashed lines) are roughly circular except for a detour around the solenoid. The angular momentum of this field corresponds to the linear momenta shown in (a) for charge q and in (b) for the induced charges on the solenoid.

cated in Fig. 2(a) could be realized by making the solenoid from a large array of small current generators, each with its own local energy source (or sink), and each at the potential established by charge q.[12] Then the current generators on the left-hand side would be sources of electromagnetic energy and those on the right-hand side, sinks.

If the solenoid is regarded as a conductor and thus an equipotential, then the flow of energy across the solenoid described in the previous paragraph is canceled by the effects of a charge distribution that is induced on the surface of the solenoid. The charge distribution induced on the solenoid cancels the electric field of charge q inside the solenoid and modifies the total field outside so that it is perpendicular to the surface of the solenoid. The electric field E_s of the induced charge distribution is illustrated in Fig. 2(b); within the solenoid the fields exactly cancel those shown in Fig. 2(a). The momentum associated with the surface charges on the solenoid is equal and opposite to that associated with the point charge. However, for a finite solenoid the magnetic field outside, though very small, is not zero, and this turns out to be critically important. We shall soon see that this external field describes the angular momentum of the system.

If the magnetic field is allowed to go to zero (by letting the current in the solenoid decay to zero), q experiences an induced electric field to the right, the impulse delivered to it being

$$\frac{q}{2\pi r}\int_{B=B}^{B=0}\frac{\partial}{\partial t}(\pi R^2 B)\,dt=\frac{-qBR^2}{2r}, \tag{6}$$

where r is the distance to q from the axis of the solenoid. This is just the product of q and the vector $BR^2/2r$ at q. It is also equal to

$$\int_\tau \epsilon_0 \mathbf{E}_q\times\mathbf{B}\,d\tau, \tag{7}$$

where $\mathbf{E}_q$ is the field of q and the integration is over the volume within the solenoid; this integral is evaluated in the Appendix.

Associated with the impulse to the right delivered to q, there is an equal impulse to the left delivered to the solenoid[3] as a consequence of the action of the induced electric field on the charge distribution on the solenoid induced by the presence of charge q.[13] (There would be no force on the solenoid if there were no shielding charges — that is, if the solenoid were a nonconductor as discussed above — but then the system would not be static.) The impulses generate a mechanical angular momentum since equal and opposite impulses act along lines separated by distance r.[14] The angular momentum must have been stored in the field before the solenoid current was turned off, and this could only be in the external field.

Consider the development of angular momentum as charge q is brought into position near an energized solenoid (rather than its being in place while the magnetic field is increased from zero). Bring q from infinity along the axis of the solenoid, considering this to be the z axis (where the x axis is to the right and the y axis upwards in Fig. 2); in this way, it can be brought from infinity to the center of the solenoid without experiencing any magnetic force. Then move q along the y axis with velocity $\mathbf{v}$ so that it experiences a magnetic force $q\,\mathbf{v}\times\mathbf{B}$ to the left. This requires the application of an external force to the right in Fig. 2(a) (where B is in the $-z$ direction) that will impart a total impulse qrB to q by the time it reaches position r, where $r\leqslant R$. As q is moved from 0 to R, an equal impulse, but to the left, must be delivered to the solenoid from an external source to counter the electromagnetic force to the right that it experiences. Thus angular momentum $qR^2B/2$ is delivered to the system by external forces as q moves from 0 to R. This angular momentum resides in electromagnetic form; if it is converted into mechanical form by letting the magnetic field go to zero at this point, the charge acquires an impulse $qRB/2$ to the right and the solenoid an impulse $qRB/2$ to the left, just the release of the angular momentum that was delivered to them by external forces as q was brought into position at $r=R$.

That equal and opposite forces must by applied to the moving charge and the solenoid to maintain the specified trajectory and position has been shown by Furry[3] in a more general treatment of the subject.[15] He showed that, for a charge q and a solenoid m surrounded by a conducting shield s, the sum of the forces on the system is zero; the relevant forces being the $\mathbf{v}\times\mathbf{B}$ force $\mathbf{F}_q$ on charge q moving in the field of the solenoid, the $\mathbf{v}\times\mathbf{B}$ forces $\mathbf{F}_s$ on the shielding charges moving in the field of the solenoid, the $\mathbf{J}\times\mathbf{B}$ force $\mathbf{F}_{mq}$ on the solenoid due to the magnetic field of the moving charge q, and the $\mathbf{J}\times\mathbf{B}$ force $\mathbf{F}_{ms}$ on the solenoid due to the magnetic field of the moving shielding charges, where $\mathbf{J}$ is the magnetization current flowing round the solenoid. Although the sum of the forces is zero, there is not pairwise cancellation. If we consider a charge q approaching a long solenoid along the y axis from $+\infty$, both q and the shield receive small impulses to the left (infinitesimally small as the solenoid becomes infinitely long) due to forces F_q and F_s. The largest impulses are to the solenoid due to F_{mq} acting to the left and F_{ms} acting to the right, their difference being small but to the right. The combined impulse to the solenoid and the shield is to the right, equal and opposite to the impulse delivered to q by the electromagnetic interaction. Due to symmetry, the $\mathbf{v}\times\mathbf{B}$ force on q due to the magnetic field of the moving shielding charges and the $\mathbf{v}\times\mathbf{B}$ forces on the moving shielding charges due to the magnetic field of q do not enter into this problem.

Going back to the previous example where q was taken along the y axis from the center of the solenoid, the charge experiences no magnetic force as it moves beyond R (in the approximation of zero magnetic field outside the long solenoid) and its angular momentum $qR^2B/2$ is conserved. Accordingly, the linear electromagnetic momentum associated with q decreases as $1/r$ and its value is $qR^2B/2r$, just the product of the vector potential $R^2B/2r$ and q. The solenoid has equal and opposite linear electromagnetic momentum. In the approximation of an infinitely long solenoid, F_q and F_s are zero and F_{mq} and F_{ms} cancel one another; however the latter two forces can be related to the decrease in linear momenta associated with the charge and with the solenoid, the angular momentum remaining constant as r increases.

The above discussion has ignored the fact that the magnetic induction outside the solenoid, though small, is not zero. Taking the weak external magnetic field into account, there is a weak magnetic force on q as it moves along the y axis beyond R, and the force decreases very slowly with distance. The magnetic induction as a function of distance r from the axis of a long solenoid of length L and radius R ($L\gg R$) is

$$B_e = \frac{2\mu_0 \alpha R^2}{L^2 \sqrt{1+4r^2/L^2}^{\,3}}, \tag{8}$$

where α is the magnetizing current per unit length flowing round the solenoid. Near the solenoid $B_e = 2\mu_0 \alpha R^2/L^2$, and it falls to about one-tenth of this value at $r = 0.95L$. The angular momentum conveyed to q in resisting this magnetic force as the particle goes to infinity along the y axis just cancels the angular momentum that was conveyed to q (and stored as emmd) as it proceeded from the axis of the solenoid to its surface, so the angular momentum at infinity is zero.

The above makes clear where the field of electromagnetic momentum exists for a charged particle q outside a long solenoid, and this is illustrated in Fig. 2(c), which shows schematically the electric field and the field of the Poynting vector. The field lines of emmd are roughly circular about q in the plane of the figure, with a detour around the solenoid. As there are no energy sources or sinks, the integral of the emmd over all space is zero; this can be seen from the continuity of the field lines of the Poynting vector. Thus the total linear momentum of the system is zero, but the angular momentum is not. Though the fields outside the solenoid are weak, they are extensive and the integrated angular momentum over all space is finite. The volume within the solenoid makes no contribution to the electromagnetic momentum. The angular momentum could be evaluated by integrating the product of the emmd and a radius vector over all space[16] but it is easier to make use of the property noted by Thomson, evaluating the angular momentum in terms of the charge q and magnetic monopoles $\pi\alpha R^2$ and $-\pi\alpha R^2$ at the ends of the solenoid. Each pole in its interaction with charge q has angular momentum $\mu_0 q \pi\alpha R^2/4\pi = q\mu_0\alpha R^2/4$, with axial component $q\mu_0\alpha R^2/4\sqrt{1+4r^2/L^2}$. Therefore, the angular momentum of the system is $q\mu_0\alpha R^2/2\sqrt{1+4r^2/L^2} \approx q\mu_0\alpha R^2/2$ for $r \ll L$. If the magnetic field is turned off with the charge and the solenoid fixed in position, the impulse delivered to each of them is $q\mu_0\alpha R^2/2r\sqrt{1+4r^2/L^2}$, to the right in the case of the charge and to the left in the case of the solenoid. This transfer of momentum is due to the decay of the field of emmd, all of which is external to the solenoid. However, as we showed earlier, these impulses can be calculated from the canceling fields inside the solenoid of the emmd for the charge q and for the induced charge distribution on the solenoid; the canceling fields of the two sources of electric field simply serve as proxies for the properties of the external field.

Much of what has been said in the context of $r > R$ is applicable for $r < R$. The angular momentum possessed by the system for $r < R$ is $qBr^2/2$. The linear momentum associated with q, expressed in terms of the electromagnetic momentum within the solenoid, is $\epsilon_0 \int_\tau \mathbf{E}_q \times \mathbf{B}\, d\tau$, where the integration is over the volume within the solenoid. The integration has been performed in the Appendix; for $r < R$, its value is $qBr/2$ in the $+x$ direction, just the product of q and the vector potential at r. The linear momentum associated with the solenoid is equal and opposite, i.e., $\epsilon_0 \int_\tau \mathbf{E}_s \times \mathbf{B}\, d\tau = -\epsilon_0 \int_\tau \mathbf{E}_q \times \mathbf{B}\, d\tau$, where $\mathbf{E}_s$ is the field of the charge distribution induced on the solenoid; this has magnitude $qBr/2$ and is in the $-x$ direction. Even though the fields of q and the charge distribution on the solenoid have different topologies, the integrals of the Poynting vector over the volume inside the solenoid are equal and opposite.

A toroidal coil with a charge q at its center is a natural extension of the picture for a long solenoid. The magnetic field is entirely within the toroid and the electric field totally outside (assuming its surface to be conducting), so the two do not overlap. The system has neither linear nor angular electromagnetic momentum, but the charge and the toroid will acquire equal and opposite linear momenta by letting the magnetic field decay to zero. The momentum of the charge can be evaluated in terms of electromagnetic momentum by considering its field to penetrate the toroid, that is, by neglecting the shielding charges; it is $\epsilon_0 \int_\tau \mathbf{E}_q \times \mathbf{B}\, d\tau = \int_\tau \mathbf{E}_q \times \mathbf{m}\, d\tau/c^2$, where $\mathbf{m}$ is the magnetic dipole moment per unit volume and τ is the volume within the solenoid. The momentum of the toroid can be evaluated in similar manner by considering the field of the shielding charges alone, that is, by neglecting the field of charge q.

III. INFINITESIMAL DIPOLE AND AN ELECTRIC CHARGE

The conversion of mechanical angular momentum into electromagnetic angular momentum and vice versa cannot be demonstrated for a magnetic monopole and a charge, but it can be for a dipole and a charge. The mechanical angular momentum that is conveyed to the system in bringing a charge from infinity into the proximity of a magnetic dipole exists in the form of electromagnetic angular momentum, and it can be released as mechanical angular momentum by letting the dipole decay to zero. The angular momentum is easily evaluated using Thomson's result for the angular momentum associated with a charge q and a magnetic monopole m. It immediately follows that the angular momentum associated with a charge q and a magnetic dipole M is

$$\mathbf{L} = -\frac{\mu_0 q M \sin\theta}{4\pi r}\hat{\theta}, \tag{9}$$

where the charge is at r,θ relative to the dipole, θ being measured from the dipole axis.

To illustrate how angular momentum is put into the system as a charge is brought from infinity into the proximity of a magnetic dipole, any path can be selected; the one that we use here serves as an example. We consider a dipole at the origin of our coordinate system with the z axis along the dipole axis. Let the coordinates of the charge be $(x,0,b)$, and let it move in the $+x$ direction from $-\infty$ with velocity v. The electromagnetic angular momentum from Thomson's relation is

$$\mathbf{L}_{em} = -\frac{q\mu_0 M \sin\theta}{4\pi r}\hat{\theta}$$

$$= \frac{q\mu_0 M}{4\pi b}(\hat{\mathbf{k}} \sin^2\theta \cos\theta + \hat{\mathbf{i}} \sin\theta \cos^2\theta). \tag{10}$$

The magnetic field in the xz plane has components:

$$B_x = -\frac{2\mu_0 M}{4\pi r^3}\sin\theta\cos\theta,$$

$$B_y = 0,$$

and

$$B_z \pm \frac{\mu_0 M}{4\pi r^3}(3\cos^2\theta - 1), \tag{11}$$

where $\theta = \tan^{-1}(-x/b)$. The magnetic force on q as it moves parallel to the x axis requires the application of an external force in the y direction to keep it on its prescribed path, and this external force is

$$\mathbf{F}_e = \frac{qv\mu_0 M}{4\pi r^3}(3\cos^2\theta - 1)\hat{\mathbf{j}}. \tag{12}$$

The force experienced by M is equal and opposite to that experienced by q, so another external force $-\mathbf{F}_e$ must be applied to M to keep it from moving. The two external forces constitute a couple that adds angular momentum to the system, the contribution during time interval dt being

$$d\mathbf{L}_{ef} = \frac{qv\mu_0 M}{4\pi r^2}(3\cos^2\theta - 1)\hat{\boldsymbol{\theta}}\, dt. \tag{13}$$

As $v\, dt = dx = -r\, d\theta/\cos\theta$ and $r = b/\cos\theta$,

$$d\mathbf{L}_{ef} = -\frac{q\mu_0 M}{4\pi b}(3\cos^2\theta - 1)(-\hat{\mathbf{k}}\sin\theta + \hat{\mathbf{i}}\cos\theta)d\theta, \tag{14}$$

and

$$\mathbf{L}_{ef} = \frac{q\mu_0 M}{4\pi b}\int_{\pi/2}^{\theta}(3\cos^2\theta - 1)(\hat{\mathbf{k}}\sin\theta + \hat{\mathbf{i}}\cos\theta)d\theta$$
$$= \frac{q\mu_0 M}{4\pi b}\{\cos\theta\sin^2\theta\hat{\mathbf{k}} + [\sin\theta\cos^2\theta - (1-\sin\theta]\hat{\mathbf{i}}\}. \tag{15}$$

There is an additional torque that must be applied to the system as q moves along its prescribed path; the magnetic field of the moving charge produces a torque on M, and an equal and opposite torque from an external source must be applied to M to stop it from turning. The magnetic field of q at M is $\mathbf{B}_q = (q\mu_0 vb/4\pi r^3)\hat{\mathbf{j}}$, so the torque exerted on M is $-(q\mu_0 vbM/4\pi r^3)\hat{\mathbf{i}}$. The angular momentum conveyed to the system in resisting this torque is

$$\int_{-\infty}^{t}\frac{q\mu_0 vbM}{4\pi r^3}dt\,\hat{\mathbf{i}} = -\frac{q\mu_0 M}{4\pi b}\int_{\pi/2}^{\theta}\cos\theta\, d\theta\hat{\mathbf{i}}$$
$$= \frac{q\mu_0 M}{4\pi b}(1-\sin\theta)\hat{\mathbf{i}}. \tag{16}$$

Thus the total angular momentum delivered to the system from external sources is

$$\mathbf{L}_e = \frac{q\mu_0 M}{4\pi b}(\hat{\mathbf{k}}\sin^2\theta\cos\theta + \hat{\mathbf{i}}\sin\theta\cos^2\theta), \tag{17}$$

in agreement with the electromagnetic angular momentum of the system as given by Eq. (10).

To release the electromagnetic angular momentum in mechanical form, let the dipole decay to zero. The impulse delivered to q can be evaluated by calculating the induced electric field. The flux of magnetic induction through a circle generated by revolution around the z axis passing through q is $(\mu_0 M/2r)\sin^2\theta$. The electric field at q is

$$\frac{1}{2\pi\sin\theta}\frac{d}{dt}\left(\frac{\mu_0 M\sin^2\theta}{2r}\right)$$

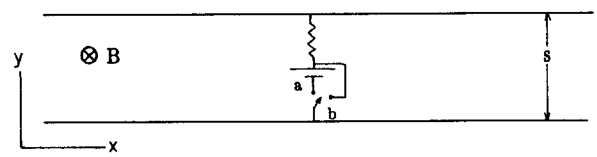

Fig. 3. A capacitor configuration to demonstrate the concept of electromagnetic momentum density. In charging the capacitor, an impulse to the left is delivered to the circuit. There is equal opposing momentum in the form of electromagnetic momentum density in the space between the capacitor plates. If a dielectric of relative permittivity ϵ_r is placed between the capacitor plates, the effect is increased by the factor ϵ_r associated with an impulse delivered to the dielectric by the polarization current.

and the integrated impulse delivered to q as the dipole is reduced to zero is

$$-\frac{q\mu_0 M\sin\theta}{4\pi r^2}\hat{\mathbf{j}}. \tag{18}$$

The easiest way to calculate the impulse delivered to the dipole is to use Costa de Beauregard's[13] relation for the force on a changing magnetic dipole in an electric field. The force is $(\mathbf{E}/c^2)\times(d\mathbf{M}/dt)$, and this yields $(q\mu_0 M\sin\theta/4\pi r^2)\hat{\mathbf{j}}$ for the impulse, equal and opposite to the impulse delivered to the charge. Alternatively and conceptually better, this result can be asserted on the basis of Furry's conclusion that the force on the shielding charges on the magnet is equal and opposite to that on charge q. Thus the angular momentum released from the electromagnetic field as a consequence of the decay of M is $(q\mu_0 M\sin\theta/4\pi r)\hat{\boldsymbol{\theta}}$, in agreement with both the electromagnetic angular momentum determined from Thomson's relation and the mechanical angular momentum added to the system by external forces as q was brought from infinity to position r,θ.

IV. THE PARALLEL-PLATE CAPACITOR

Further insight into the nature of electromagnetic momentum can be obtained by considering the charge and discharge of a parallel-plate capacitor in a magnetic field, the magnetic field being parallel to the plates. The example that we consider is illustrated in Fig. 3. The parallel-plate capacitor is arranged so that moving a switch to position "a" charges the capacitor, and moving it to position "b" discharges it. A is the area of the plates, s is the distance between them, V is the potential difference supplied by the battery, and the electric field between the capacitor plates is $E = -V/s$ when the capacitor is fully charged. The role of resistor R is to make the charging and discharging processes slow enough so that radiation can be ignored. The constant magnetic field B is in the $-z$ direction, directed into the plane of the figure. When the switch is moved to position "a," a current I flows to charge the capacitor and a force to the left (the $-x$ direction) is exerted on the circuit joining the capacitor plates due to the interaction between the charging current and the magnetic field. The impulse delivered to the circuit in the $+x$ direction is $-Bs\int I\, dt = -\epsilon_0 AsEB$, neglecting edge effects, where B and E are both negative in Fig. 3.

Something must have acquired equal and opposite momentum, and it can be considered to be the electromagnetic field within the volume of space between the capacitor plates; this volume has acquired emmd $\mathbf{g}=\epsilon_0\mathbf{E}\times\mathbf{B}$, which is directed towards the right in Fig. 3. The total electromagnetic momentum between the plates is $\epsilon_0 AsEB$, equal and opposite to the impulse delivered to the circuit. The rate of creation of emmd is $(\partial/\partial t)(\epsilon_0\mathbf{E}\times\mathbf{B})=\mathbf{J}_d\times\mathbf{B}$, where $\mathbf{J}_d=\epsilon_0\partial\mathbf{E}/\partial t$ is the displacement current in vacuum; the rate is also equal to the stress per unit volume indicated by the magnetic portion of the Maxwell stress tensor.[17] In general, the presence of a displacement current in vacuum in a region of space in which there is a steady magnetic field indicates that emmd is being created.

It is conceptually better to regard the momentum required to balance the momentum delivered to the capacitor as having been delivered to the source of the magnetic field; the use of electromagnetic momentum is simply a device to avoid the necessity of examining the details. Coleman and Van Vleck[18] have shown that the total linear momentum of the entire system must be zero. The location of some of the momentum may be difficult to find, and it may at times be categorized as hidden momentum. In this example, the electromagnetic momentum between the capacitor plates describes momentum possessed by the source of the magnetic field.

The example provides additional insight if the volume between the plates is filled by a nonmagnetic dielectric of relative permittivity ϵ_r, in which case the charge acquired by the capacitor and the impulse delivered to the circuit are correspondingly larger by the factor ϵ_r. What is the emmd in this case? Practice varies as to how it should be defined, $\epsilon_0\mathbf{E}\times\mathbf{B}$ or $\mathbf{D}\times\mathbf{B}=\epsilon_r\epsilon_0\mathbf{E}\times\mathbf{B}$, the difference between the two being the momentum per unit volume delivered to the dielectric, described next. We consider the dielectric to be supported separately from the capacitor plates and the connecting circuit so that the effects of the forces acting on each can be considered separately.

With the dielectric in place, the displacement current includes a polarization term, and $\mathbf{J}_d=\epsilon_0\partial\mathbf{E}/\partial t+\partial\mathbf{P}/\partial t$. The first term is indicative of the same rate of creation of emmd that existed without the dielectric, $\epsilon_0\partial\mathbf{E}/\partial t\times\mathbf{B}$; it can be thought of as indicating the rate of production of pure-field momentum. The second term involves a force per unit volume, $\partial\mathbf{P}/\partial t\times\mathbf{B}$, delivered to the dielectric as a consequence of the polarization current. It is this momentum delivered to the dielectric that is considered to be part of the emmd when it is defined as $\mathbf{D}\times\mathbf{B}=\epsilon_0\partial\mathbf{E}/\partial t\times\mathbf{B}+\partial\mathbf{P}/\partial t\times\mathbf{B}$. Further, the example provides an argument that it is more appropriate from a physical point of view to consider the emmd to be given by $\mathbf{D}\times\mathbf{B}$ rather than by $\epsilon_0\mathbf{E}\times\mathbf{B}$, with the mechanical momentum treated separately in the latter case. During the discharge of the capacitor the emmd collapses and the circuit receives an impulse to the right $\epsilon_r\epsilon_0 AsEB$ and the dielectric an impulse to the left $(\epsilon_r-1)\epsilon_0 AsEB$, irrespective of the momentum state of the dielectric at the time when the discharge commences. During the charging of the capacitor, the dielectric did receive an impulse to the right. Whether the dielectric retains this momentum or delivers it to something else (the dielectric support structure, for example), an impulse to the left is delivered to the dielectric when the capacitor is discharged.[19] It is not simply a matter of momentum being stored in the dielectric while the capacitor is charged; rather it is a property of the electromagnetic field that an impulse be delivered to the dielectric whenever there is a change in its polarization, the impulse being $(\epsilon_r-1)=\chi_e$ times the change in $\epsilon_0\mathbf{E}\times\mathbf{B}$, where χ_e is the electrical susceptibility. This argues that the emmd, including its interaction with the dielectric, should be regarded as a property of the electromagnetic field and equal to $\mathbf{D}\times\mathbf{B}$.

The emmd between the capacitor plates in this example can be produced by a different sequence of events. Suppose that the capacitor without the dielectric is charged in the absence of any magnetic field and that the magnetic field is then applied slowly with $\partial B/\partial t$ negative (for the axes chosen in Fig. 3) and constant until it reaches a specified negative value B. While the magnetic field is changing there is an induced electric field described by $\nabla\times\mathbf{E}_i=\partial\mathbf{B}/\partial t$. This gives rise to a force to the left on the charges on both capacitor plates, the total force in the $+x$ direction being $Qs\partial B/\partial t=\epsilon_0 AsE\partial B/\partial t$, where $Q=-\epsilon_0 AE$ is the charge on the capacitor plates and E is the electric field when B is not changing.[20] ($\partial B/\partial t$ and E are negative and Q is positive.) The total impulse that has been delivered to the plates when the magnetic field reaches the value B is $-\epsilon_0 AsEB$; this is associated with the generation of emmd between the capacitor plates, the total amount of field momentum between the plates being $+\epsilon_0 AsEB$, as before.

The picture can also be described in terms of the electric portion of the Maxwell stress tensor, which is a useful device for calculating electromagnetic forces. First, it is necessary to evaluate the electric field while the magnetic field is changing. The induced electric field E_i causes a redistribution of the charges on the capacitor plates, driving charges to the left until the resultant field within the plates is zero. (Note that the plates are not equipotentials while the magnetic field is changing.) The resultant field E_r between the plates is $\mathbf{E}_r=\mathbf{E}-\hat{\mathbf{j}}\,(\partial B/\partial t)\,x$, where the origin is at the center of the capacitor. The only term in the stress tensor that contributes to a force in the x direction is the T_{xx} term, and $T_{xx}=\epsilon_0 E_r^2/2=\epsilon_0(E-x\partial B/\partial t)^2/2$. The electromagnetic stress per unit volume in the region between the capacitor plates is $-(\epsilon_0/2)(\partial/\partial x)(E-x\partial B/\partial t)^2=\epsilon_0 E_r\partial B/\partial t$, and the total electromagnetic stress exerted on the volume between the plates is $\int\epsilon_0 E_r\partial B/\partial t\,d\tau=\epsilon_0 EAs\partial B/\partial t$, in the $+x$ direction. The reaction force is exerted on the capacitor plates. Thus the impulse delivered to the capacitor plates as B changes from 0 to B, as indicated by the stress tensor, is $-\epsilon_0 AsEB$.

If this sequence, in which the magnetic field is changed from zero, is repeated with the dielectric in place, the impulse to the left delivered to the plates is larger by the factor ϵ_r, the increase being associated with an impulse to the right in the amount $(\epsilon_r-1)\epsilon_0 AsEB$ delivered to the dielectric. The impulse to the dielectric is due to the action of the induced electric field on the bound charge density $(\epsilon_r-1)\epsilon_0 E$ on the surfaces of the dielectric (or on the surfaces of volume elements of the dielectric), the charge being negative on the upper surface of the dielectric slab (or of the volume elements).[21] Thus the force is proportional to the polarization of the dielectric. Alternatively, it can be seen in terms of the negative gradient of the electric field energy density $-(\partial/\partial x)(\epsilon_r\epsilon_0 E_r^2/2)$, which is larger than the pure-field value by the factor ϵ_r.

If B and E change in proportion, the contributions to emmd from $\partial E/\partial t$ and $\partial B/\partial t$ are equal, no matter what their ratio may be, but the nature of the forces exerted on the medium by the two terms is different. The force on the dielectric associated with the variation in E is due to the polarization current interacting with the magnetic field, thus conveying momentum to the medium; this is also reflected in the spatial distribution of B as perturbed by the polarization current and in the gradient of magnetic field energy density. The force associated with the time variation of B is due to an electrical force on the polarization surface charges of volume elements of the dielectric, and this is reflected in the spatial distribution of E and the gradient of the electric field energy density; the force per unit volume on the dielectric is the negative gradient of the electric field energy density $\epsilon_r\epsilon_0E^2/2$ less the negative gradient of the pure electric field energy density $\epsilon_0E^2/2$, or $(\epsilon_r-1)\epsilon_0E^2/2$. As this force per unit volume due to the time variation in B is not easily visualized, it might be overlooked, which would lead to a factor of 2 error in evaluating the mechanical momentum delivered to the medium.

The above discussion has not considered edge effects and fields outside the volume between the capacitor plates. They can be seen to be inconsequential to the arguments presented above by considering the equivalent of a guard ring. Visualize the capacitor as being simply an element of a much larger capacitor consisting of closely spaced coaxial cylinders with the outer surface grounded. On the other hand, if the source of the magnetic field is considered to be a large solenoid with a relatively small capacitor along its axis, the edge effects and the interactions between the charging current and shielding currents induced on the surface of the solenoid are important, and it makes an interesting exercise to discuss them.

V. SUMMARY

The angular electromagnetic momentum possessed by a magnetic dipole and an electric charge, and by a long solenoid and an electric charge, provide two examples of convincing evidence of the physical significance of circulating energy fields as indicated by the Poynting vector. If the magnetic field is turned off, the circulating energy flow stops and its associated angular momentum is converted to mechanical angular momentum; this is manifested in equal and opposite impulses being delivered to the charge and to the source of the magnetic field.

When a system (such as a charged capacitor in a magnetic field) possesses linear electromagnetic momentum, this is just a proxy for momentum existing elsewhere in the system, and that momentum involves motion (usually the currents producing the magnetic field).

Consideration of a parallel-plate capacitor in a magnetic field shows that the linear electromagnetic momentum possessed by the system consists of a pure-field component, $\epsilon_0\mathbf{E}\times\mathbf{B}$, plus an additional amount $\chi_e\epsilon_0\mathbf{E}\times\mathbf{B}$ associated with the dielectric. The additional amount depends on the polarization of the dielectric, not on its mechanical momentum; it is a property of the electromagnetic field that an amount of linear momentum dependent upon the polarization will appear as mechanical momentum when the capacitor is discharged, irrespective of whatever mechanical momentum the dielectric happens to possess when the discharge commences. The momentum transfer to or from the dielectric always accompanies any change in the pure-field momentum and depends on the polarization of the dielectric. The reasons for regarding $\chi_e\epsilon_0\mathbf{E}\times\mathbf{B}$ as part of the emmd are about the same as those for including the polarization current $\epsilon_r\epsilon_0\partial\mathbf{E}/\partial t$ in the definition of the displacement current

$$\mathbf{J}_d=\epsilon_0\partial\mathbf{E}/\partial t+\partial\mathbf{P}/\partial t\,.$$

Emmd is created whenever the electric field changes in the presence of a magnetic field and whenever the magnetic field changes in the presence of an electric field, according to

$$\frac{\partial}{\partial t}(\epsilon_r\epsilon_0\mathbf{E}\times\mathbf{B})=\epsilon_r\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}+\epsilon_r\epsilon_0\mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}\,.$$

The first term on the right is just $\mathbf{J}_d\times\mathbf{B}$, with $\mathbf{J}_d=\epsilon_r\epsilon_0\partial\mathbf{E}/\partial t=\epsilon_0\partial\mathbf{E}/\partial t+\partial\mathbf{P}/\partial t$; $\epsilon_0\partial\mathbf{E}/\partial t\times\mathbf{B}$ is the rate of change of pure-field momentum, and $\mathbf{P}/\partial t\times\mathbf{B}=\mathbf{J}_p\times\mathbf{B}$ is the rate at which momentum is conveyed to the dielectric, where $\mathbf{J}_p$ is the polarization current. The second term can also be broken up into two terms, one the rate of change of pure-field momentum, $\epsilon_0\mathbf{E}\times\partial\mathbf{B}/\partial t$, and the other, $\chi_e\epsilon_0\mathbf{E}\times\partial\mathbf{B}/\partial t=\mathbf{P}\times\partial\mathbf{B}/\partial t$, the rate at which momentum is conveyed to the dielectric by the induced electric field acting on the polarization charges. The presence of a changing electric field in a magnetic field and the the presence of a changing magnetic field in an electric field both provide evidence that emmd is being produced.

APPENDIX

Here, the volume integral $\mathbf{P}$ of the emmd $\epsilon_0\mathbf{E}\times\mathbf{B}$ is evaluated explicitly for the case of a point charge q located a distance r from the axis of an infinitely long, nonconducting solenoid. The solenoid has radius R, and the magnetic field within it has the constant value $\mathbf{B}=B\mathbf{k}$; outside the solenoid the magnetic field is zero. The electric field inside the solenoid is taken to be the undisturbed electric field of q, ignoring any effects of shielding charges on the surface of the solenoid, as prescribed by Konopinski.[11] Taking the origin of the coordinate system to coincide with q, we have

$$E_{x,y,z}=\frac{q}{4\pi\epsilon_0}\frac{[x,y,z]}{\sqrt{x^2+y^2+z^2}^{\,3}} \tag{A1}$$

and the solenoid's axis is along the line $x=0$, $y=r$. From symmetry, the only nonvanishing component of $\mathbf{P}$ integrated over the volume within the solenoid is

$$P_x=\int_\tau \epsilon_0E_yB_z\,d\tau$$
$$=\epsilon_0B\int_{-R}^{R}dx\int_{y_{\min}}^{y_{\max}}dy\int_{-\infty}^{\infty}dz\,E_y, \tag{A2}$$

where $y_{\min}=r-\sqrt{R^2-x^2}$, $y_{\max}=r+\sqrt{R^2-x^2}$, and E_y is obtained from Eq. (A1). Upon the substitution $\tan\theta=z/\sqrt{x^2+y^2}$, the integration over z reduces to elementary form, with the result that

$$P_x=\frac{qB}{2\pi}\int_{-R}^{R}dx\int_{y_{\min}}^{y_{\max}}\frac{y}{x^2+y^2}\,dy. \tag{A3}$$

This in turn is easily integrated to give

$$P_x=\frac{qB}{2\pi}\int_{-R}^{R}\frac{1}{2}\ln\frac{r^2+R^2+2r\sqrt{R^2-x^2}}{r^2+R^2-2r\sqrt{R^2-x^2}}\,dx \tag{A4}$$

$$=\frac{qB}{2\pi}2R\int_0^1 \frac{1}{2}\ln\frac{r^2+R^2+2rR\sqrt{1-x^2}}{r^2+R^2-2rR\sqrt{1-x^2}}\,dx$$

$$=\frac{qB}{2\pi}2R\int_0^1 \frac{1}{2}\ln\frac{1+\alpha\sqrt{1-x^2}}{1-\alpha\sqrt{1-x^2}}\,dx, \qquad \text{(A5)}$$

where

$$\alpha=\alpha(r)=\frac{2rR}{R^2+r^2}. \qquad \text{(A6)}$$

Integrating Eq. (A5) by parts,

$$P_x=\frac{qBR}{2\pi}\left(x\ln\frac{1+\alpha\sqrt{1-x^2}}{1-\alpha\sqrt{1-x^2}}\right)_0^1-\frac{qBR}{2\pi}\times\int_0^1 x\,d\ln\frac{1+\alpha\sqrt{1-x^2}}{1-\alpha\sqrt{1-x^2}}$$

$$=0+\frac{qBR}{2\pi}\int_0^1\frac{\alpha x^2}{1+\alpha\sqrt{1-x^2}}\times\left(1+\frac{1+\alpha\sqrt{1-x^2}}{1-\alpha\sqrt{1-x^2}}\right)\frac{dx}{\sqrt{1-x^2}}$$

$$=\frac{qBR}{\pi}\alpha\int_0^1\frac{x^2}{1-\alpha^2+\alpha^2x}\frac{dx}{\sqrt{1-x^2}}. \qquad \text{(A7)}$$

The integrals appearing in Eqs. (A5) and (A7) are not found in standard tables. However, the integral in Eq. (A7) can be evaluated using the tabulated results of Gradshteyn and Ryzhik:[22]

$$\int_0^{\pi/2}\ln(1+a\sin^2 x)\sin^2 x\,dx=\frac{\pi}{2}\left(\ln\frac{1+\sqrt{1+a}}{2}-\frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right) \qquad \text{(A8)}$$

and

$$\int_0^{\pi/2}\ln(1+a\sin^2 x)\cos^2 x\,dx=\frac{\pi}{2}\left(\ln\frac{1+\sqrt{1+a}}{2}+\frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right) \qquad \text{(A9)}$$

for $a>-1$. Adding Eqs. (A8) and (A9) and differentiating both sides of the result with respect to the parameter a, we find

$$\int_0^{\pi/2}\frac{\sin^2 x}{1+a\sin^2 x}\,dx=\frac{\pi}{2}\frac{1}{1+a+\sqrt{1+a}}. \qquad \text{(A10)}$$

The substitution $x=\sin^{-1}u$ then yields

$$\int_0^1\frac{u^2}{1+au^2}\frac{du}{\sqrt{1-u^2}}=\frac{\pi}{2}\frac{1}{1+a+\sqrt{1+a}}. \qquad \text{(A11)}$$

After a simple rescaling and redefinition of variables, this becomes

$$\int_0^1\frac{x^2}{a^2+b^2x^2}\frac{dx}{\sqrt{1-x^2}}=\frac{\pi}{2}\frac{1}{a^2+b^2+a\sqrt{a^2+b^2}}. \qquad \text{(A12)}$$

The integral appearing in Eq. (A7) is a special case of this result, obtained by setting $a^2=1-\alpha^2(r)$ and $b^2=\alpha^2(r)$. Thus

$$P_x=\frac{qBR}{2}\frac{\alpha(r)}{1+\sqrt{1-\alpha^2(r)}} \qquad \text{(A13)}$$

$$=\frac{qBR}{2}\frac{2rR}{R^2+r^2\pm(R^2-r^2)}$$

$$=qBr/2,\ \text{or}\ qBR^2/2r. \qquad \text{(A14)}$$

For $r<R$, symmetry demands that $P_x\to 0$ as $r\to 0$, and the first solution, obtained by choosing the positive square root, is applicable. For $r>R$, it is necessary that $P_x\to 0$ as $r\to\infty$, and the second solution applies.

[1]J. J. Thomson, *Elements of the Mathematical Theory of Electricity and Magnetism*, 3rd ed. (Cambridge University, London, 1904).

[2]J. J. Thomson, *Electricity and Matter* (Charles Scribner's Sons, New York, 1904), pp. 26–35.

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[5]E. M. Pugh and G. E. Pugh, "Physical significance of the Poynting vector in static fields," Am. J. Phys. **35**, 153–156 (1967).

[6]R. H. Romer, "Angular momentum of static electromagnetic fields," Am. J. Phys. **34**, 772–778 (1966), "Electromagnetic angular momentum," *ibid.* **35**, 445–446 (1967). The first of these papers suffers from the lack of consideration of the external field of the solenoid, something that is dealt with in the second paper.

[7]E. B. Moullin, *The Principles of Electromagnetism* (Oxford University, London, 1932), p. 218; Y. Aharonov and D. Bohm, "Significance of electromagnetic potentials in quantum theory," Phys. Rev. **115**, 485–491 (1959); E. J. Konopinski, "What the electromagnetic vector potential means," Am. J. Phys. **46**, 499–502 (1978); E. J. Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw-Hill, New York, 1981).

[8]M. G. Calkin, "Linear momentum of quasistatic electromagnetic fields," Am. J. Phys. **34**, 921–925 (1966).

[9]E. J. Konopinski, Ref. 7 (1978).

[10]L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, MA, 1951); p. 42; Y. Aharonov and D. Bohm, Ref. 7; R. K. Wangsness, *Introduction to Theoretical Physics* (Wiley, New York, 1963), p. 399.

[11]E. J. Konopinski, Ref. 7 (1981), p. 160.

[12]It might be supposed that the equivalent of a nonconducting solenoid could be realized using nonconducting permanent magnets, but S. Coleman and J. H. Van Vleck ["Origin of hidden momentum forces on magnets," Phys. Rev. 171, 1370–1375 (1968)] have shown that this is not true, that there is a relativistic counter flow of energy whose momentum just cancels the electromagnetic momentum.

[13]An alternative approach to the evaluation of the force on the solenoid as B is reduced to zero has been provided by O. Costa de Beauregard ["A new law of electrodynamics," Phys. Lett. **24A**, 177–178 (1967)], but it is not conceptually correct. He states as a new law of electrodynamics that the force on a slowly varying current i in an electric potential field V is $(di/dt)\oint V\,d\mathbf{l}/c^2$ and that the force on a varying magnetic dipole is $(1/c^2)\mathbf{E}\times d\mathbf{M}/dt$. Furry (Ref. 3) has shown that the force in question is actually that on the shielding charges induced on the solenoid or on the surface of a permanent magnet, and it is zero in their absence. However, with this reservation, Costa de Beauregard's relations are handy in evaluating the forces.

[14]This problem has been treated by others. Konopinski (Ref. 8) did not discuss the force on the solenoid, and Calkin (Ref. 9) incorrectly stated that it was zero. Calkin ignored the shielding charges; see Furry (Ref. 4). J. J. Thomson (Ref. 2, p. 32) stated correctly that a charge and a magnet (both stationary) possessed equal and opposite electromagnetic momenta, but for a charge and an electric circuit he stated incorrectly that the combination had linear momentum and that the circuit possessed no momentum; in this he failed to recognize the significance of shielding charges.

[15]Furry's treatment may be regarded as the basis for the reaction concept for the electromagnetic force exerted by one system upon another, that the forces are equal and opposite even if they are not in line. The reaction concept was also presented by V. H. Rumsey in lectures at the University if Illinois in 1956.

[16]Furry (Ref. 3) has done this for an arbitrary magnetic dipole [his Eq. (26)], showing result to be independent of the particular structure assumed for the dipole.

[17]The force per unit volume indicated by the stress tensor is regarded as real only to the extent that it describes the force on charged particles, that is, to the extent that it includes the Lorentz force. The portion of the force per unit volume that is equal to the rate of formation of emmd in vacuum is not considered to be a real force.

[18]Coleman and Van Vleck, Ref. 12.

[19]If the dielectric in the capacitor example is fixed to the capacitor plates, then the impulse delivered to the capacitor–dielectric combination is just the change in $\epsilon_0\mathbf{E}\times\mathbf{B}$, the same impulse that would occur without the dielectric in place, something that has been noted by P. Lorrain ["The Abraham force; comments on two recent experiments," Can. J. Phys. 3, 233–245 (1980)]. This interesting fact should not divert attention from the real nature of the interactions.

[20]There may be in addition a torque due to equal and opposite forces in the x direction on the two capacitor plates if the induced field is symmetrical about some point other than the center of the capacitor; this does not affect the net force in the x direction on the capacitor plates.

[21]Lorrain (Ref. 19) has described forces on the medium that reduce for the simple system considered here to $\mu_0(\partial\mathbf{P}/\partial t)\times\mathbf{H}$ and $\mu_0\mathbf{P}\times(\partial\mathbf{H}/\partial t)$, which are equivalent to the expressions used here.

[22]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980), p. 594, 4.399 1.

Magnetically imploded soft drink can

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A demonstration apparatus is described suitable for use in a large lecture hall, in which the "pinch effect" applied to an empty aluminum soft drink can provides a dramatic and instructive example of induction and of the repulsion of oppositely directed current elements. The can is placed in a coil into which a short pulse of electrical current is driven from a charged capacitor. The current in the primary coil induces an oppositely directed current in the can wall, and the portion of the can under the coil is driven violently inward, pinching the can down to form a waist, or with a larger charge on the storage capacitor, separating it into two pieces.

I. INTRODUCTION

An impressive demonstration of the phenomenon of magnetic induction, and of the repulsion of two oppositely directed current elements is provided by the "Can Crusher," one of the most popular and instructive of the many physics lecture demonstrations used at the University of Maryland (see Fig. 1). An ordinary empty soft drink can is inserted into a close fitting coil through which a large amplitude but short duration pulse of current is driven from a charged capacitor. Induced current in the can is repelled from the primary current in the coil, and the wall of the can is impulsively driven radially in, pinching the can more or less uniformly to a smaller diameter under the coil. An example of the result appears at the top of Fig. 1. With a larger charge on the capacitor, the can may be pinched so vigorously that it separates violently into two pieces. This provides a very memorable demonstration of the force of repulsion between two current carrying conductors having oppositely directed currents, and makes use of familiar and readily available cost-free supplies.

The action is known as the "pinch effect," and may be equally well described by the concept of pressure exerted by a magnetic field $\mathbf{B}$.[1] This pressure is $B^2/2\mu_0$, exerted perpendicular to any surface lying parallel to the field. For a rapidly rising current in the driver coil, the induced current in the can wall excludes magnetic field from inside of the can, so the field is confined to the narrow annulus between the can and coil, and the pressure it exerts is therefore not balanced by a corresponding pressure from the inside. Under the intense pressure of the magnetic field outside the can, the wall accelerates radially inward, buckling to accommodate the smaller radius. The coils we have used are about 1 cm wide, so the can is pinched down only in a narrow waist.

Such strong pulsed magnetic fields find application in industry, where the ability to exert pressure from different directions offers unique possibilities.[2,3] Larger versions of the pinch apparatus that utilize considerably higher voltage on the capacitor are used in plasma physics research to create hot dense plasmas. In that case, the can is replaced by a glass tube filled to a low pressure with some test gas. The induced azimuthal electric field causes the gas to break down into a plasma, which is pinched to the center of the tube and heated intensely by the compression.[4]

II. CIRCUIT ANALYSIS

The circuit analysis is interesting and has many ramifications. Without the can in place, it is a simple *LRC* circuit, with initial condition that the current is zero, and the capacitor charged to a potential V_0. The inductance is $L_p + L_c$, where L_p represents the sum of primary circuit inductances due to the spark gap, buswork, and internal in-

