Lecture 25 2/4/09

Electric dipole radiation

Last time

$$\boxed{V(r,\theta,t) = \frac{-p_0 \omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}$$

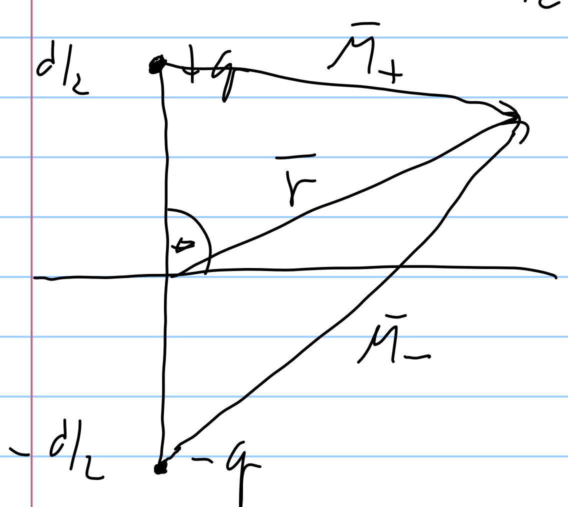
Vector Potential

$$\bar{I}(t) = \frac{dq}{dt}\hat{z}$$

$$= -q_0\omega\sin(\omega t)\,\hat{z}$$

$$\bar{A}(\bar{r},t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{\bar{I}(t_r)}{\eta}\,dz$$

$$= \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{-q_0\omega\sin\left[\omega\left(t-\eta/c\right)\right]\hat{z}}{\eta}\,dz$$



to 1st order, approximate the mean value of the integral as

(d) × (value of the integrand at the center)

$$\Rightarrow \boxed{\vec{A}(\vec{r},t) = -\frac{q_0 d \mu_0 \omega}{4\pi} \frac{\sin[\omega(t - r/c)]}{r} \hat{z}}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} \cos\left[\omega\left(t - \frac{r}{c}\right)\right] \left(\cos\theta\, \hat{r} - \sin\theta\, \hat{\theta}\right)$$

since $p_0 = q_0 d$ and $\hat{z} = \cos\theta\, \hat{r} - \sin\theta\, \hat{\theta}$

$$\nabla V = \frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta}$$

$$= \frac{-p_0 \omega}{4\pi \epsilon_0 c} \left[\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r} \sin\left[\omega\left(t - \frac{r}{c}\right)\right]\right) \hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\cos\theta}{r} \sin\left[\omega\left(t - \frac{r}{c}\right)\right]\right) \hat{\theta} \right]$$

$$\nabla V = \frac{-p_0 \omega}{4\pi \epsilon_0 c} \left[\left(-\frac{\cos\theta}{r^2} \sin\left[\omega\left(t - \frac{r}{c}\right)\right] - \frac{\omega \cos\theta}{rc} \cos\left[\omega\left(t - \frac{r}{c}\right)\right] \right) \hat{r} - \frac{\sin\theta}{r^2} \sin\left[\omega\left(t - \frac{r}{c}\right)\right] \hat{\theta} \right]$$

drop terms of $1/r^2$ since we're interested radiation

$$\Rightarrow \quad \nabla V \approx \frac{p_0 \omega^2}{4\pi \epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}$$

$$\frac{1}{c^2} = \mu_0 \epsilon_0$$

$$\boxed{\begin{aligned} \bar{E} &= -\nabla V - \frac{\partial \bar{A}}{\partial t} \\ &= -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta} \end{aligned}}$$

$\hat{r}$ terms cancel

$$\nabla \times \bar{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r A_\theta\right) - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi}$$

$$= \frac{-p_0 \mu_0 \omega}{4\pi r}\left(\sin\theta \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\frac{\omega}{c} + \frac{\sin\theta}{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\hat{\phi}$$

drop $1/r^2$ term

$$\Rightarrow \quad \boxed{\bar{B} \approx -\frac{p_0 \mu_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\phi}}$$

$$|\bar{B}| = \frac{|\bar{E}|}{c} \qquad \bar{B} \perp \bar{E}$$

energy radiated by the oscillating dipole

$$\bar{S} = \frac{1}{\mu_0}\left(\bar{E} \times \bar{B}\right)$$

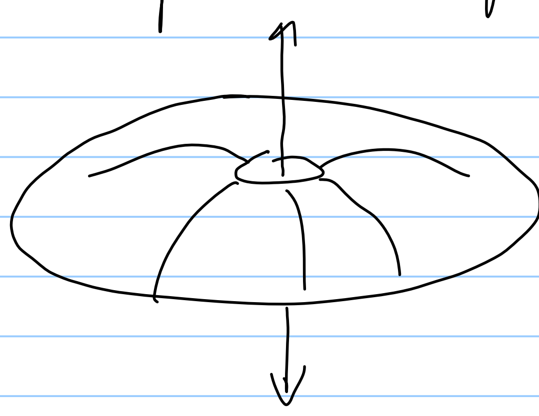
$$= \frac{1}{\mu_0 c}\left(\frac{\mu_0 p_0 \omega^2}{4\pi} \frac{\sin\theta}{r} \cos\left[\omega\left(t - \frac{r}{c}\right)\right]\right)^2 \hat{r}$$

$$\boxed{\langle \bar{S} \rangle = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \frac{\sin^2\theta}{r^2} \hat{r}}$$

$\langle \bar{S} \rangle = 0$ along the dipole axis

$\theta = 0, \pi$

θ - dependence of the radiation



total Power radiated

$$P = \int \langle \bar{S} \rangle \cdot d\bar{a}$$

$$= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi$$

$$\Rightarrow \boxed{P = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}} = \frac{\mu_0 p_0^2 c^3}{12\pi \lambda^4}$$

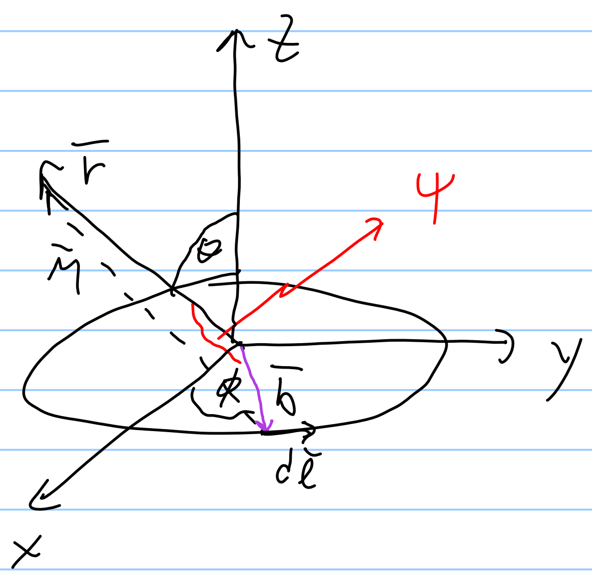
more power at shorter wavelengths.

<u>Magnetic dipole radiation</u>

consider a wire loop of radius b

with current

$$I(t) = I_0 \cos(\omega t)$$



$\bar{r} = r\sin\theta \, \hat{x} + r\cos\theta \, \hat{z}$ in $x-z$ plane

$V = 0$ since loop is neutral

$$\bar{b} = b\cos\phi\,\hat{x} + b\sin\phi\,\hat{y}$$

$$\bar{A} = \frac{\mu_0}{4\pi}\int \frac{I_0\cos[\omega(t - \eta/c)]}{\eta}\,d\bar{l}$$

magnetic dipole

$$\bar{m}(t) = \pi b^2 I(t)\,\hat{z} = m_0\cos(\omega t)\,\hat{z}$$

where $m_0 = \pi b^2 I_0$

ψ = angle between $\bar{r}$ and $\bar{b}$

$$\eta^2 = (r^2 + b^2 - 2rb\cos\psi)$$

$$= (r^2 + b^2 - 2rb\sin\theta\cos\phi)$$

make similar approximations as for electric dipole

$$\Rightarrow \bar{A} = -\frac{\mu_0 m_0\omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t - \tfrac{r}{c})]\,\hat{\phi}$$

$$\boxed{\bar{E} = -\frac{\partial\bar{A}}{\partial t} = \frac{\mu_0 m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - \tfrac{r}{c})]\,\hat{\phi}}$$

$$\boxed{\bar{B} = \nabla \times \bar{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}}$$

$$\langle \bar{S} \rangle = \frac{\mu_0 m_0 \omega^4}{32\pi^2 c^3} \frac{\sin^2\theta}{r^2} \hat{r}$$

$$P_{mag} = \int \langle \bar{S} \rangle \cdot d\bar{a} = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

$$P_{elec} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \Rightarrow \frac{P_{mag}}{P_{elec}} = \left(\frac{m_0}{p_0 c}\right)^2$$

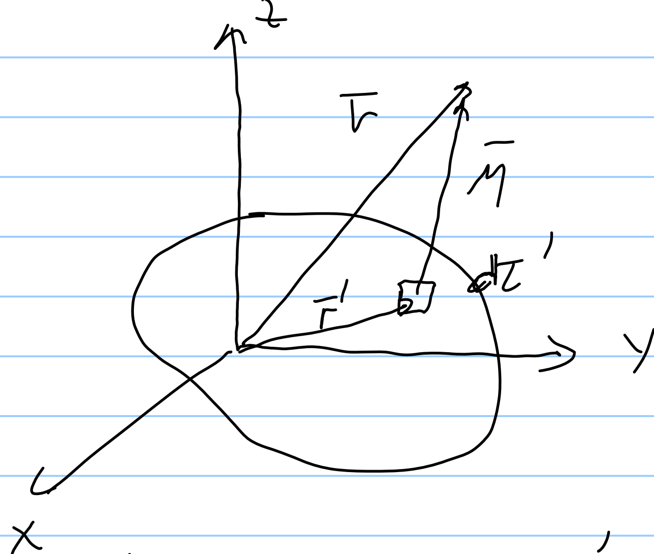
$$= \left(\frac{\pi b^2 I_0}{q_0 d c}\right)^2$$

let $I_0 \approx q_0 \omega$ and $d \simeq \pi b$

$$\Rightarrow \frac{P_{mag}}{P_{elec}} \approx \left(\frac{\omega b}{c}\right)^2 = \left(\frac{b}{\lambda}\right)^2 \ll 1$$

Radiation from an Arbitrary Source

$$V(\bar{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\bar{r}', t - \mathscr{r}/c)}{\mathscr{r}}\, d\tau'$$



$$\mathscr{r} = \left(r^2 + r'^2 - 2\,\bar{r}\cdot\bar{r}'\right)^{1/2}$$

Approx. 1 $\quad r' \ll r$

$$\mathscr{r} = r\left(1 + \frac{r'}{r^2} - \frac{2\,\bar{r}\cdot\bar{r}'}{r^2}\right)^{1/2}$$

$$\approx r\left(1 - \frac{\bar{r}\cdot\bar{r}'}{r^2}\right) = r - \hat{r}\cdot\bar{r}'$$

$$\frac{1}{\mathscr{r}} \approx \frac{1}{r}\left(1 + \frac{\bar{r}\cdot\bar{r}'}{r^2}\right)$$

$$\rho(\bar{r}', t - \mathscr{r}/c) \approx \rho\left(\bar{r}', t - \frac{r}{c} + \frac{\hat{r}\cdot\bar{r}'}{c}\right)$$

let $t_0 = t - \frac{r}{c}$ and expand around t_0

$$\rho(\bar{r}', t - \mathscr{r}/c) \approx \rho(\bar{r}', t_0) + \dot{\rho}(\bar{r}', t_0)\left(\frac{\hat{r}\cdot\bar{r}'}{c}\right) + \frac{1}{2}\ddot{\rho}\left(\frac{\hat{r}\cdot\bar{r}'}{c}\right)^2 + \ldots$$

<u>Approx 2</u> assume $\frac{|\dot{\rho}|}{|\ddot{\rho}|} \gg \frac{\hat{r}\cdot\bar{r}'}{c}$

timescale for fluctuations in the source $\gg$ timescale for signal to travel along the line of sight

$$\Rightarrow \quad |\dot{\rho}| \gg |\ddot{\rho}|\left(\frac{\hat{r}\cdot\bar{r}'}{c}\right)$$

therefore keep terms to $O(\dot{\rho})$

$$\Rightarrow V \approx \frac{1}{4\pi\epsilon_0}\int \frac{1}{r}\left(1 + \frac{\bar{r}\cdot\bar{r}'}{r^2}\right)\left[\rho(\bar{r}', t_0) + \dot{\rho}(\bar{r}', t_0)\left(\frac{\hat{r}\cdot\bar{r}'}{c}\right)\right]d\tau'$$

drop 2nd order terms

$$V \simeq \frac{1}{4\pi\epsilon_0} \frac{1}{r}\Big[\int \rho(\bar{r}', t_0)\, d\tau' +$$

$$\frac{\hat{r}}{r} \cdot \int \bar{r}' \rho(\bar{r}', t_0)\, d\tau' +$$

$$\frac{\hat{r}}{c} \cdot \int \bar{r}' \dot{\rho}(\bar{r}', t_0)\, d\tau' \Big]$$

note $\quad \bar{p}(t_0) = \int \bar{r}' \rho(\bar{r}', t_0)\, d\tau'$

is the dipole moment

$$\Rightarrow \boxed{V(\bar{r}, t) = \frac{1}{4\pi\epsilon_0}\Big[\underbrace{\frac{Q}{r}}_{\text{monopole}} + \underbrace{\frac{\hat{r}\cdot\bar{p}(t_0)}{r^2} + \frac{\hat{r}\cdot\dot{\bar{p}}(t_0)}{rc}}_{\text{dipole terms}} \Big]}$$

Multipole expansion for V

Vector Potential

$$\bar{A}(\bar{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\bar{J}(\bar{r}', t - \eta/c)}{\eta}\, d\tau'$$

proceed as we did with the ρ expansion

$$\bar{A}(\bar{r},t) \approx \frac{\mu_0}{4\pi}\int \frac{1}{r}\left(1+\frac{\bar{r}\cdot\bar{r}'}{r^2}\right)\left[\bar{J} + \dot{\bar{J}}\left(\frac{\hat{r}\cdot\bar{r}'}{c}\right)\right]d\tau'$$

$$\approx \frac{\mu_0}{4\pi}\frac{1}{r}\int \bar{J}(\bar{r}',t_0)\,d\tau' + \text{higher order terms.}$$

we can neglect these since

$$\nabla\cdot(x\bar{J}) = x(\nabla\cdot\bar{J}) + \bar{J}\cdot(\nabla x)$$

$$\nabla x = \hat{x} \Rightarrow \bar{J}\cdot\nabla x = J_x$$

$$\Rightarrow J_x = \nabla\cdot(x\bar{J}) - x(\nabla\cdot\bar{J})$$

$$\int J_x\,d\tau = \int \nabla\cdot(x\bar{J})\,d\tau - \int x(\nabla\cdot\bar{J})\,d\tau$$

but $\displaystyle\int_V \nabla\cdot(x\bar{J})\,d\tau = \oint_S x\bar{J}\cdot d\bar{a} = 0$

Since $\bar{J}$ is confined to the volume and we are free to make the volume as large as we want.

$$\Rightarrow \int_V J_x \, d\tau = -\int x (\nabla \cdot \bar{J}) \, d\tau$$

For x, y, z components

$$\int \bar{J} \, d\tau = -\int \bar{r} (\nabla \cdot \bar{J}) \, d\tau$$

$$= \int \bar{r} \frac{\partial \rho}{\partial t} \, d\tau$$

$$= \frac{\partial}{\partial t} \int \bar{r} \rho \, d\tau = \dot{\bar{p}}$$

$$\Rightarrow \boxed{\bar{A}(\bar{r}, t) \approx \frac{\mu_0}{4\pi} \frac{\dot{\bar{p}}}{r}}$$

To get E & B fields, we keep only terms of $1/r$

$$E = -\nabla V - \frac{\partial \bar{A}}{\partial t}$$

$$\frac{\partial \bar{A}}{\partial t} = \frac{\mu_0}{4\pi} \frac{\ddot{\bar{p}}(t_0)}{r}$$

since $t_0 = t - \frac{r}{c}$

$$\Rightarrow \frac{\partial t_0}{\partial t} = 1$$

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\bar{p}(t_0)}{r^2} + \frac{\hat{r}\cdot\dot{\bar{p}}(t_0)}{rc}\right]$$

keeping only $1/r$ terms and neglecting the monopole term.

$$\nabla V \simeq \frac{1}{4\pi\epsilon_0}\nabla\left(\frac{\hat{r}\cdot\dot{\bar{p}}(t_0)}{r}\right)$$

$$\simeq \frac{1}{4\pi\epsilon_0}\left(\frac{\hat{r}\cdot\ddot{\bar{p}}}{r}\right)(\nabla t_0)$$ again, here keeping only $1/r$ term

$$t_0 = t - \frac{r}{c} \Rightarrow \nabla t_0 = -\frac{1}{c}\nabla r = -\frac{\hat{r}}{c}$$

$$\Rightarrow \nabla V \simeq -\frac{1}{4\pi\epsilon_0 c^2}\left(\frac{\hat{r}\cdot\ddot{\bar{p}}}{r}\right)\hat{r}$$

$$\bar{E} \simeq \frac{1}{4\pi\epsilon_0}\frac{1}{c^2}\left(\frac{\hat{r}\cdot\ddot{\bar{p}}}{r}\right)\hat{r} - \frac{\mu_0\ddot{\bar{p}}}{4\pi r}$$

$$\boxed{\bar{E} = \frac{\mu_0}{4\pi r}\left[(\hat{r}\cdot\ddot{\bar{p}})\hat{r} - \ddot{\bar{p}}\right]} \qquad \frac{1}{c^2} = \mu_0\epsilon_0$$

but $\bar{E} = \frac{\mu_0}{4\pi r}\left[(\hat{r}\cdot\ddot{\bar{p}})\hat{r} - (\hat{r}\cdot\hat{r})\ddot{\bar{p}}\right]$

$$\Rightarrow \boxed{\bar{E} = \frac{\mu_0}{4\pi r}\left[\hat{r} \times \left(\hat{r} \times \ddot{\bar{p}}(t_0)\right)\right]}$$

more useful form

$$\bar{B} = \nabla \times \bar{A} = \frac{\mu_0}{4\pi}\frac{1}{r}\left[\nabla \times \dot{\bar{p}}(t_0)\right]$$

keeping $1/r$ terms

$$\left(\nabla \times \dot{\bar{p}}\right)_x = \frac{\partial \dot{\bar{p}}_z}{\partial y} - \frac{\partial \dot{\bar{p}}_y}{\partial z}$$

$$= \ddot{\bar{p}}_z \frac{\partial t_0}{\partial y} - \ddot{\bar{p}}_y \frac{\partial t_0}{\partial z}$$

$$= \left(\nabla t_0 \times \ddot{\bar{p}}\right)_x$$

but $\nabla t_0 = -\frac{\hat{r}}{c}$

$$\Rightarrow \left(\nabla \times \dot{\bar{p}}\right)_x = -\frac{1}{c}\left(\hat{r} \times \ddot{\bar{p}}\right)_x$$

$$\Rightarrow \boxed{\bar{B} = -\frac{\mu_0}{4\pi r c}\left[\hat{r} \times \ddot{\bar{p}}\right]}$$

If $\bar{p}$ in z-direction

$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$\bar{E} = \frac{\mu_0}{4\pi r}\left[\hat{r}\times(\hat{r}\times\ddot{\bar{p}})\right]$$

$$= \frac{-\mu_0 \ddot{p}}{4\pi r}(\hat{r}\times\hat{\phi})\sin\theta$$

$$\Rightarrow \boxed{\begin{aligned} \bar{E} &= \frac{\mu_0 \ddot{p}(t_0)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\theta} \\ \bar{B} &= \frac{\mu_0 \ddot{p}(t_0)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\phi} \end{aligned}}$$

as before

<u>Note</u> 1) the dipole term in the multipole expansion is the lowest order that produces radiation

2) If the dipole moment is zero radiation is still possible from higher order terms (eg quadrupole)