Why the speed of light is reduced in a transparent medium

Mary B. James and David J. Griffiths
Physics Department, Reed College, Portland, Oregon 97202

(Received 26 April 1991; accepted 25 July 1991)

It is well known from optics that the speed of light in a transparent medium is reduced by a factor of n (the index of refraction) as compared with vacuum. Maxwell's electrodynamics provides a simple account of this phenomenon, and relates n to the electric susceptibility of the material. But the conventional analysis does little to illuminate the mechanism involved. This paper offers some elucidation of the "miracle" by which the radiation from many induced molecular dipoles conspires to produce a single wave propagating at the reduced speed.

I. INTRODUCTION

From Maxwell's equations

$$\text{(i) } \nabla\cdot\mathbf{E} = (1/\epsilon_0)\rho, \quad \text{(iii) } \nabla\cdot\mathbf{B} = 0,$$

$$\text{(ii) } \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{(iv) } \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}, \tag{1}$$

it follows that electromagnetic waves propagate through the vacuum at speed

$$c = 1/\sqrt{\epsilon_0\mu_0}. \tag{2}$$

For if we apply the curl to Eq. (1) (ii), invoke the mathematical identity $\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}$, and set $\rho = 0$, $\mathbf{J} = 0$, we obtain the wave equation

$$\nabla^2\mathbf{E} = \frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2}, \tag{3}$$

with $v = 1/\sqrt{\epsilon_0\mu_0}$.

In a linear dielectric medium of susceptibility χ_e, Maxwell's equations can be written in the form

$$\text{(i) } \nabla\cdot\mathbf{E} = (1/\epsilon)\rho_f, \quad \text{(iii) } \nabla\cdot\mathbf{B} = 0,$$

$$\text{(ii) } \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{(iv) } \nabla\times\mathbf{B} = \mu_0\mathbf{J}_f + \mu_0\epsilon\frac{\partial \mathbf{E}}{\partial t}, \tag{4}$$

where ρ_f and $\mathbf{J}_f$ are the *free* charge and current densities, and

$$\epsilon = (1 + \chi_e)\epsilon_0. \tag{5}$$

The same argument as before, using $\rho_f = 0$ and $\mathbf{J}_f = 0$, leads again to the wave equation, but this time

$$v = 1/\sqrt{\epsilon\mu_0} = c/\sqrt{1 + \chi_e}. \tag{6}$$

In this way classical electrodynamics accounts for the fact—familiar from geometrical optics—that the speed of light in a transparent medium is reduced by a factor of n (the index of refraction); evidently

$$n = \sqrt{1 + \chi_e}. \tag{7}$$

That argument is quick and beautiful, but it does little to elucidate the *mechanism* involved. *Why should* light travel slower in glass or water than in vacuum? Well, when an electromagnetic wave strikes dielectric material, the electric field induces an oscillating electric dipole in each molecule, and these oscillating dipoles radiate new electric and magnetic fields. By a marvelous coincidence, these secondary fields combine just right with the primary fields to produce a single wave propagating at the reduced velocity.

This story is perfectly *correct*, as far as it goes, but scarcely *satisfying*, since it relies on a seemingly miraculous conspiracy of the induced dipole fields. It becomes a little more plausible when reformulated as follows: an electric field ($\mathbf{E}$) produces a polarization ($\mathbf{P}$) in the material:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E}, \tag{8}$$

and a changing polarization creates a current ($\mathbf{J}_p$):[1]

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}. \tag{9}$$

Inserting (8) and (9) into Maxwell's equation (1) (iv) with total current $\mathbf{J} = \mathbf{J}_f + \mathbf{J}_p$ yields

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}_f + \mu_0\frac{\partial}{\partial t}(\epsilon_0\chi_e\mathbf{E}) + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$$

$$= \mu_0\mathbf{J}_f + \mu_0\epsilon\frac{\partial \mathbf{E}}{\partial t}. \tag{10}$$

So the change from ϵ_0 to ϵ [between (1) and (4)] is indeed attributable to fields generated by currents associated with the polarization of the material.

Still, it would be nice to track the mechanism step by step, and see just how the "miraculous conspiracy" occurs. In this paper we study the process as a perturbation expansion in powers of χ_e. Imagine that a plane wave is incident on a piece of dielectric material. In zeroth order it simply continues along without modification. But this zeroth-order field polarizes the medium, and the resulting polarization currents give rise to a first-order field. This first-order field, in turn, further polarizes the medium, and the resulting currents generate a second-order field...and so on. We will show explicitly that the sum of *all* these fields is a wave propagating at speed c/n within the medium. Meanwhile, *outside* the dielectric the higher-order fields combine to form the reflected wave.

There are no surprises here—only a comforting confirmation that the story we have told is consistent, and perhaps a somewhat deeper understanding of the mechanism by which the speed of light is reduced in a dielectric medium. Before we begin—pursuant to Feynman's famous injunction (never start a calculation until you know the answer)—we shall briefly review the standard (nonperturbative) approach to the problem.

II. TRANSMITTED AND REFLECTED WAVES AT A DIELECTRIC BOUNDARY

A monochromatic plane wave incident from vacuum ($x < 0$) on a transparent dielectric medium ($x > 0$) gives rise to a transmitted wave and a reflected wave:[2]

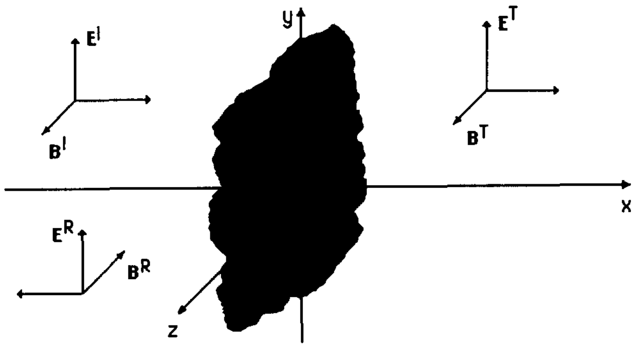

Fig. 1. Plane wave incident from vacuum ($x<0$) on dielectric ($x>0$).

incident:

$$\mathbf{E}^I(x,t) = E_0 e^{i(kx-\omega t)}\hat{j}, \quad \mathbf{B}^I(x,t) = (1/c)E_0 e^{i(kx-\omega t)}\hat{k}, \qquad (x<0),$$

reflected:

$$\mathbf{E}^R(x,t) = E_R e^{i(-kx-\omega t)}\hat{j}, \quad \mathbf{B}^R(x,t) = -(1/c)E_R e^{i(-kx-\omega t)}\hat{k}, \qquad (x<0), \tag{11}$$

transmitted:

$$\mathbf{E}^T(x,t) = E_T e^{i(k'x-\omega t)}\hat{j}, \quad \mathbf{B}^T(x,t) = (n/c)E_T e^{i(k'x-\omega t)}\hat{k}, \qquad (x>0),$$

where

$$k=\omega/c \text{ and } k'=n\omega/c \tag{12}$$

incident:

$$\mathbf{E}^I(x,t) = E_0 e^{i(kx-\omega t)}\hat{j}, \quad \mathbf{B}^I(x,t) = (1/c)E_0 e^{i(kx-\omega t)}\hat{k}, \qquad (x<0),$$

reflected:

$$\mathbf{E}^R(x,t) = E_R e^{i(-kx-\omega t)}\hat{j}, \quad \mathbf{B}^R(x,t) = -(1/c)E_R e^{i(-kx-\omega t)}\hat{k}, \qquad (x<0), \tag{11}$$

transmitted:

$$\mathbf{E}^T(x,t) = E_T e^{i(k'x-\omega t)}\hat{j}, \quad \mathbf{B}^T(x,t) = (n/c)E_T e^{i(k'x-\omega t)}\hat{k}, \qquad (x>0),$$

where

$$k=\omega/c \text{ and } k'=n\omega/c \tag{12}$$

(see Fig. 1). At $x=0$ the fields must satisfy the usual electromagnetic boundary conditions:[2]

$$\text{(i) } \epsilon_1 E_{1_\perp} = \epsilon_2 E_{2_\perp}, \quad \text{(iii) } \mathbf{E}_{1_\parallel} = \mathbf{E}_{2_\parallel},$$
$$\text{(ii) } B_{1_\perp} = B_{2_\perp}, \quad \text{(iv) } (1/\mu_1)\mathbf{B}_{1_\parallel} = (1/\mu_2)\mathbf{B}_{2_\parallel}. \tag{13}$$

In this case (normal incidence) $E_\perp$ and $B_\perp$ are zero, and $\mu_1=\mu_2=\mu_0$, so the boundary conditions simply require that **E** and **B** be continuous at $x=0$:

$$E_0+E_R=E_T, \quad E_0-E_R=nE_T \tag{14}$$

from which it follows that

$$E_T=[2/(n+1)]E_0, \quad E_R=-[(n-1)/(n+1)]E_0. \tag{15}$$

Thus the transmitted and reflected electric fields are

$$\mathbf{E}^T(x,t) = [2/(n+1)]E_0 e^{i(k'x-\omega t)}\hat{j}, \quad (x>0),$$
$$\mathbf{E}^R(x,t) = -[(n-1)/(n+1)]E_0 e^{i(-k'x-\omega t)}\hat{j}, \quad (x<0). \tag{16}$$

For future reference we express these fields in terms of the susceptibility, using Eqs. (7) and (12):

$$\mathbf{E}^T(x,t) = \frac{2}{(\sqrt{1+\chi_e}+1)} E_0 e^{i(\sqrt{1+\chi_e}kx-\omega t)}\hat{j}, \quad (x>0),$$
$$\mathbf{E}^R(x,t) = -\left(\frac{\sqrt{1+\chi_e}-1}{\sqrt{1+\chi_e}+1}\right) E_0 e^{i(-kx-\omega t)}\hat{j}, \quad (x<0). \tag{17}$$

In particular, expanding in powers of χ_e:

$$\mathbf{E}^T(x,t) = \left(\frac{2}{(\sqrt{1+\chi_e}+1)} e^{i(\sqrt{1+\chi_e}-1)kx}\right) E_0 e^{i(kx-\omega t)}\hat{j}$$
$$= [1-\tfrac{1}{4}\chi_e(1-2ikx)+\tfrac{1}{8}\chi_e^2 \times(1-2ikx-k^2x^2)+\cdots] \times E_0 e^{i(kx-\omega t)}\hat{j}, \tag{18}$$

$$\mathbf{E}^R(x,t) = -\tfrac{1}{4}\chi_e(1-\tfrac{1}{2}\chi_e+\tfrac{5}{16}\chi_e^2+\cdots)E_0 e^{i(-kx-\omega t)}\hat{j}. \tag{19}$$

III. PERTURBATIVE APPROACH

We now attack the same problem from the perturbative perspective outlined in the Introduction. To zeroth order the incident wave continues on to the right:

$$\mathbf{E}^0(x,t) = E_0 e^{i(kx-\omega t)}\hat{j}, \quad (x>0) \tag{20}$$

[this is the first term in Eq. (18)], producing a polarization in the medium

$$\mathbf{P}^1(x,t) = \epsilon_0\chi_e\mathbf{E}^0 = \epsilon_0\chi_e E_0 e^{i(kx-\omega t)}\hat{j}, \quad (x>0). \tag{21}$$

The resulting polarization current is

$$\mathbf{J}_p^1(x,t) = \frac{\partial \mathbf{P}}{\partial t} = -i\omega\epsilon_0\chi_e E_0 e^{i(kx-\omega t)}\hat{j}, \quad (x>0). \tag{22}$$

To calculate the field generated by this polarization current we chop the dielectric into slabs of infinitesimal thickness dx' (see Fig. 2). In the Appendix we show that the electric field at a distance s from a neutral plane surface current $\mathbf{K}(t)$ is[3]

$$\mathbf{E} = -(\mu_0 c/2)\mathbf{K}(t-s/c). \tag{23}$$

So the field at x due to $\mathbf{J}_p^1$ is

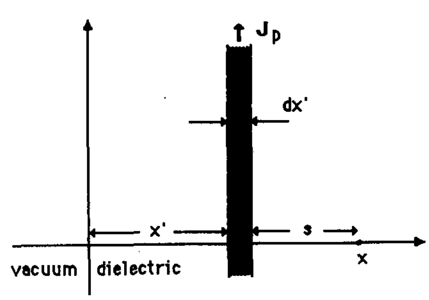

Fig. 2. Chopping the dielectric into current sheets.

$$\mathbf{E}^1(x,t) = \left(-\frac{\mu_0 c}{2}\right)(-i\omega\epsilon_0\chi_e E_0\hat{j})$$
$$\times\left(\int_0^x e^{i[kx' - \omega(t-(x-x')/c)]}dx' + \int_x^\infty e^{i[kx' - \omega(t-(x'-x)/c)]}dx'\right)$$
$$= \frac{ik}{c}\chi_e E_0\hat{j}\left(e^{i(kx-\omega t)}\int_0^x dx' + e^{i(-kx-\omega t)}\int_x^\infty e^{2ikx'}dx'\right),$$

which simplifies to[4]

$$\mathbf{E}^1(x,t) = (\chi_e/4)(1-2ikx)\mathbf{E}^0(x,t), \quad (x>0) \tag{24}$$

[reproducing the second term in Eq. (18)].

This first-order field further polarizes the dielectric:

$$\mathbf{P}^2(x,t) = \epsilon_0\chi_e\mathbf{E}^1(x,t),$$

resulting in an additional polarization current

$$\mathbf{J}_p^2(x,t) = i\omega\epsilon_0\chi_e\mathbf{E}^1(x,t),$$

which, in turn, generates the second-order field (calculated as before):

$$\mathbf{E}^2(x,t) = (\chi_e^2/2)(1-2ikx-k^2x^2)\mathbf{E}^0(x,t), \quad (x>0) \tag{25}$$

[consistent with the third term in Eq. (18)]. The progression is now clear: the nth-order field is of the form

$$\mathbf{E}^n(x,t) = (-\chi_e/2)^n Q_n(z)\mathbf{E}^0(x,t), \quad (x>0), \tag{26}$$

where Q_n is a polynomial of degree n in the variable $z=-2ikx$. [$Q_0=1$, $Q_1=(1+z)/2$, $Q_2=(1+z+z^2/4)/2$]. The resulting polarization current is

$$\mathbf{J}^{n+1}(x,t) = -i\omega\epsilon_0\chi_e\mathbf{E}^n(x,t), \tag{27}$$

and this produces a field[5]

$$\mathbf{J}^{n+1}(x,t) = \frac{1}{2}(-\chi_e/2)^{n+1}\mathbf{E}^0(x,t)$$
$$\times\left(\int_0^z Q_n(z')dz' + e^z\int_z^\infty Q_n(z')e^{-z'}dz'\right). \tag{28}$$

Evidently the recursion relation for Q_n is

$$Q_{n+1}(z) = \frac{1}{2}\left(\int_0^z Q_n(z')dz' + e^z\int_z^\infty Q_n(z')e^{-z'}dz'\right). \tag{29}$$

It follows that

$$\frac{dQ_{n+1}}{dz} = \frac{1}{2}e^z\int_z^\infty Q_n(z')e^{-z'}dz'$$
$$= Q_{n+1} - \frac{1}{2}\int_0^z Q_n(z')dz',$$

and hence

$$\frac{d^2Q_{n+1}}{dz^2} - \frac{dQ_{n+1}}{dz} + \frac{1}{2}Q_n = 0. \tag{30}$$

Now, the *total* transmitted wave is

$$\mathbf{E}^T(x,t) = \sum_{n=0}^\infty \mathbf{E}^n(x,t) = f(z)\mathbf{E}^0(x,t), \quad (x>0), \tag{31}$$

where

$$f(z) = \sum_{n=0}^\infty\left(-\frac{\chi_e}{2}\right)^n Q_n(z). \tag{32}$$

From Eq. (30) it follows that

$$\frac{d^2f}{dz^2} - \frac{df}{dz} + \frac{1}{2}\sum_{n=0}^\infty\left(-\frac{\chi_e}{2}\right)^n Q_{n-1} = 0,$$

or, redefining the summation index $n\to n+1$:

$$\frac{d^2f}{dz^2} - \frac{df}{dz} - \frac{\chi_e}{4}f = 0. \tag{33}$$

This is a differential equation for $f(z)$; the general solution is

$$f(z) = A(\chi_e)e^{(1-\sqrt{1+\chi_e})z/2} + B(\chi_e)e^{(1+\sqrt{1+\chi_e})z/2}. \tag{34}$$

It remains to determine $A(\chi_e)$ and $B(\chi_e)$.

For $\chi_e=0$ we get $f(z)=A(0)+B(0)e^z$; however, from (32) we know that $f(z)=Q_0(z)=1$ when $\chi_e=0$. Evidently $A(0)=1$ and $B(0)=0$. Similarly, differentiating (34) with respect to χ_e and setting $\chi_e=0$ we find $f'(z)=A'(0)-z/4+B'(0)e^z$, whereas (32) gives $f'(z)=-\frac{1}{2}Q_1(z)=-\frac{1}{4}-z/4$. So $A'(0)=-\frac{1}{4}$ and $B'(0)=0$. As we continue in this way the derivatives of B will always be accompanied by a factor of e^z, which is absent in (32), and hence *all* the derivatives of B must vanish. Conclusion:

$$B(\chi_e)=0. \tag{35}$$

To determine $A(\chi_e)$ we proceed as follows: From (29) we have

$$Q_{n+1}(0) = \frac{1}{2}\int_0^\infty e^{-z'}Q_n(z')dz', \quad \text{for } n\geqslant 0. \tag{36}$$

So (32) yields

$$f(0) = \sum_{n=0}^\infty\left(-\frac{\chi_e}{2}\right)^n Q_n(0)$$
$$= Q_0(0) + \sum_{n=1}^\infty\left(-\frac{\chi_e}{2}\right)^n\frac{1}{2}\int_0^\infty e^{-z'}Q_{n-1}(z')dz'$$
$$= 1 - \frac{\chi_e}{4}\int_0^\infty e^{-z'}\left(\sum_{n=1}^\infty\left(-\frac{\chi_e}{2}\right)^n Q_{n-1}(z')\right)dz'$$
$$= 1 - \frac{\chi_e}{4}\int_0^\infty e^{-z'}f(z')dz'. \tag{37}$$

Inserting (34), with $B=0$, we have

$$A(\chi_e) = 1 - \frac{\chi_e}{4}A(\chi_e)\int_0^\infty e^{-z'}e^{(1-\sqrt{1+\chi_e})z'/2}dz'$$
$$= 1 - \frac{\chi_e}{4}A(\chi_e)\frac{2}{1+\sqrt{1+\chi_e}}.$$

Or, solving for A:

$$A(\chi_e) = 2/(1+\sqrt{1+\chi_e}), \tag{38}$$

and hence

$$f(z) = [2/(1+\sqrt{1+\chi_e})]e^{(1-\sqrt{1+\chi_e})z/2}. \tag{39}$$

Returning, finally, to Eq. (31)—and recalling that $z=-2ikx$—we conclude that

$$\mathbf{E}^T(x,t) = \frac{2}{1+\sqrt{1+\chi_e}} e^{-ikx(1-\sqrt{1+\chi_e})} E_0 e^{i(kx-\omega t)}\hat{j}$$
$$= \frac{2}{1+\sqrt{1+\chi_e}} E_0 e^{i(\sqrt{1+\chi_e}kx-\omega t)}\hat{j}, \qquad (40)$$

which is precisely what we got in Sec. II [Eq. (17)] by more traditional means.

IV. THE REFLECTED WAVE

The reflected wave (in the region $x<0$) can be recovered by the same procedure. In zeroth order there is *no* reflection; to first order we have the field generated by $\mathbf{J}_p^1$ [Eq. (22)]:

$$\mathbf{E}^1(x,t)$$
$$= \left(-\frac{\mu_0 c}{2}\right)(-\omega\epsilon_0\chi_e E_0\hat{j})\int_0^\infty e^{i[kx'-\omega(t-(x'-x)/c]}dx'$$
$$= \frac{ik}{2}\chi_e E_0 e^{i(-kx-\omega t)}\hat{j}\int_0^\infty e^{2ikx'}dx'$$
$$= -\frac{\chi_e}{4}E_0 e^{i(-kx-\omega t)}\hat{j}, \qquad (41)$$

which is consistent with the first term in Eq. (19). The second-order field is generated by $\mathbf{J}_p^2$:

$$\mathbf{E}^2(x,t) = \left(-\frac{\mu_0 c}{2}\right)(-i\omega E_0\chi_e)\left(-\frac{\chi_e}{4}E_0\hat{j}\right)$$
$$\times\int_0^\infty (1-2ikx')e^{i[kx'-\omega(t-(x'-x)/c]}dx'$$
$$= \left(-\frac{ik\chi_e^2}{8}\right)E_0 e^{i(-kx-\omega t)}\hat{j}$$
$$\times\int_0^\infty (1-2ikx')e^{2ikx'}dx'$$
$$= \frac{\chi_e^2}{8}E_0 e^{i(-kx-\omega t)}\hat{j}, \qquad (42)$$

consistent with the second term in Eq. (19).

In general, $\mathbf{E}^{n+1}$ is generated by $\mathbf{J}_p^{n+1}$ [Eq. (27)]:

$$\mathbf{E}^{n+1}(x,t) = (-\mu_0 c/2)(-i\omega E_0\chi_e)(-\chi_e/2)^n E_0$$
$$\times e^{i(-kx-\omega t)}\hat{j}\left(\frac{1}{-2ik}\right)\int_0^\infty Q_n(z')e^{-z'}dz'$$
$$= E_0 e^{i(-kx-\omega t)}\hat{j}\left(-\frac{\chi_e}{2}\right)^{n+1}\frac{1}{2}$$
$$\times\int_0^\infty Q_n(z')e^{-z'}dz'$$
$$= E_0 e^{i(-kx-\omega t)}\hat{j}\,(\chi_e/2)^{n+1}Q_{n+1}(0)$$

[we used Eq. (36) in the last step]. Thus the total reflected wave is

$$\mathbf{E}^R(x,t) = \sum_{n=1}^\infty \mathbf{E}^n(x,t)$$
$$= \left(\sum_{n=1}^\infty\left(-\frac{\chi_e}{2}\right)^n Q_n(0)\right)E_0 e^{i(-kx-\omega t)}\hat{j}$$
$$= [f(0)-1]E_0 e^{i(-kx-\omega t)}\hat{j}$$
$$= \left(\frac{1-\sqrt{1-\chi_e}}{1+\sqrt{1+\chi_e}}\right)E_0 e^{i(-kx-\omega t)}\hat{j}, \qquad (43)$$

which is what we got in Sec. II [Eq. (17)] by the conventional method.

V. CONCLUSION

We do not, of course, pretend that the perturbative approach developed here is superior to the traditional one. But it is instructive (and comforting) to see in detail how the induced polarization currents in a dielectric medium give rise both to a reflected wave and to a transmitted wave traveling at the reduced speed c/n.

ACKNOWLEDGMENT

We thank Ray Mayer for help in analyzing the summation in Eq. (32).

APPENDIX: THE ELECTRIC FIELD OF A NEUTRAL PLANE CURRENT SHEET

Suppose that the yz plane carries a uniform (but time-dependent) surface current $\mathbf{K}(t)$. The retarded vector potential at a point a distance x above the plane is given by[6]

$$\mathbf{A}(x,t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{K}(t_r)}{\sqrt{r^2+x^2}}da, \qquad (A1)$$

where $t_r = t-\sqrt{r^2+x^2}/c$ is the retarded time, and $da = 2\pi r\,dr$ is an infinitesimal element of area (see Fig. 3). Thus

$$\mathbf{A}(x,t) = \frac{\mu_0}{2}\int_0^\infty \mathbf{K}(t-\sqrt{r^2+x^2}/c)\frac{r}{\sqrt{r^2+x^2}}dr. \qquad (A2)$$

To simplify the integral, let $u = (1/c)(\sqrt{r^2+x^2}-x)$, so that

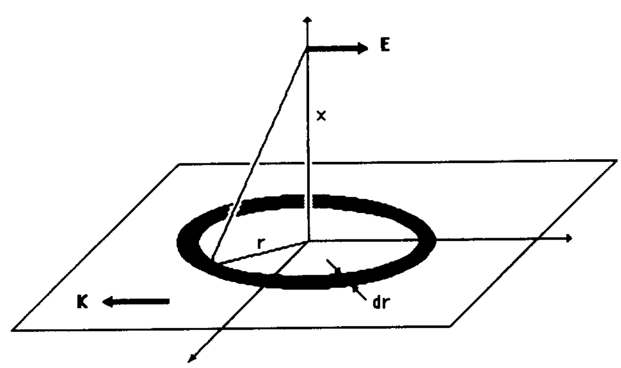

Fig. 3. The electric field of a current sheet.

$$du = \frac{1}{c}\frac{r}{\sqrt{r^2+x^2}}dr \quad \text{and} \quad t - \frac{\sqrt{r^2+x^2}}{c} = t - \frac{x}{c} - u.$$

Then

$$\mathbf{A}(x,t) = \frac{\mu_0 c}{2}\int_0^\infty \mathbf{K}\left(t - \frac{x}{c} - u\right)du. \qquad \text{(A3)}$$

From the vector potential it is easy to compute the electric field:[7]

$$\begin{aligned}\mathbf{E}(x,t) = \frac{\partial \mathbf{A}}{\partial t} &= -\frac{\mu_0 c}{2}\int_0^\infty \left[\frac{\partial}{\partial t}\mathbf{K}\left(t - \frac{x}{c} - u\right)\right]du \\ &= \frac{\mu_0 c}{2}\int_0^\infty \left[\frac{\partial}{\partial u}\mathbf{K}\left(t - \frac{x}{c} - u\right)\right]du \\ &= \frac{\mu_0 c}{2}\left[\mathbf{K}\left(t - \frac{x}{c} - u\right)\right]\Bigg|_0^\infty \\ &= -\frac{\mu_0 c}{2}\left[\mathbf{K}\left(t - \frac{x}{c}\right) - \mathbf{K}(-\infty)\right].\end{aligned}$$

Assuming the current goes to zero in the distant past, we can drop the second term, leaving

$$\mathbf{E}(x,t) = -(\mu_0 c/2)\mathbf{K}(t - x/c). \qquad \text{(A4)}$$

[1]P. Lorrain, D. P. Corson, and F. Lorrain, *Electromagnetic Fields and Waves* (Freeman, New York, 1988), 3rd ed., Eq. 9-9.

[2]Reference 1, Chap. 30.

[3]Equation (23) presupposes that the current goes to zero in the distant past (see the Appendix). Although this is not strictly true for the sinusoidal current in Eq. (22), it better describes the physical circumstance, in which the incident light source was turned on at some finite time. As a formal device we can handle the problem by attaching a small imaginary part to the frequency ($\omega \to \omega + i\epsilon$), which attenuates the field at large negative t, and taking the limit $\epsilon \to 0$ at the end of the calculation.

[4]The upper limit on the second integral yields a term of the form $e^{2ikL}e^{i(-kx-\omega t)}$ (with $L\to\infty$), representing a wave reflected back from the far side of the dielectric (at $x = L$). Obviously, we do not want such a term here; it is, again, an artifact of the pure sinusoidal current (22), and disappears if we attenuate the incident wave, as in Ref. 3: the factor e^{2ikL} then goes to zero in the limit $L\to\infty$.

[5]Actually, the upper limit on the second integral is $2L(\epsilon/c - ik)$, with $L\to\infty$. But the integrand is nonsingular, and a suitable rotation in the complex plane brings it around to the real axis.

[6]Reference 1, Sec. 37.4.

[7]Reference 1, Eq. 23.46. Note that since the surface is neutral, the *scalar* potential is zero.

Thomas precession: Where is the torque?

Richard A. Muller

Department of Physics and Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720

(Received 1 March 1991; accepted 24 April 1991)

Special relativity appears to violate the conservation of angular momentum **L** since it predicts that an accelerated gyroscope will precess, i.e., **L** will change in the absence of any applied torque. The paradox is resolved in a simple example by demonstrating that there is a torque present. The mass distribution in the gyroscope undergoes a relativistic distortion, and the center of mass is displaced away from the position of the accelerating force. The resulting torque $\boldsymbol{\tau} = d\mathbf{L}/dt$. The model also shows the physical origins of spin-orbit coupling and of the "oscillating term." A related calculation shows why a moving magnetic dipole has an *electric* dipole moment.

I. INTRODUCTION

According to the special theory of relativity, a gyroscope that moves along a curved path will also precess, i.e., the direction of its spin angular momentum will change. This effect is known as the "Thomas precession" after L. H. Thomas, who showed how this effect could resolve a paradox in atomic physics.[1] The standard derivation[2] of Thomas precession uses the fact that the product of successive Lorentz transformations is equivalent to a single Lorentz transformation plus a rotation. The most straightforward derivation known to the author is reproduced in the Appendix.

Even those who feel comfortable with one of the standard derivations may be at a loss to explain how angular momentum can be conserved. Angular momentum **L** is related to torque $\boldsymbol{\tau}$ through the vector relation

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}.$$

Yet the angular momentum of the gyroscope is changing, in the apparent absence of torque. How can that happen?

The resolution of the paradox is simple: There *is* a torque applied to the gyroscope, by the same force that accelerates the gyroscope along the curved path. The torque exists because of a relativistic distortion of the mass distribution in the gyroscope that moves the center of mass away from the axle.