Whence

$$[a^2]-[a]^2=1/g_{11}^2(1/4+1/2n-1/4)=1/2ng_{11}^2,$$

so that

$$\text{P.E.}(a)=1/g_{11}(2n)^{\frac{1}{2}},$$

or

$$\text{P.E.}(1/g_{11})=1/g_{11}(2/n)^{\frac{1}{2}},$$

the announced formula. The required P.E. is thus inversely proportional to the *square root* of the number n of color matchings. Similarly, we have, of course,

$$g_{22}\text{P.E.}(1/g_{22})=g_{33}\text{P.E.}(1/g_{33})=(2/n)^{\frac{1}{2}}.$$

Acknowledgment

My thanks are due Mr. Walter R. J. Brown, of these Laboratories, for having drawn my attention to the errors in the original derivation of the formula.

II

To every covariant tensor g_{ik} corresponds a contravariant tensor g^{ik}, the *conjugate* or the *supplement* of g_{ik}. If D be the determinant of g_{ik} and D_{ik} its minors, the "supplement" is defined by

$$g^{ik}=D_{ik}/D=(\partial/\partial g_{ik})\log D.$$

And, what is important in the present connection, *the supplement of the supplement* is the *original tensor*, i.e., g_{ik} is the supplement of g^{ik}. This holds for any number of dimensions.

Now, the equations for the most probable g_{ik} derived from n free color matchings in the threefold are

$$(\partial/\partial g_{ik})(1/2\log D-ag_{11}-dg_{32}-dg_{23}-\text{etc.})=0,$$

where D is the determinant of the g_{ik}. Although the g_{ik} are known to be components of a symmetrical tensor, confusion and errors in the determination of the conjugate tensors can be best avoided by treating g_{ik} and g^{ik} as general, non-symmetrical tensors until the final results are obtained, when it can be recalled that $g_{ik}=g_{ki}$. Therefore, $1/2g^{11}=a$, etc., $1/2g^{23}=d$, $1/2g^{32}=d$, etc., or

$$g^{11}=2a,\ \text{etc.},\quad g^{23}=2d,\ \text{etc.},\quad g^{32}=2d,\ \text{etc.} \qquad (1)$$

This, then is the (most probable) *conjugate* colorimetric tensor g^{ik}, which may itself be useful (*viz.*, in writing down the equations of the color geodesics). Written out fully, the conjugate tensor is

$$g^{ik}=\begin{vmatrix} 2a, & 2f, & 2e \\ 2f, & 2b, & 2d \\ 2e, & 2d, & 2c \end{vmatrix}. \qquad (1)$$

The determinant of this tensor is

$$|g^{ik}|=8E=8\begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix}. \qquad (1a)$$

Now, our required tensor g_{ik} is the conjugate of g^{ik}, that is to say,

$$g_{ik}=(\partial/\partial g^{ik})(\log E),$$

$$\begin{gathered} g_{11}=(bc-d^2)/2E,\quad g_{22}=(ca-e^2)/2E,\quad g_{33}=(ab-f^2)/2E, \\ g_{23}=g_{32}=(ef-ad)/2E,\quad g_{31}=g_{13}=(fd-be)/2E, \\ g_{12}=g_{21}=(de-cf)/2E, \end{gathered} \qquad (2)$$

where, by (1a),

$$E=abc-(ad^2+be^2+cf^2)+2def. \qquad (2a)$$

We may note that the determinant $D=|g_{ik}|$, which may be needed in practice, is the reciprocal of $|g^{ik}|$, i.e.,

$$D=1/8E. \qquad (3)$$

The formulas (2) agree with solutions found and communicated to us by Mr. Anthony Marriage, of Kodak Ltd., Harrow, England.

The formulas (2) are much simpler than my original, incorrect formulas (16), Phil. Mag. 136 (February 1946), and in J. Opt. Soc. Am. **36**, 465 (1946).

Acknowledgment

My thanks are due Mr. Marriage for calling this matter to my attention and indicating the correct solution.

Whose Absorption Law?

FRED H. PERRIN
Kodak Research Laboratories, Rochester, New York
September 29, 1947

THE exponential law relating the absorptance of radiation to the thickness of the absorbing medium is attributed, especially in this country, to that great photometric pioneer, Johann Lambert, just often enough so that when a writer* ventures to attribute it to a predecessor, he is greeted with raised eyebrows and a demand that he make good his assertion. This has happened often enough to indicate that there may be some little interest in the genesis of this law.

The founder of the modern science of photometry was Pierre Bouguer (1698–1758), Royal Professor of Hydrography at le Havre. In 1729 he described his fundamental researches (a more extensive treatment was published posthumously in 1760) in a small 164-page book entitled *Essai d'Optique, sur la Gradation de la Lumiere.* He starts his first section with a proof of the inverse-square law and then describes briefly what is probably the first recorded photometric head with a halfway decent type of field. He then describes experiments on the reduction of the "force" of the light after being attenuated by various numbers of glass plates. The theoretical work of immediate interest appears in the second section of the book, and, for those who like their authority at first hand, let us note the pertiment passages from the copy in the library of this laboratory. On pages 44–45 we find, translating from the old French:

"... now we must study the law according to which it [the light] diminishes in its passage [through transparent media]. One's first thought ... is that if one imagines a transparent medium to be divided into parallel layers of equal thickness, all the layers will intercept the same number of rays, so that the light, receiving in the passage of each layer an exactly equal diminution, would decrease in arithmetical progression, ..."

On the contrary, he had found that, if a certain pair of transparent plates reduced the intensity by 50 percent, a second pair did not reduce the intensity by an equal additional amount and thus produce extinction; indeed, he found that an appreciable amount of light was transmitted by even five such pairs. Returning to his theoretical argument, he points out that, if the second pair is to intercept the same number of rays, the same number must have been present originally, and then he continues (page 46):

"But since, at this layer, there may be present only a third or a quarter of the total number of rays, because all the others have already been interrupted, it is certain that this layer must also intercept three or four times fewer rays than the first. Thus the equal layers must not destroy equal quantities but only proportional quantities ... it is evident that the light will always diminish in geometrical progression."

Apparently Bouguer was unfamiliar with the possibilities of the forty-year old method of fluxions in the present connection, but he knew, as he states on page 48, that the ordinates of a logarithmic curve vary in geometrical progression. For parallel rays, therefore, in his own words,

"... les forces qu'a la lumiere, après avoir traversé differentes épaisseurs, peuvent être representées par les ordonnées d'une logarithmique qui a pour axe l'épaisseur du corps."

Translated,

"... the intensity of the light, after traversing different thicknesses, can be represented by the ordinates of a logarithmic [curve] which has the thickness of the medium as its axis [of abscissae]."

Except that the inverse relationship is not stated explicitly, this is exactly the form that Lambert was to give later, as will be seen presently, and clearly entitles Bouguer to priority.

In 1760, two years after Bouguer's death, Johann Heinrich Lambert (1728–1777) published his opus entitled *Photometria, sive de Mensura et Gradibus Luminis, Colorum et Umbrae.* It has been translated into German by Anding and was published in 1892 as Nos. 31–33 of Ostwald's *Klassiker der exakten Wissenschaften* by Engelmann in Leipzig. Anding in his notes discusses Lambert's debt to his predecessors and admits on page 59 of Vol. III (No. 33) that Bouguer had discovered the exponential law although he had not expressed it in a concise form. Lambert himself frequently mentions Bouguer's work. Referring to the copy of *Photometria* in the Derr Collection of the Massachusetts Institute of Technology library, we find that Part V, Chapter I starts on page 388 as follows:

"§. 865. Both above [in present work] and more especially in the book of the famous Bouguer, which has been frequently quoted already, there occur many [passages] which lead to the determination of the weakening of light in transparent media, ..."

On pages 390–1, Lambert derives the exponential law in the modern manner as follows, to use his own words:

"§. 875. Sit iam medium diaphanum CB (Fig. 79), lumen in istud incidat secundum directionem AB, sitque densitas incidentis$=1$. Dum vero in P peruenit sit densitas residua$=v$, via percursa $AP=x$, spatiolum $Pp=dx$, densitas obstaculorum in hoc spatiolo$=\delta$, debilitatio luminis ipsi debita$=-dv$, atque erit (§. 467.)

$$-dv=v\delta\cdot dx$$

adeoque

$$\log(1/v)=\int\delta dx."$$

In English:

"§. 875. Let CB be the transparent medium, let the light fall on it in the direction AB, and let the intensity of the incident light equal unity. Now, when the light reaches P, let the remaining intensity be v, the path already traversed be x, the elementary layer Pp be dx, the density of the obstacles in this layer be δ [and] the diminution of the light caused by these obstacles be $-dv$; then we have

$$-dv=v\delta\cdot dx$$

and, furthermore,

$$\log(1/v)=\int\delta dx."$$

Lambert then goes on to point out that, if the elementary layers are equally thick (meaning apparently that their thickness is not a function of x), the last equation integrates into the familiar expression

$$\log(1/v)=\delta x.$$

A century later, August Beer made his contribution to the subject. To determine the exact nature and scope of this contribution, let us go to his paper in Poggendorff's Annalen 86, 78–88 (1852), the title of which is "Bestimmung der Absorption des rothen Lichts in farbigen Flüssigkeiten." For his measurements, he used an outfit like the now familiar Martens photometer. After explaining the principle of the instrument, he defines absorption coefficient as the diminution (*Schwächung*) in amplitude that results from the traversing of a unit thickness of absorbing medium. Then on page 83 he points out that copper sulfate is chemically inert to water and continues

"... first a tube one decimeter long was filled with a solution of copper sulfate which contained one volume of solution [having a] concentration of 13.5° [that is, which read 13.5° on his photometer] to nine volumes of water; the resulting solution thus had a dilution of 1/9 [10 percent concentration in modern terminology]. Another tube two decimeters long was filled with a solution of 1/19 dilution [5 percent concentration]. The latter thus contains as much of the concentrated solution as the former; it also . . . showed in red light practically the same value for the angle α as the former. Specifically, the value of this angle for the short tube was 3°28′10″, for the longer 3°23′0″, which give for the absorption coefficient of the concentrated solution the approximately equal values 0.065 and 0.063."

In Beer's table on page 87, he records enough other examples to establish indubitably his law of the reciprocal relation between thickness and concentration. It is to be noted, however, that his law has nothing to do with Bouguer's exponential law and concerns the reciprocal relation between thickness and concentration alone, although he used Bouguer's law for his computations when necessary.

* For example, Arthur C. Hardy and Fred H. Perrin in *The Principles of Optics* (McGraw-Hill Book Company, Inc., New York, 1932), p. 24.

The Measurement of Strongly Reflecting Transmission Samples with the Commerical Recording Spectrophotometer

W. E. KNOWLES MIDDLETON
Optics Section, National Research Laboratories, Ottawa, Canada
October 27, 1947

THE recording spectrophotometer originated by Hardy,[1] and since made available commercially,[2] is so well known that no description is needed here. The photometric errors of this instrument have been treated by Pineo.[3] A very large error not mentioned by Pineo can occur, however, in the measurement of transmission samples which have a high specular reflectance. Outstanding examples are the recently developed interference filters which transmit a narrow region of the spectrum and reflect (instead of absorbing) the remainder.[4] These filters have the superficial appearance of excellent mirrors.

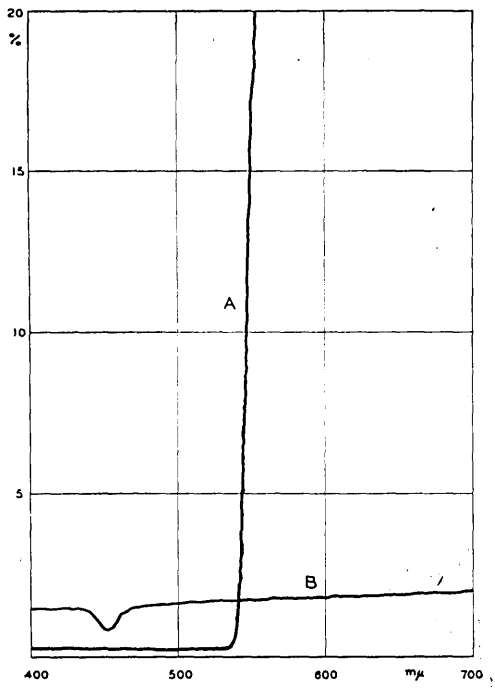

FIG. 1.

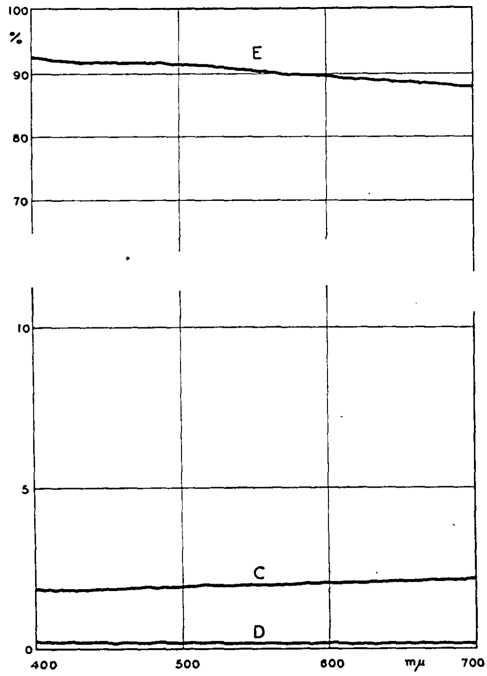

FIG. 2.

The error in question was drawn to the attention of the author by the extraordinary result shown in Fig. 1, which would have shaken one's faith in the rules of photometry if it had gone without explanation. Curve A of Fig. 1 shows the transmission of a sharp cut-off orange filter up to the point where it reaches 20 percent, and was made on the recording spectrophotometer with the 20 percent cam. Curve B shows the transmission of the same filter with an interference filter, having a maximum transmission of 28 percent at 452 mμ, placed in front of it. If the curves could be accepted at their face value, the interference filter would increase the transmission of blue light by the orange filter! It was further noticed that the values obtained for the interference filter alone were many times greater in the excluded regions of the spectrum than those obtained with a König-Martens spectrophotometer.

To investigate the source of this error, "transmission" measurements were made on a good first-surface aluminized mirror, provided with an opaque backing. Curve C of Fig. 2 shows the apparent transmittance of this completely opaque sample when placed in the transmission compartment against the wall in the usual position, i.e., normal to the beam. Curve D shows what happened when the mirror was slightly inclined so that no part of the reflected beam