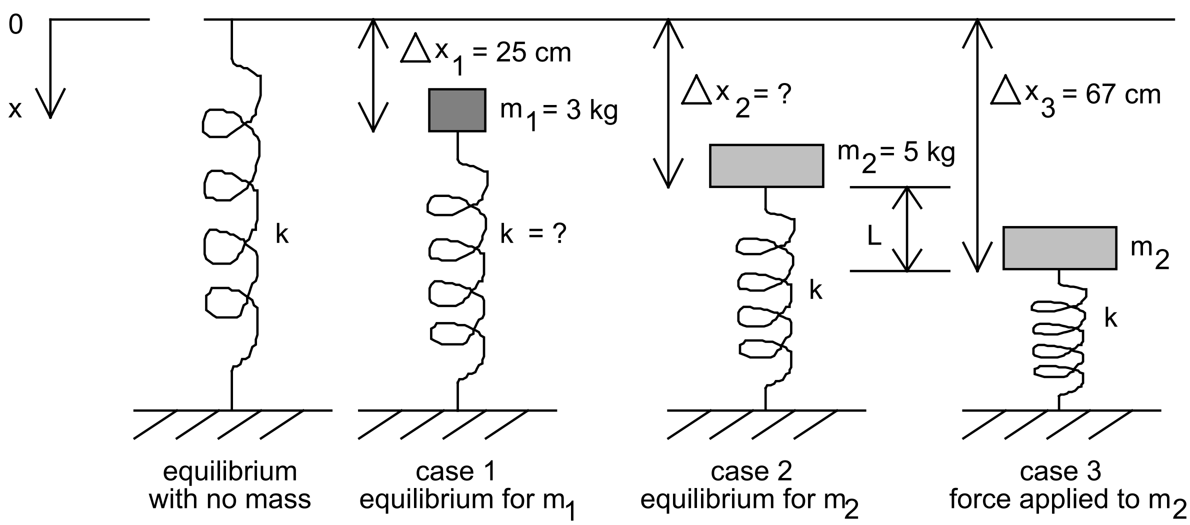
PHY 140Y – FOUNDATIONS OF PHYSICS
2001-2002
Tutorial Questions #9 – Solutions
November 12/13

Conservation of Energy and Springs

1. One end of a massless spring is placed on a flat surface, with the other end pointing upward, as shown below. A mass of 3.0 kg is placed on top of the spring, compressing it by 25 cm. The 3.0 kg mass is removed and replaced by a 5.0 kg mass. Then the spring is compressed by hand so that the end of the spring is 67 cm lower than the position of the spring with no mass attached. The spring is then released. What is the maximum kinetic energy of the 5.0 kg mass?

Solution:
Define the x axis as shown, positive downwards with x = 0 at the initial equilibrium with no mass attached.



For case 1, apply Newton's Second Law to mass m_1 in order to determine the value of the spring constant:

$$F_{net} = F_g - F_s = 0 \quad \text{(at rest)}$$

$$m_1 g - k\Delta x_1 = 0$$

$$k = \frac{m_1 g}{\Delta x_1} = \frac{(3.0\ \text{kg})(9.81\ \text{m/s}^2)}{0.25\ \text{m}} = 117.72\ \text{N/m} = 120\ \text{N/m} \quad \text{(sig. figs.)}$$

For case 2, apply Newton's Second Law to mass m_2 in order to determine the equilibrium position for m_2:

$$F_{net} = F_g - F_s = 0 \quad \text{(at rest)}$$

$$m_2 g - k\Delta x_2 = 0$$

$$\Delta x_2 = \frac{m_2 g}{k} = m_2 g \frac{\Delta x_1}{m_1 g} = \frac{m_2}{m_1}\Delta x_1 = \frac{5.0 \text{ kg}}{3.0 \text{ kg}} \times 0.25 \text{ m} = 0.42 \text{ m}$$

For case 3, apply Conservation of Mechanical Energy in order to determine the total energy:

$$E_m = K_3 + U_3 = 0 + \tfrac{1}{2}kL^2$$

where m_2 is at rest and L is the displacement from the equilibrium position for m_2.

Thus: $E_m = \tfrac{1}{2}kL^2 = \tfrac{1}{2}k(\Delta x_3 - \Delta x_2)^2 = \tfrac{1}{2}(117.72 \text{ N/m})(0.67 \text{ m} - 0.42 \text{ m})^2 = 3.7 \text{ J}$

The maximum kinetic energy of the 5.0 kg mass is: $K_{max} = E_m = 3.7 \text{ J}$

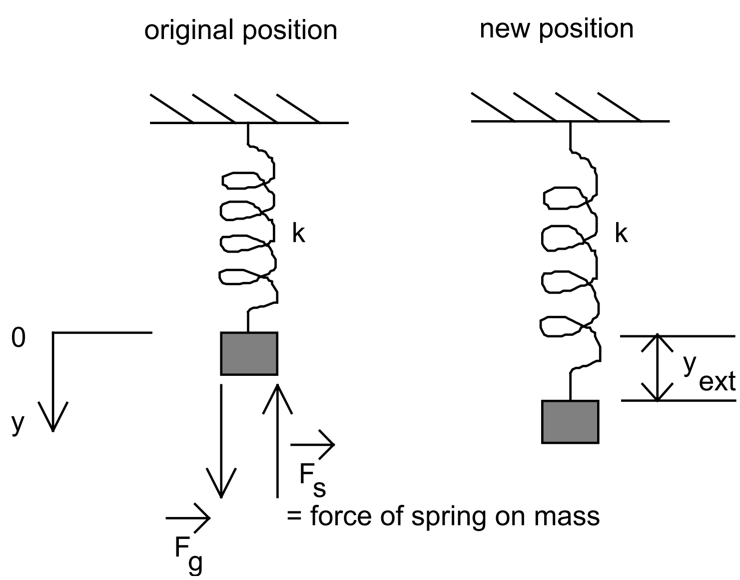
which will occur when all of the mechanical energy is kinetic. This happens when the potential energy of m_2 is zero, i.e., when L = 0, the equilibrium position for m_2.

Simple Harmonic Motion

2. A 340-g mass is attached to a vertical spring and lowered slowly until it rests at a new equilibrium position, which is 30. cm below the spring's original equilibrium position. The system is then set into simple harmonic motion. What is the period of the motion?

Solution:

Now, let's define the y axis as shown, positive downwards with y = 0 at the initial equilibrium position.



Apply Newton's Second Law in the y direction:

$$\vec{F}_{net} = \vec{F}_g + \vec{F}_s = m\vec{a}$$

$$mg - ky_{ext} = ma_y = 0 \quad \text{(at rest in new equilibrium position)}$$

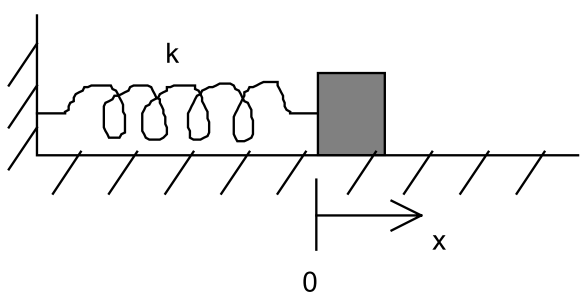
$$\frac{k}{m} = \frac{g}{y_{ext}}$$

Perturb the system to get simple harmonic motion. For a spring in SHM: $\omega = \sqrt{\frac{k}{m}}$

Thus: $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{y_{ext}}{g}} = 2\pi\sqrt{\frac{0.30\ \text{m}}{9.81\ \text{m/s}^2}} = 1.1\ \text{seconds}$$

3. A 0.500 kg block on a frictionless horizontal surface is attached to an ideal spring and is found to complete one oscillation every 2.00 s. The range of the oscillation is measured to be 0.40 m, and the block has zero speed when t = 0.
(a) Determine the period, frequency, and angular frequency of the motion.
(b) Find the spring constant of this spring.
(c) Determine the amplitude and phase shift of the oscillation.
(d) Find x(t), where x = 0 m is the equilibrium position of the mass on the spring.
(e) Determine $v_x(t)$ and the maximum speed.
(f) Determine $a_x(t)$ and the magnitude of the maximum acceleration.

Solution:



(a) The block executes one complete oscillation during 2.00 s. The period of the oscillation is the time interval for one complete oscillation and so is just: $T = 2.00\ \text{seconds}$

The frequency is: $$f = \frac{1}{T} = \frac{1}{2.00\ \text{s}} = 0.500\ \text{Hz}$$

The angular frequency is: $$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi\ \text{rad}}{2.00\ \text{s}} = 3.14\ \text{rad/s}$$

(b) The spring constant is: $k = m\omega^2 = (0.500\ \text{kg})(3.14\ \text{rad/s})^2 = 4.93\ \text{N/m}$
(Show that units of N/m [always used for k] are equivalent to kg/s^2.)

(c) The range of the oscillation is given as 0.40 m. This is the distance between the minimum and maximum positions of the block and so is twice the amplitude. Thus: $A = 0.20\ \text{m}$.

The mass has zero speed when t = 0 s, so we can use this information to determine the phase shift δ of the oscillation in $x(t) = A\cos(\omega t + \delta)$.

The velocity is thus: $v_x(t) = \dfrac{dx(t)}{dt} = -A\omega\sin(\omega t + \delta)$

Substitute v=0 when t=0:

$$-A\omega\sin(\omega 0 + \delta) = 0$$
$$\sin\delta = 0$$
$$\delta = 0 \text{ radians}$$

(d) The position of the mass at any time is: $x(t) = A\cos(\omega t) = (0.20\ \text{m})\cos[(3.14\ \text{rad/s})t]$

(e) The velocity at any time is:

$$\begin{aligned} v_x(t) &= -A\omega\sin(\omega t) \\ &= -(0.20\ \text{m})(3.14\ \text{rad/s})\sin[(3.14\ \text{rad/s})t] \\ &= -(0.63\ \text{m/s})\sin[(3.14\ \text{rad/s})t] \end{aligned}$$

The maximum speed.occurs when $\sin(\omega t) = \pm 1$ and thus has magnitude $A\omega = 0.63\ \text{m/s}$.

(f) The accleration is:

$$\begin{aligned} a_x(t) &= \frac{dv_x(t)}{dt} = -A\omega^2\cos(\omega t) \\ &= -(0.20\ \text{m})(3.14\ \text{rad/s})^2\cos[(3.14\ \text{rad/s})t] \\ &= -(2.0\ \text{m/s}^2)\cos[(3.14\ \text{rad/s})t] \end{aligned}$$

The maximum acceleration occurs when $\cos(\omega t) = \pm 1$ and has magnitude $A\omega^2 = 2.0\ \text{m/s}^2$.

4. You can measure the acceration due to gravity, g, with a simple pendulum. Set up a pendulum of length 1.75 m. With a stopwatch, you should find that it executes 25 complete small oscillations in 66.4 seconds.
(a) What value of g do these data imply?
(b) If the oscillation amplitude were reduced to half the original value, what would be the period of the pendulum's motion?

<u>Solution:</u>

(a) Since 25 oscillations take 66.4 seconds, the time required for one oscillation (the period) is:

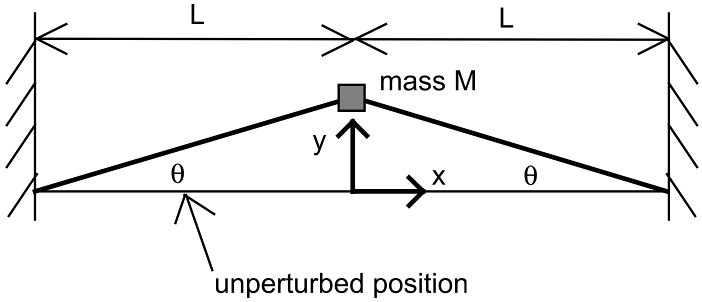
$$T = \frac{66.4\ \text{s}}{25} = 2.66\ \text{seconds}$$

For a pendulum, $\omega = \sqrt{\dfrac{g}{L}}$

Thus: $$g = L\omega^2 = L\left(\frac{2\pi}{T}\right)^2 = 4\pi^2\frac{L}{T^2} = 4\pi^2\frac{1.75\ \text{m}}{(2.66\ \text{s})^2} = 9.80\ \text{m/s}^2$$

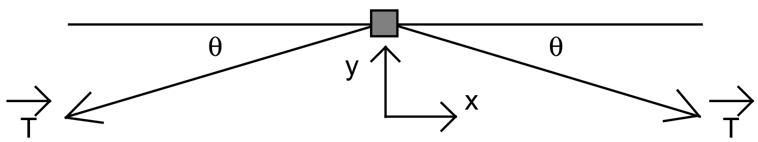
(b) If the oscillation amplitude were reduced to half the original value, the period of the pendulum's motion would be unchanged because it is independent of the amplitude for small oscillations.

5. A mass M is connected to two rubber bands of length L. Each rubber band has a constant tension T. The mass is displaced by a very small distance y and is released. It then exhibits simple harmonic motion. What is the angular frequency of this motion? (Only consider the tension forces – ignore gravity.)



Solution:

First draw the force diagram for mass M.



Apply Newton's Second Law in the y direction:

$$\vec{F}_{net} = \vec{T}_1 + \vec{T}_2 = m\vec{a}$$

$$-T\sin\theta - T\sin\theta = ma_y = m\frac{d^2y}{dt^2}$$

$$-2T\sin\theta = m\frac{d^2y}{dt^2}$$

Now: $\sin\theta = \dfrac{y}{\sqrt{y^2 + L^2}}$. Assume that y << L, so that: $\sin\theta \approx \dfrac{y}{L}$

The equation of motion then becomes:

$$-2T\frac{y}{L} \approx m\frac{d^2y}{dt^2}$$

$$-\frac{2T}{mL}y \approx \frac{d^2y}{dt^2}$$

This is clearly the equation defining SHM, $\dfrac{d^2y}{dt^2} = -\omega^2 y$,

and so the angular frequency is: $\omega = \sqrt{\dfrac{2T}{mL}}$.