# PHY 140Y – FOUNDATIONS OF PHYSICS
2001-2002
Tutorial Questions #8 – Solutions
November 5/6

Conservation of Energy

1. The masses shown below are connected by a massless string over a frictionless, massless pulley and are released from rest. Use energy conservation to find
    (a) the velocity of the 7.0-kg mass just before it hits the floor,
    (b) the maximum height reached by the 4.0-kg mass, and
    (c) the fraction of the system's initial mechanical energy lost when the 7.0 kg mass comes to rest on the floor.
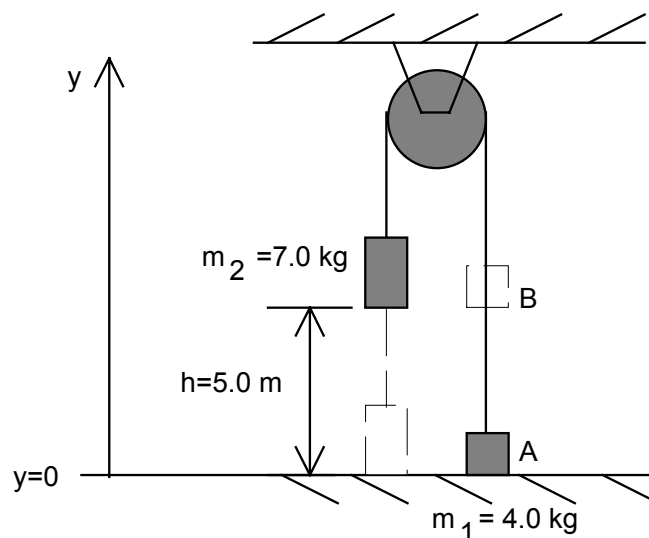
Solution:

Note that both blocks start from rest. Ignore friction and air resistance. Define the floor as the "zero" of potential energy. Also note that since the masses are connected, their speeds will be the same when both are off the floor.

(a) Define "A" the position when the 4.0-kg (m_1) block is on the floor. Define "B" as the position when the 7.0-kg (m_2) block is just touching the floor.

Then: $v(m_1) = v(m_2) = v_B$ = the velocity of the 7.0-kg mass just before it hits the floor

By Conservation of Energy:

$$U_A + K_A = U_B + K_B$$

$$U_A(m_1) + U_A(m_2) + K_A(m_1) + K_A(m_2) = U_B(m_1) + U_B(m_2) + K_B(m_1) + K_B(m_2)_B$$

$$m_1 g(0) + m_2 gh + \tfrac{1}{2}m_1(0)^2 + \tfrac{1}{2}m_2(0)^2 = m_1 gh + m_2 g(0) + \tfrac{1}{2}m_1 v_B^2 + \tfrac{1}{2}m_2 v_B^2$$

$$m_2 gh = m_1 gh + \tfrac{1}{2}m_1 v_B^2 + \tfrac{1}{2}m_2 v_B^2$$

$$v_B = \sqrt{\frac{2(m_2 - m_1)gh}{m_1 + m_2}} = \sqrt{\frac{2(7.0\text{kg} - 4.0\text{kg})(9.81\text{m/s}^2)(5.0\text{m})}{7.0\text{kg} + 4.0\text{kg}}}$$

$$= 5.172 \text{ m/s} = 5.2 \text{ m/s}$$

(b) The 4.0-kg block will continue to rise above h=5.0 m with upward speed 5.2 m/s.

$$a = -g$$

Define t_o=0. Then:

$$v(t) = v_o + at = v_B - gt$$

$$y(t) = y_o + v_o t + \tfrac{1}{2}at^2 = h + v_B t - \tfrac{1}{2}gt^2$$

At the maximum height, t=t':

$$v(t') = 0$$

$$\therefore v_B - gt' = 0$$

$$t' = \frac{v_B}{g}$$

Therefore the maximum height reached by the 4.0-kg mass is:

$$y(t') = h + v_B t' - \tfrac{1}{2}gt'^2$$

$$= h + v_B\left(\frac{v_B}{g}\right) - \tfrac{1}{2}g\left(\frac{v_B}{g}\right)^2$$

$$= h + \frac{v_B^2}{2g} = 5.0 \text{ m} + \frac{(5.172 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)}$$

$$= 6.4 \text{ m}$$

Note that the total mechanical energy (U + K) of both masses is no longer conserved after the 7.0-kg block hits the floor because a nonconservative contact force acts to stop it.

(c) When the 7.0 kg mass comes to rest on the floor, its kinetic energy is lost.

$$K = \tfrac{1}{2}m_2 v_B^2 = \tfrac{1}{2}(7.0 \text{ kg})(5.172 \text{ m/s})^2 = 93.64 \text{ J} = 94 \text{ J}$$

Compare this with the initial energy of the system:

$$E_A = m_2 gh = (7.0 \text{ kg})(9.81 \text{ m/s}^2)(5.0 \text{ m}) = 343.35 \text{ J} = 340 \text{ J}$$

Thus the fraction of the system's initial mechanical energy lost is:

$$\frac{93.64}{343.35} \times 100\% = 27\% \quad \text{(rounding to two significant figures)}$$

2. A biologist uses a spring-loaded dart gun to shoot a 50-g tranquilizing dart into an elephant 21 m away. The gun's spring has spring constant k = 690 N/m and is pulled back 14 cm to launch the dart. The dart embeds itself 2.2 cm in the elephant. Assume that the dart's trajectory is nearly horizontal.
    (a) What is the average stopping force exerted on the dart by the elephant's flesh?
    (b) How long does it take the dart to reach the elephant?

Solution:
(a) Assume that there is no change in the gravitational potential energy of the dart. Also assume that the stopping force is the only non-negligible non-conservative force.

The work done by this non-conservative force is:　　$W_{nc} = -F\Delta x = \Delta U + \Delta K = 0 - \frac{1}{2}kx^2$

where Δx is the distance that the dart is embedded and x is the distance that the spring is compressed.

The stopping force is then:　　$F = \dfrac{kx^2}{2\Delta x} = \dfrac{(690 \ N/m)(0.14 \ m)^2}{2(0.022 \ m)} = 307.4 \ N = 310 \ N$

(b) The horizontal speed of the dart as it leaves the spring is determined using:

$$\Delta U + \Delta K = -\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = 0$$

$$\therefore v = \sqrt{\frac{kx^2}{m}} = \sqrt{\frac{(690 \ N/m)(0.14 \ m)^2}{50 \times 10^{-3} \ kg}} = 16.44 \ m/s = 16 \ m/s$$

This horizontal velocity remains constant, so the distance travelled to reach the elephant is:
　　$d = d_o + vt = vt$　　assuming $d_o = 0$ at $t_o=0$.

The time taken for the dart to reach the elephant is:　$t = \dfrac{d}{v} = \dfrac{21 \ m}{16.44 \ m/s} = 1.3 \ s$

3. A ball of mass m = 100 g slides inside a bowl whose cross section has circular arcs at each side and a flat horizontal central portion between points A and B of length L = 30 cm, as shown below. The curved sides of the bowl are frictionless, and for the flat bottom, the coefficient of kinetic friction is $\mu_k = 0.15$. The ball is released from rest at the rim, which is 15 cm above the flat part of the bowl.
    (a) What is the speed of the ball at A?
    (b) What is the speed of the ball at B?
    (c) Where does the ball finally come to rest?
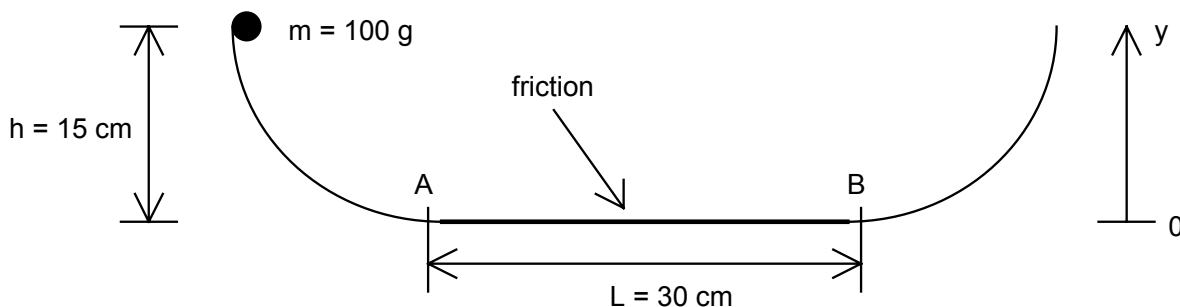
Solution:
Choose the coordinate system with y=0 at the bottom of the bowl.

(a) For motion from the initial position to point A, apply Conservation of Energy:

$$U_i + K_i = U_f + K_f$$

$$mgh + \tfrac{1}{2}mv_i^2 = mg(0) + \tfrac{1}{2}mv_A^2$$

$$v_A = \sqrt{2gh} = \sqrt{2(9.81 \ \text{m/s})(0.15 \ \text{m})}$$

$$= 1.7 \ \text{m/s}$$

(b) For motion from point A to point B, with kinetic friction acting and U constant, we have:

$$W_{net} = -F_k L = K_f - K_i$$

$$\therefore -(\mu_k mg)L = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2$$

$$v_B^2 - v_A^2 = -2\mu_k gL$$

$$v_B = \sqrt{v_A^2 - 2\mu_k gL} = \sqrt{2gh - 2\mu_k gL}$$

$$= \sqrt{2(9.81 \ \text{m/s})\big[0.15 \ \text{m} - (0.15)(0.30 \ \text{m})\big]}$$

$$= 1.4 \ \text{m/s}$$

(c) For motion from the initial point to the final stopping point, let D represent the total horizontal distance over which the friction force acts. Because kinetic friction opposes the motion, we have:

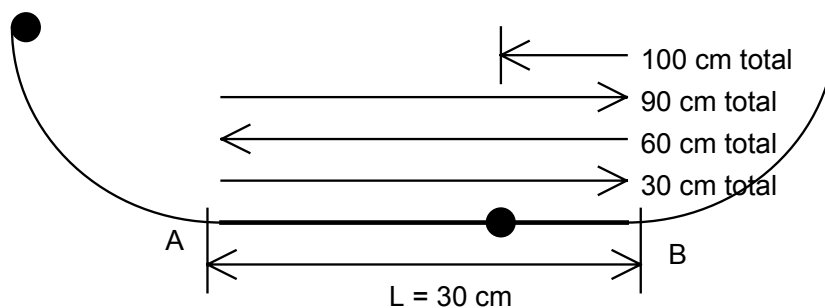$$W_{net}^{friction} = -F_k D = \Delta U + \Delta K \qquad \text{for non-conservative forces like friction}$$

Therefore:
$$-(\mu_k mg)D = (U_f - U_i) + (K_f - K_i)$$
$$= (0 - mgh) + (0 - 0)$$

since the ball is initially and finally at rest, and is at y=0 when stopped.

$$\mu_k mgD = -mgh \qquad\qquad \therefore D = \frac{h}{\mu_k} = \frac{0.15\text{m}}{0.15} = 1.0\text{m}$$

Now, the distance between points A and B is L=30 cm. Thus, the ball will travel 3L + 10 cm and will finally come to rest 10 cm to the left of point B (20 cm to the left of point A).

Force and Potential Energy

4.   (a) Derive an expression for the potential energy of an object subject to a force $F_x = ax - bx^3$, where a = 5.0 N/m and b = 2.0 N/m^3, taking U = 0 at x = 0.

    (b) Graph the potential energy curve for x > 0 and use it to find the turning points for an object whose total energy is –1.0 Joule.
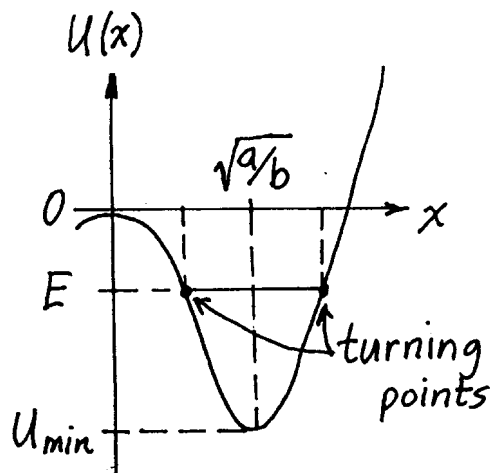
Solution:

(a) The force is conservative (any one-dimensional force given by an integrable function of position is), so the potential energy can be found using:

$$U(x_2) - U(x_1) = -\int_{x_1}^{x_2} F_x dx = -\int_{x_1}^{x_2} (ax - bx^3) dx$$

$$= -\left[\frac{ax^2}{2} - \frac{bx^4}{4} \right]_{x_1}^{x_2} = -\frac{a}{2}(x_2^2 - x_1^2) + \frac{b}{4}(x_2^4 - x_1^4)$$

With the zero of potential energy at x=0, we have: $U(x) = -\dfrac{ax^2}{2} + \dfrac{bx^4}{4}$

(b) A graph of U(x) for x > 0 is shown below, when a = 5.0 N/m, b = 2.0 N/m^3, and *x* is in m. Note that the potential energy is symmetric, U(-x)=U(x), but that only positive displacements are considered in this problem.

The conservation of energy can be written in terms of the total energy:

$$E = K + U = \tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x), \quad \text{so that:} \qquad \frac{dx}{dt} = \pm\sqrt{2[E - U(x)]/m}$$

The maximum speed occurs when U(x) is a minimum; i.e., $\dfrac{dU}{dx} = 0$ and $\dfrac{d^2U}{dx^2} > 0$.

Taking the derivative, we get:
$$\frac{dU(x)}{dx} = \frac{d}{dx}\left(-\frac{ax^2}{2} + \frac{bx^4}{4}\right) = -ax + bx^3 = 0$$

This has solutions: $\qquad x = 0 \qquad$ and $\qquad x = \pm\sqrt{\frac{a}{b}} = \pm\sqrt{\frac{5.0 \text{ N/m}}{2.0 \text{ N}/m^3}} = \pm 1.6 \text{ m}$

The second derivative is: $\qquad \frac{d^2U(x)}{dx^2} = \frac{d^2}{dx^2}\left(-\frac{ax^2}{2} + \frac{bx^4}{4}\right) = -a + 3bx^2$

This is negative for x=0, which is a local maximum, but is positive for $x = \pm\sqrt{a/b}$, which are minima with:

$$U_{min} = U\left(\pm\sqrt{a/b}\right) = -\frac{a\left(\sqrt{a/b}\right)^2}{2} + \frac{b\left(\sqrt{a/b}\right)^4}{4} = -\frac{a^2}{4b} = -\frac{(5.0 \text{ N/m})^2}{4(2.0 \text{ N}/m^3)} = -3.1 \text{ J}$$

There is real physical motion $K \geq 0$ for total energy $E \geq U_{min}$.

The turning points (where $\frac{dx}{dt} = 0$) can be found from the equation $U(x) = E$.

There are four solutions (two positive) for energies with $U_{min} < E < 0$ and two solutions (one positive) for $E > 0$. The equation $U(x) - E = 0$ is equivalent to: $\quad x^4 - \frac{2a}{b}x^2 - \frac{4E}{b} = 0$.

The quadratic formula gives: $\qquad x = \pm\sqrt{\frac{a}{b} \pm \sqrt{\left(\frac{a}{b}\right)^2 + 4\frac{E}{b}}} \qquad$ for $U_{min} < E < 0$,

and: $\qquad x = \pm\sqrt{\frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 + 4\frac{E}{b}}} \qquad$ for $E > 0$.

For the particular values given (E=-1.0 J), the positive turning points are:

$$x = \sqrt{\frac{5.0}{2.0} \pm \sqrt{\left(\frac{5.0}{2.0}\right)^2 + 4\frac{(-1.0)}{2.0}}} = 0.66\text{m} \quad \text{and} \quad 2.1\text{m}$$

as shown in the graph.