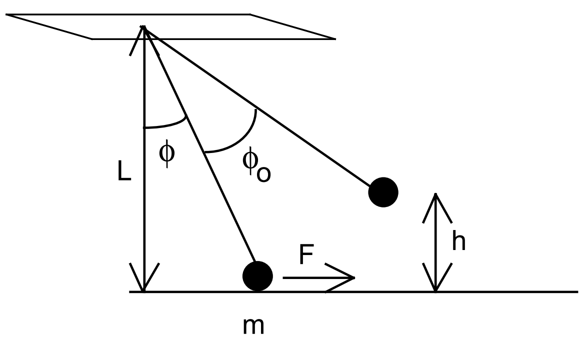
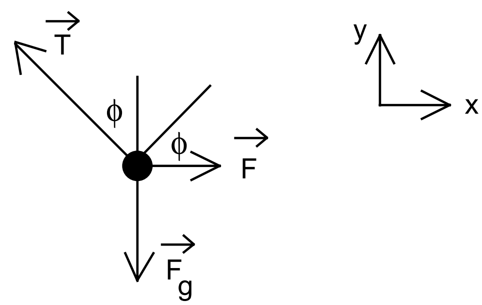
PHY 140Y – FOUNDATIONS OF PHYSICS
2001-2002
Tutorial Questions #7 – Solutions
October 29/30

Work, Energy, and Power

1. A particle of mass m is suspended from a massless string of length L. The particle is displaced along a circular path of radius L from $\phi=0$ to $\phi=\phi_o$, as shown below, by applying a force $\vec{F}$ that is always horizontal (for example by pulling horizontally with another string attached to the particle). The particle is thus displaced a vertical distance h. Assume that there is no acceleration, so that the motion is very slow.
(a) What is the magnitude F?
(b) What is the work done by the applied force as the mass moves from $\phi=0$ to $\phi=\phi_o$?
(c) What is the work done by the applied force as the mass moves from $\phi=0$ to $\phi=\phi_o$ if $\vec{F}$ is always directed along the arc rather than horizontally?





Answer:
(a) Draw the free-body diagram and choose a coordinate system as above (right).

Apply Newton's Second Law: $\vec{F}_{net} = \vec{F} + \vec{F}_g + \vec{T} = m\vec{a}$

x direction:
$$\vec{F}_{net,x} = F + 0 - T\sin\phi = ma_x = 0$$
$$\therefore F = T\sin\phi$$

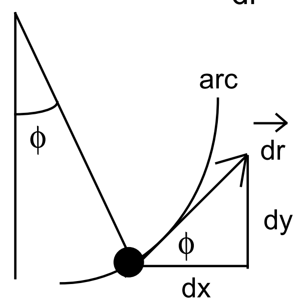
y direction:
$$\vec{F}_{net,y} = 0 - mg + T\cos\phi = ma_y = 0$$
$$\therefore T = \frac{mg}{\cos\phi}$$

Thus:
$$F = T\sin\phi = \frac{mg}{\cos\phi}\sin\phi = mg\tan\phi$$

So the applied force F varies with angle ϕ along the arc.

(b) The work done by force F is: $W = \int_{\phi=0}^{\phi=\phi_o} \vec{F} \bullet d\vec{r}$

Now, the displacement $d\vec{r}$ is always along the arc and so it depends on angle ϕ. The angle between $d\vec{r}$ and $\vec{F}$ is also just ϕ, with: $\sin\phi = \frac{dy}{dr}$ $\tan\phi = \frac{dy}{dx}$, as shown below.



$$W = \int_{\phi=0}^{\phi=\phi_o} \vec{F} \bullet d\vec{r} = \int_{r(0)}^{r(\phi_o)} F\cos\phi dr = \int_{r(0)}^{r(\phi_o)} (mg\tan\phi)\cos\phi dr$$

Therefore:
$$= \int_{r(0)}^{r(\phi_o)} mg\sin\phi dr = \int_{r(0)}^{r(\phi_o)} mg\left(\frac{dy}{dr}\right)dr$$

$$= mg\int_{y(0)}^{y(\phi_o)} dy = mg\int_0^h dy = mg[y]_0^h = mgh$$

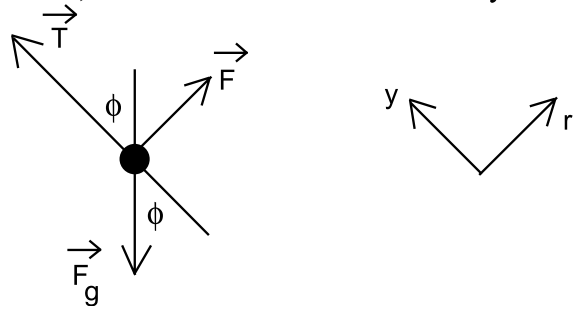
Alternatively:
$$W = \int_{r(\phi=0)}^{r(\phi=\phi_o)} \vec{F} \bullet d\vec{r} = \int_{x=0,y=0}^{x=(L-h)\tan\phi,y=h} (F_x dx + F_y dy) = \int_{x=0,y=0}^{x=(L-h)\tan\phi,y=h} (mg\tan\phi dx + 0)$$

$$= \int_{x=0,y=0}^{x=(L-h)\tan\phi,y=h} mg\left(\frac{dy}{dx}\right)dx = mg\int_0^h dy = mg\int_0^h dy = mgh$$

(c) If $\vec{F}$ is always directed along the arc, then the work done from $\phi=0$ to $\phi=\phi_o$ is:

$$W = \int_{\phi=0}^{\phi=\phi_o} \vec{F} \bullet d\vec{r} = \int_{\phi=0}^{\phi=0} Fdr$$ because $\vec{F} \parallel d\vec{r}$ now.

Reapply Newton's Second Law, but use a different coordinate system:



r direction:

$$\vec{F}_{net,r} = F - mg\sin\phi + 0 = ma_r = 0$$
$$\therefore F = mg\sin\phi$$

y direction:

$$\vec{F}_{net,y} = 0 - mg\cos\phi + T = ma_y = 0$$
$$\therefore T = mg\cos\phi$$

$$W = \int_{\phi=0}^{\phi=\phi_o} \vec{F}\bullet d\vec{r} = \int_{r(0)}^{r(\phi_o)} F dr = \int_{r(0)}^{r(\phi_o)} mg\sin\phi dr \quad \text{(same as above)} = ... = mgh$$

Note that both of these results for the work done by F in raising mass m vertically through height h are the same.

2. By measuring oxygen uptake, sports physiologists have found that the power output of long-distance runners is given approximately by P = m(bv-c), where m and v are the runner's mass and speed, respectively, and b and c are constants given by b = 4.27 J kg^{-1} m^{-1} and c = 1.83 W kg^{-1}.
(a) Determine the average power output and work done by a 65-kg runner who runs a 10-km race at a speed of 5.2 m/s.
(b) If the same runner starts at speed v_o = 4.8 m/s and accelerates to 6.1 m/s over a 25-s interval, what is the runner's power output as a function of time?
(c) How much work does the runner do during the acceleration period in part (b)?

Answer:

(a) The average power output of a 65-kg runner who runs a 10-km race at a speed of 5.2 m/s is:

$$\begin{aligned}\overline{P} &= m(bv - c)\\ &= 65kg(4.27J/kgm \times 5.2m/s - 1.83W/kg)\\ &= 1324 \text{ J/s} = 1.3 \text{ kW}\end{aligned}$$

The average work done this runner is:

$$\begin{aligned}\Delta W &= \overline{P}\Delta t = 1324J/s \times \frac{10\times10^3 m}{5.2m/s}\\ &= 1324 \text{ J/s} \times 1923s\\ &= 2.546 \text{ MJ} = 2.5 \text{ MJ}\end{aligned}$$

(b) Given v_o and v_f. Therefore we can calculate the (constant) rate of acceleration:

$$a = \frac{v_f - v_o}{\Delta t} = \frac{6.1m/s - 4.8m/s}{25s} = 0.052 \text{ m/s}^2$$

The velocity as a function of time is then: $v(t) = v_o + at$

Therefore, the power output as a function of time is:

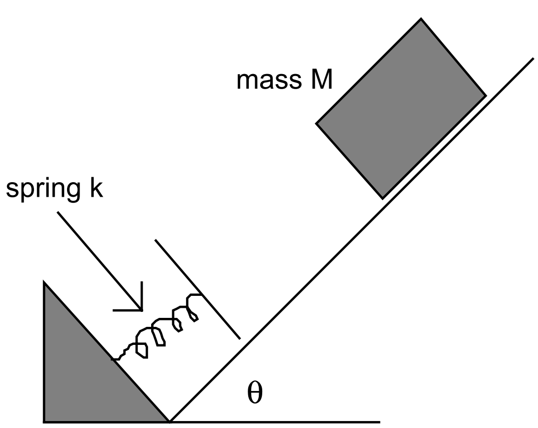
$$P(t) = m[bv(t) - c] = m[b(v_o + at) - c]$$

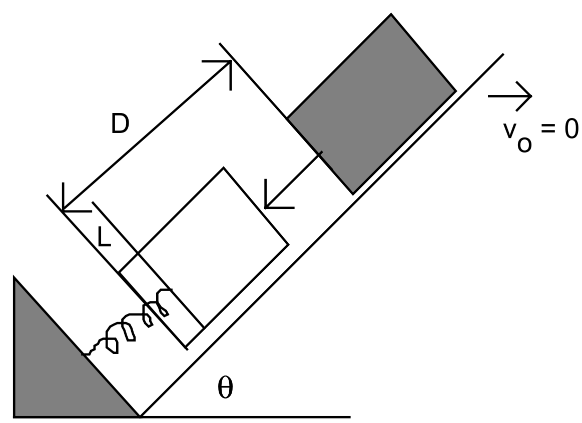
(c) The work done is:

$$
\begin{aligned}
W &= \int_o^{\Delta t} P(t)dt = \int_o^{\Delta t} m[b(v_o + at) - c]dt = m\int_o^{\Delta t}[bv_o + bat - c]dt \\
&= m\left[bv_o t + \frac{bat^2}{2} - ct\right]_0^{\Delta t} = m\left[bv_o\Delta t + \frac{b\Delta t^2}{2}\left(\frac{v_f - v_o}{\Delta t}\right) - c\Delta t\right] \\
&= m\Delta t\left[bv_o + \frac{b}{2}(v_f - v_o) - c\right] \\
&= m\Delta t\left[\frac{1}{2}b(v_f + v_o) - c\right] \\
&= 65\text{kg} \times 25\text{s}\left[\frac{1}{2}(4.27\text{J}/\text{kgm})(6.1\text{m}/\text{s} + 4.8\text{m}/\text{s}) - 1.83\text{W}/\text{kg}\right] \\
&= 35 \ \text{kJ}
\end{aligned}
$$

Conservation of Energy

3. A block of mass M is released from rest near the top of a frictionless incline, as shown below. The angle of the incline is θ. The block comes to rest momentarily after it has compressed a spring by a distance L. The spring constant is k.
(a) How far has the block moved down the incline when the spring is compressed by distance L?
(b) What is the speed of the block just as it touches the spring?
(c) What is the distance along the incline between the point of first contact and the point where the block's speed is the greatest?





Answer:
Treat the height of the block when the spring is compressed as the "zero" of potential energy.

(a) Apply Conservation of Energy: $U_i + K_i = U_f + K_f$

But $K_i = 0$ because mass M starts from rest and $K_f = 0$ because M comes to rest after sliding down the slope.

Now U_i = potential energy at the initial position (due to gravity): $U_i = mgD \sin\theta$
where D = distance that the block moves down the inclined plane.

Similarly, U_f = potential energy at the final position (due to the spring): $U_f = \frac{1}{2}kL^2$

Therefore:

$$U_i = U_f$$
$$mgD \sin\theta = \tfrac{1}{2}kL^2$$
$$D = \frac{kL^2}{2mg \sin\theta}$$

(b) As the block touches the spring: $\text{total energy} = U + K = mgL \sin\theta + \frac{1}{2}mv^2$

This must equal the total energy calculated in part (a).

$$mgL \sin\theta + \tfrac{1}{2}mv^2 = \tfrac{1}{2}kL^2$$
$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}kL^2 - mgL \sin\theta$$
$$v = \sqrt{\frac{kL^2}{m} - 2gL \sin\theta}$$

or

$$mgL \sin\theta + \tfrac{1}{2}mv^2 = mgD \sin\theta$$
$$\tfrac{1}{2}mv^2 = mgD \sin\theta - mgL \sin\theta$$
$$v = \sqrt{2g(D - L)\sin\theta}$$

(c) Let L_o = distance from the point of first contact to the point where the speed of the block is the greatest.

The total energy when the spring is compressed by L_o is:

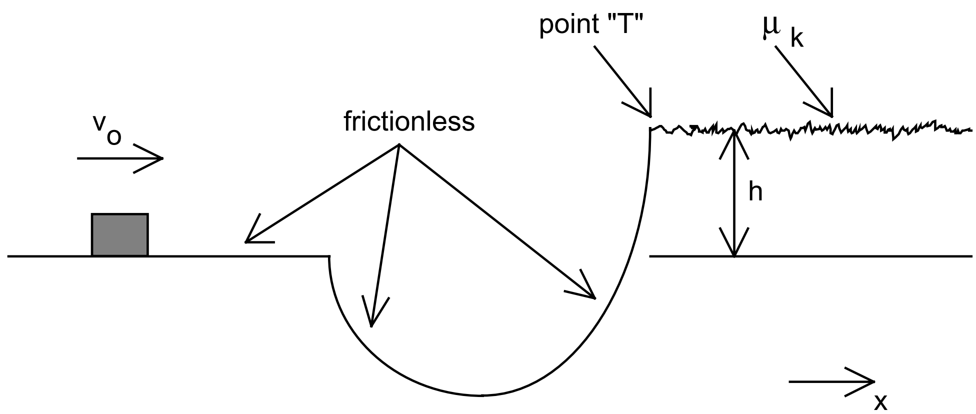
$$U + K = mg(L - L_o)\sin\theta + \tfrac{1}{2}kL_o^2 + \tfrac{1}{2}mv^2 = \tfrac{1}{2}kL^2$$
$$\therefore \tfrac{1}{2}mv^2 = \tfrac{1}{2}kL^2 - mg(L - L_o)\sin\theta - \tfrac{1}{2}kL_o^2$$

When the speed is a maximum, then: $\left.\frac{dv}{dL}\right|_{L=L_o} = 0$

Could solve for v first, or just use:

$$\frac{d}{dL}\left(\tfrac{1}{2}mv^2\right) = \frac{d}{dL}\left[\tfrac{1}{2}kL^2 - mg(L - L_o)\sin\theta - \tfrac{1}{2}kL_o^2\right] = 0$$
$$\therefore kL - mg \sin\theta = 0 \qquad \text{for} \quad L = L_o$$
$$L_o = \frac{mg \sin\theta}{k}$$

4. A block slides along a track from one level to a higher level by moving through an intermediate valley. The track is frictionless until the block reaches the higher level. At the higher level a friction force stops the block in a distance d. If the block's initial speed is V_o, the height difference is h, and the coefficient of kinetic friction is μ_k, what is d?



Answer:
Treat the lower (initial) level as the "zero" of potential energy.

At the initial point: $E_i = U_i + K_i = 0 + \frac{1}{2}mv_o^2 = \frac{1}{2}mv_o^2$

Call "T" the point at the start of the top level, where: $E_T = U_T + K_T = mgh + \frac{1}{2}mv_T^2$

Apply Conservation of Energy:

$$E_T = E_i$$
$$mgh + \frac{1}{2}mv_T^2 = \frac{1}{2}mv_o^2$$
$$v_T^2 = v_o^2 - 2gh$$

Next, we need to apply Newton's Second Law on the upper level (just consider the x component).

$$F_{net,x} = F_k = -\mu_k N$$
$$\therefore ma_x = -\mu_k N = -\mu_k mg$$
$$a_x = -\mu_k g$$

$$v_x(t) = v_{ox} + a_x(t - t_o)$$
$$= v_T - \mu_k gt$$

where $v_x = v_T$ at $t = t_o = 0$

$$x(t) = x_o + v_{ox}(t - t_o) + \frac{1}{2}a_x(t - t_o)^2$$
$$= v_T t - \frac{1}{2}\mu_k gt^2$$

where $x = x_o = 0$ at $t = t_o = 0$

Define d as the distance travelled before the block comes to rest, say at t = t'.

$$v_x(t') = v_T - \mu_k g t' = 0$$

$$\therefore\ t' = \frac{v_T}{\mu_k g}$$

$$d = x(t') = v_T t' - \tfrac{1}{2}\mu_k g t'^2 = v_T\left(\frac{v_T}{\mu_k g}\right) - \tfrac{1}{2}\mu_k g\left(\frac{v_T}{\mu_k g}\right)^2$$

$$= \frac{v_T^{\,2}}{\mu_k g} - \tfrac{1}{2}\frac{v_T^{\,2}}{\mu_k g} = \frac{v_T^{\,2}}{2\mu_k g}$$

$$= \frac{1}{2\mu_k g}\left(v_o^{\,2} - 2gh\right) = \frac{v_o^{\,2}}{2\mu_k g} - \frac{h}{\mu_k}$$