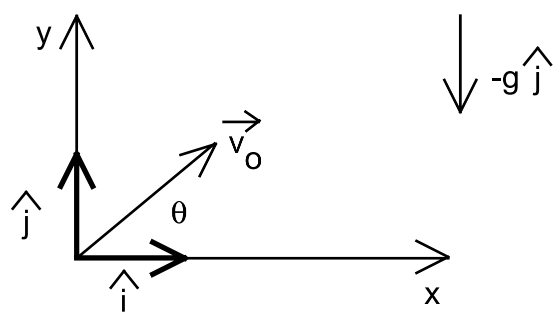
PHY 140Y - FOUNDATIONS OF PHYSICS
2001-2002
Tutorial Questions #3 - Solutions
October 1/2

Motion in More than One Dimension and Projectile Motion

1. A projectile is launched over flat ground. At what angles with respect to the ground should the launcher be oriented so that the projectile's range is half its maximum range? Neglect air resistance.

Answer:

First define the coordinate system.



Next, define the initial conditions.

$t_o = 0$ $\quad x_o = 0$ $\quad y_o = 0$ $\quad \vec{v}(t_o) = \vec{v}_o$

Consider the x and y components separately.

x direction:
$$a_x = 0$$
$$v_x(t) = v_{ox} + a_x(t - t_o) = v_o \cos\theta$$
$$x(t) = x_o + v_{ox}(t - t_o) + \tfrac{1}{2}a_x(t - t_o)^2 = v_o t \cos\theta$$

y direction:
$$a_y = -g$$
$$v_y(t) = v_{oy} + a_y(t - t_o) = v_o \sin\theta - gt$$
$$y(t) = y_o + v_{oy}(t - t_o) + \tfrac{1}{2}a_y(t - t_o)^2 = v_o t \sin\theta - \tfrac{1}{2}gt^2$$

In order to determine the range, we need to find $x(t_R)$ for which $y(t_R) = 0$.

$$y(t_R) = v_o t_R \sin\theta - \tfrac{1}{2} g t_R^2 = 0$$

Thus:

$$t_R = \frac{2v_o}{g}\sin\theta$$

The range is then;

$$x(t_R) = v_o t_R \cos\theta = v_o\left(\frac{2v_o}{g}\sin\theta\right)\cos\theta = \frac{2v_o^2}{g}\sin\theta\cos\theta$$

$$= \frac{v_o^2}{g}\sin 2\theta$$

The maximum range is $x(t_R)$ for which $\sin 2\theta$ is a maximum, i.e., for which:

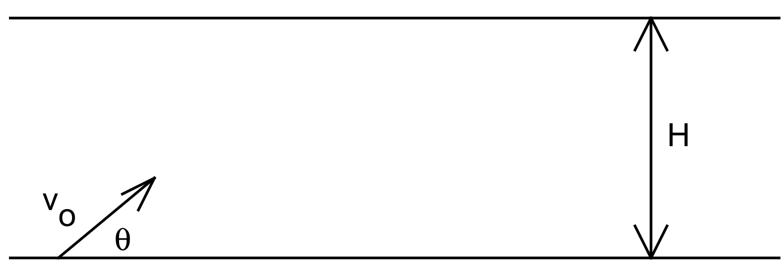
$$\sin 2\theta = 1$$
$$2\theta = 90^\circ$$
$$\theta = 45^\circ$$

and so: $$x_{max}(t_R) = \frac{v_o^2}{g}\sin(2\times 45^\circ) = \frac{v_o^2}{g}$$

Now, we want to find the angle(s) θ that will give: $x(t_R) = \tfrac{1}{2} x_{max}(t_R)$

$$x(t_R) = \tfrac{1}{2} x_{max}(t_R)$$
$$\frac{v_o^2}{g}\sin 2\theta = \frac{v_o^2}{2g}$$
$$\sin 2\theta = \tfrac{1}{2}$$
$$2\theta = 30^\circ \quad \text{or} \quad 150^\circ$$
$$\theta = 15^\circ \quad \text{or} \quad 75^\circ$$

2. A ball is shot with an initial speed v_o from the floor of a large gymnasium of height H, as shown below. What is the maximum horizontal distance it can travel (in terms of H and v_o) without touching the roof? Derive ALL the relevant equations. Neglect air resistance.



Answer:

The coordinate system and initial conditions are the same as those in Question 1.

$t_o = 0$ $\quad x_o = 0$ $\quad y_o = 0$ $\quad \vec{v}(t_o) = \vec{v}_o$

So we can also use the equations for x(t) and y(t) derived in Question 1.

$x(t) = v_o t \cos\theta$ and $y(t) = v_o t \sin\theta - \frac{1}{2} g t^2$

First, we need to derive an expression for the maximum height.

The maximum height will occur at time t', when: $\dfrac{dy(t')}{dt} = 0$

$$\frac{d}{dt}\left(v_o t' \sin\theta - \tfrac{1}{2} g t'^2\right) = 0$$

Therefore: $v_o \sin\theta - g t' = 0$

$$t' = \frac{v_o}{g}\sin\theta$$

The maximum height, $y_{max} = H$, is then:

$$\begin{aligned} y_{max} &= y(t') = v_o t' \sin\theta - \tfrac{1}{2} g t'^2 \\ &= v_o\left(\frac{v_o}{g}\sin\theta\right)\sin\theta - \tfrac{1}{2} g\left(\frac{v_o}{g}\sin\theta\right)^2 \\ &= \frac{v_o^{\,2}}{g}\sin^2\theta - \frac{1}{2}\frac{v_o^{\,2}}{g}\sin^2\theta \\ &= \frac{v_o^{\,2}}{2g}\sin^2\theta \end{aligned}$$

We can rearrange this to get an expression for angle θ in terms of H:

$$\frac{v_o^{\,2}}{2g}\sin^2\theta = H$$

$$\sin\theta = \frac{\sqrt{2gH}}{v_o}$$

and hence

$$\begin{aligned} \cos\theta &= \sqrt{1 - \sin^2\theta} \\ &= \sqrt{1 - \frac{2gH}{v_o^{\,2}}} \\ &= \frac{1}{v_o}\sqrt{v_o^{\,2} - 2gH} \end{aligned}$$

From Question #1, we know that the equation for the range is

$$x(t_R) = \frac{v_o^{\,2}}{g}\sin 2\theta = \frac{2v_o^{\,2}}{g}\sin\theta\cos\theta$$

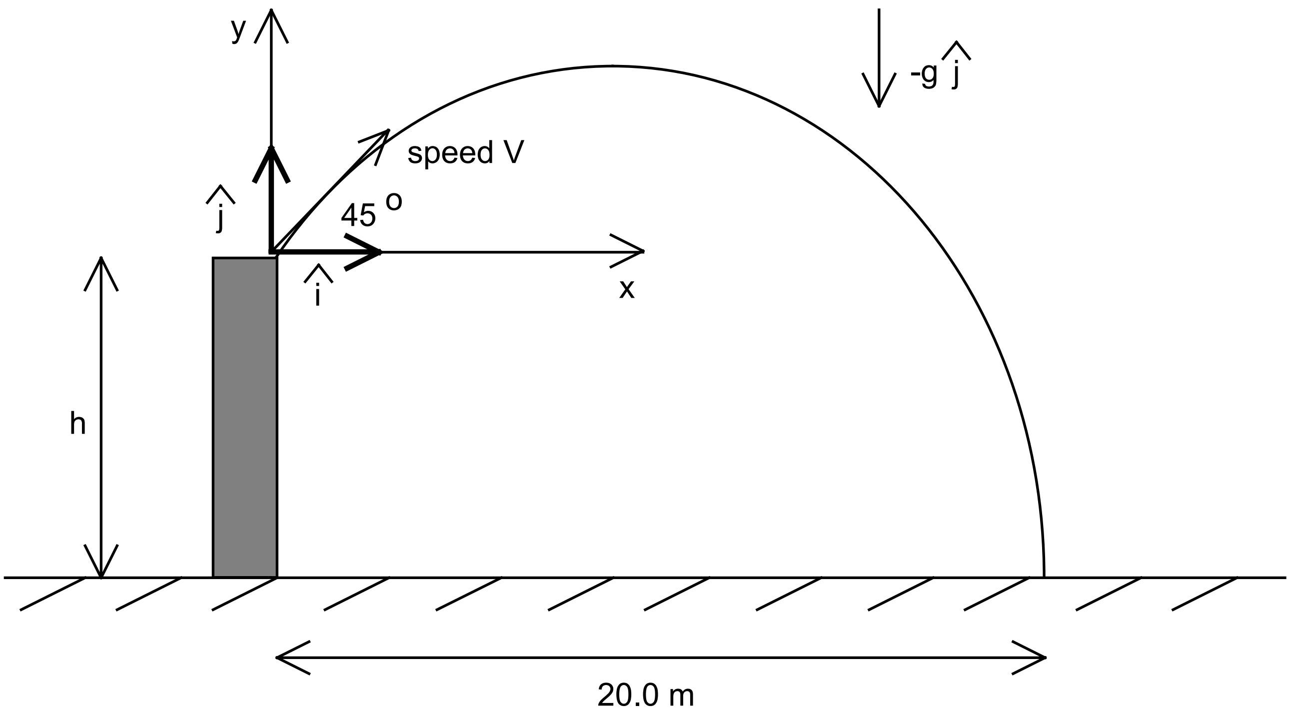
And so the maximum horizontal distance that can be travelled without hitting the roof is:

$$x(t_R) = \frac{2}{g}\sqrt{2gH\left(v_o^{\,2} - 2gH\right)}$$

3. A person throws a rock from the top of a tower at speed V and at an angle of 45° up from the horizontal. The rock is in flight for 4.00 seconds and hits the ground 20.0 m from the base of the building. Neglect air resistance.
(a) What is the speed V?
(b) How high off the ground is the top of the tower?
(c) What is the speed of the rock just before it hits the ground?

Answer:

First, sketch the geometry.



Set up the equations of motion for the x and y components.

x direction:

$$a_x = 0$$
$$v_x(t) = v_{ox} + a_x(t - t_o) = V\cos 45^\circ = V/\sqrt{2}$$
$$x(t) = x_o + v_{ox}(t - t_o) + \tfrac{1}{2}a_x(t - t_o)^2 = Vt/\sqrt{2}$$

y direction:

$$a_y = -g$$
$$v_y(t) = v_{oy} + a_y(t - t_o) = V\sin 45^\circ - gt = V/\sqrt{2} - gt$$
$$y(t) = y_o + v_{oy}(t - t_o) + \tfrac{1}{2}a_y(t - t_o)^2 = Vt/\sqrt{2} - \tfrac{1}{2}gt^2$$

(a) Given that x = 20.0 m at t = 4.00 s, we can say that:

$$x(t) = Vt/\sqrt{2}$$
$$20.0\text{m} = V(4.00\text{s})/\sqrt{2}$$
$$V = 20.0\sqrt{2} / 4.00 \quad \text{m / s}$$
$$= 5.00\sqrt{2} \quad \text{m / s}$$
$$= 7.07 \quad \text{m / s}$$

(b) We also know that y = -h at t = 4.00 s, so:

$$y(t) = Vt/\sqrt{2} - \tfrac{1}{2}gt^2$$
$$-h = \frac{(5.00\sqrt{2}\text{m / s})(4.00\text{s})}{\sqrt{2}} - \frac{(9.81\text{m / s}^2)(4.00\text{s})^2}{2}$$
$$h = 58.4 \quad \text{m}$$

(c) When the rock hits the ground at t = 4.00 s:

$$v_x(t) = V/\sqrt{2} = (5.00\sqrt{2}\text{m / s})/\sqrt{2} = 5.00 \quad \text{m / s}$$

and

$$v_y(t) = V/\sqrt{2} - gt = (5.00\sqrt{2}\text{m / s})/\sqrt{2} - (9.81\text{m / s}^2)(4.00\text{s}) = -34.2 \quad \text{m / s}$$

So the speed of the rock just before it hits the ground is:

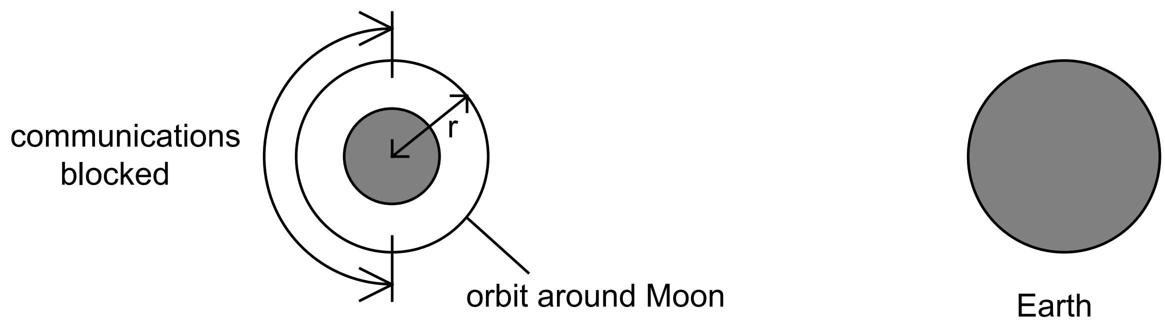
$$\text{speed} = \sqrt{v_x{}^2 + v_y{}^2} = \sqrt{(5.00)^2 + (-34.2)^2} \quad \text{m / s} = 34.6 \quad \text{m / s}$$

Uniform Circular Motion

4. When Apollo astronauts landed on the Moon, they left one astronaut behind in a circular orbit around the Moon. For the half of the orbit spent over the far side of the Moon, that individual was completely cut off from communication with the rest of humanity. How long did this lonely state last? Assume a sufficiently low orbit that you can use the Moon's surface gravitational acceleration (1.62 m/s^2) for the spacecraft; say the radius of the orbit is 1.74×10^6 m.

Answer:

Given: orbital radius r = 1.74×10^6 m
lunar gravity a = 1.62 m/s^2



This is an example of uniform circular motion for which: $a = \dfrac{v^2}{r}$

So we can use: $v = \sqrt{ar}$.

The period for one orbit is: $T = \dfrac{2\pi r}{v}$

So this can be evaluated as: $T = \dfrac{2\pi r}{\sqrt{ar}} = 2\pi\sqrt{\dfrac{r}{a}} = 2\pi\sqrt{\dfrac{1.74 \times 10^6 m}{1.62 m / s^2}} = 6510 \quad s$

Communications will be blocked for one-half of the period, i.e., for 3260 seconds or 54.3 minutes.

5. How long would a day last if the Earth were rotating so fast that the acceleration of an object on the equator were equal to g?

Answer:

From Question #4, we know that Earth's period of rotation can be expressed as $T = 2\pi\sqrt{\dfrac{r}{a}}$.

In this case: Earth's equatorial radius $R_E = r = 6378$ km $= 6.378 \times 10^6$ m
centripetal acceleration $a_c = g = 9.81$ m/s^2

Thus, the period is: $T = 2\pi\sqrt{\dfrac{r}{a}} = 2\pi\sqrt{\dfrac{6.378 \times 10^6 m}{9.81 m / s^2}} = 5070 \quad s$

which is equivalent to 84.4 minutes or 1 hour and 24.4 minutes. Why isn't this 24 hours?! Compare with Example 4-8 in the textbook.