PHY 140Y - FOUNDATIONS OF PHYSICS
2001-2002
Solutions for Tutorial Questions #1
September 17/18

Units and Dimensional Analysis

1. The metre is now defined as the length of the path travelled by light in vacuum during a time interval of 1/299,792,458 of a second.
(a) What is the speed of light in m/sec?
(b) What is the speed of light in km/hour?
(c) What is the speed of light in miles/hour?

Answer
Recall that 1 metre = distance travelled in 1/299,792,458 seconds. Use the Chain Rule to do the unit conversion.

(a) $c = \dfrac{1 \quad m}{1/299{,}792{,}458 \quad s} = 299{,}792{,}458 \quad m/s$

(b) $c = 299{,}792{,}458 \quad \dfrac{m}{s} \times \dfrac{60 \quad s}{min} \times \dfrac{60 \quad min}{hour} \times \dfrac{1 \quad km}{1000 \quad m} = 1{,}079{,}252{,}849 \quad km/hr$

(c) $c = 1{,}079{,}252{,}849 \quad km/hr \times \dfrac{1 \quad mile}{1.609 \quad km} = 670{,}616{,}629 \quad miles/hr$

2. The speed of waves in deep water depends only on the gravitational acceleration g, which has dimensions L/T^2, and on the wavelength λ, which has dimension L. Which of the following could be the correct formula for the wave speed?
(a) $v = \sqrt{g/\lambda}$ (b) $v = \lambda g^2$ (c) $v = \sqrt{\lambda g}$

Answer
Apply dimensional analysis to each of the three possibilities.

(a) $\sqrt{\dfrac{g}{\lambda}}$ has dimensions of $\sqrt{\dfrac{L/T^2}{L}} = \dfrac{1}{T}$

(b) λg^2 has dimensions of $L\left(\dfrac{L}{T^2}\right)^2 = \dfrac{L^3}{T^4}$

(c) $\sqrt{\lambda g}$ has dimensions of $\sqrt{L\dfrac{L}{T^2}} = \dfrac{L}{T}$

Since speed has dimensions of L/T, option (c) <u>could</u> be the correct formula.

3. Energy has dimensions ML^2/T^2. The energy U stored in a stretched spring is given by an equation of the form $U = \frac{1}{2}kx^\alpha$, where k is a constant with the dimensions M/T^2, and x is the stretch. What should be the value of the exponent α?

Answer

Apply dimensional analysis to the equation $U = \frac{1}{2}kx^\alpha$.

LHS "U" has dimensions $\frac{ML^2}{T^2}$ (given). RHS $\frac{1}{2}kx^\alpha$ has dimensions $\frac{ML^\alpha}{T^2}$. Thus $\alpha = 2$ for consistency of the dimensions.

4. The number 3.6 has two significant figures, and represents a value that is closer to 3.6 than to 3.5 or 3.7. Thus, the actual value lies between 3.55 and 3.65. Show that the percentage uncertainty implied by two significant figure accuracy varies with the value of the number, being smaller for numbers beginning with 9 and larger for numbers beginning with 1. What is the percentage uncertainty implied by the following numbers?
(a) 1.1 (b) 5.0 (c) 9.9

Answer

Since 3.6 lies between 3.55 and 3.65, the % uncertainty in this number is

$$\frac{0.05(0)}{3.6} \times 100\% = 1.4\%.$$

For the general case of a two-digit number, N = a.b, N lies between (a.b – 0.050) and (a.b + 0.050) (or between N – 0.050 and N + 0.050). The % uncertainty in N is

$$\frac{0.050}{a.b} \times 100\% = \frac{5.0}{a.b}\% = \frac{5.0}{N}\%.$$

Thus, the uncertainty will be smaller for large N, and larger for small N.

e.g., Say b = 0 and a = 1 and 9. Then for N = 1.0, the uncertainty will be 5/1.0 = 5 %, while for N = 9.0, the uncertainty will be 5/9.0 = 0.6%

(a) Uncertainty = $\frac{0.050}{1.1} \times 100\% = 4.5\%$

(b) Uncertainty = $\frac{0.050}{5.0} \times 100\% = 1.0\%$

(c) Uncertainty = $\frac{0.050}{9.9} \times 100\% = 0.51\%$

5. (a) Compare the values obtained for the following expressions by maintaining extra significant figures until the end and by rounding off at the intermediate step:
$(17.8 + 0.06) \times 4.93$ and $(17.8 \times 4.93) + (0.06 \times 4.93)$.
(b) Do the same for the following expression. How many significant figures should be maintained in the final answer? $(0.9 + 0.06) \times 6.71$

Answer
Recall the rules for dealing with significant figures:
(1) When multiplying or dividing, the answer should have the same number of significant figures as the least accurate of the quantities entering the calculation.
(2) When adding or subtracting, the number of digits to the right of the decimal point should equal that of the term in the sum or difference that has the smallest number of digits to the right of the decimal point.
But also avoid rounding off to the correct number of significant figures until the final answer has been calculated (this is what your calculator will do).

(a) $(17.8 + 0.06) \times 4.93$
$= 17.8(6) \times 4.93 = 88.0498... = 88.0$ right - keeping extra significant figures until the end
$= 17.9 \times 4.93 = 88.2470... = 88.2$ wrong - rounding off before the end

$(17.8 \times 4.93) + (0.06 \times 4.93)$
$= 87.7(540) + 0.2(958) = 88.0498... = 88.0$ right (again!) - keeping extra significant figures until the end
$= 87.8 + 0.3 = 88.1$ wrong (again) - rounding off before the end

(b) $(0.9 + 0.06) \times 6.71$
$= 0.9(6) \times 6.71 = 6.4416... = 6$ right - keeping extra significant figures until the end
$= 1 \times 6.71 = 6.71 = 7$ wrong - rounding off before the end
Should only maintain one significant figure in the final answer.

Scalars and Vectors

6. Two vectors are given by $\vec{a} = 4\hat{i} - 3\hat{j}$ and $\vec{b} = -\hat{i} + \hat{j} + 4\hat{k}$. Find
   (a) $\vec{a} + \vec{b}$
   (b) $\vec{a} - \vec{b}$
   (c) the magnitudes of $\vec{a}$ and $\vec{b}$, and the direction of $\vec{a}$ with respect to the x axis; sketch $\vec{a}$ and $\vec{b}$ in an x-y-z co-ordinate system
   (d) a vector $\vec{c}$ such that $\vec{a} - 3\vec{b} + 2\vec{c} = 0$.

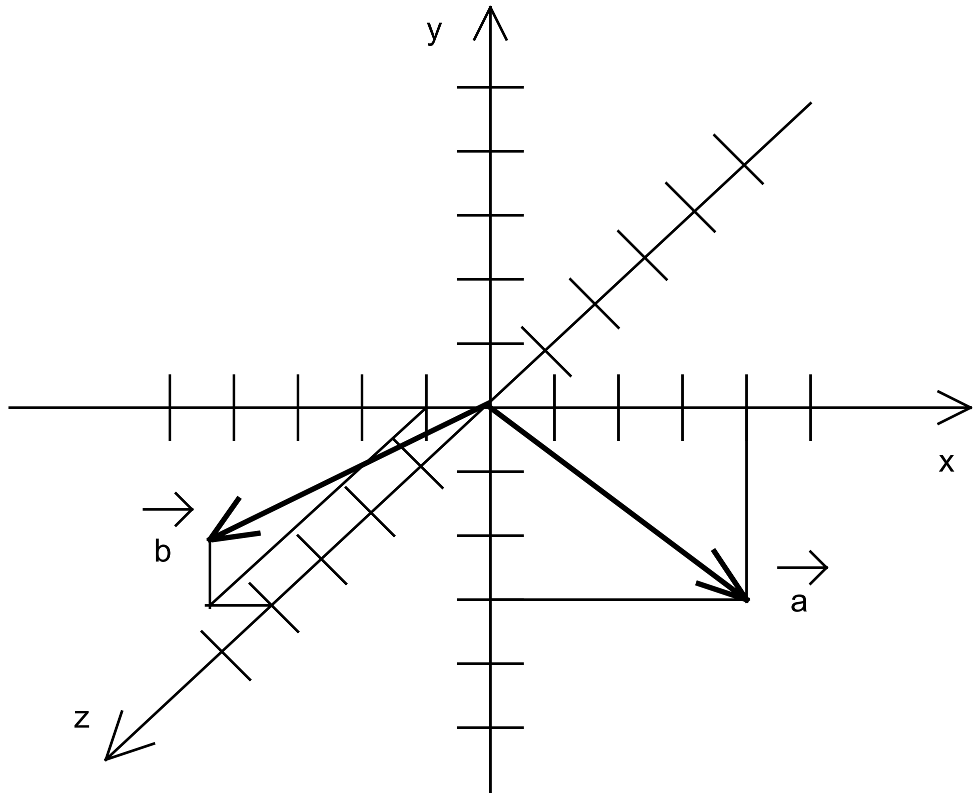
Answer
(a) $\vec{a} + \vec{b} = (4\hat{i} - 3\hat{j}) + (-\hat{i} + \hat{j} + 4\hat{k}) = (4 - 1)\hat{i} + (-3 + 1)\hat{j} + 4\hat{k} = 3\hat{i} - 2\hat{j} + 4\hat{k}$
(b) $\vec{a} - \vec{b} = (4\hat{i} - 3\hat{j}) - (-\hat{i} + \hat{j} + 4\hat{k}) = (4 + 1)\hat{i} + (-3 - 1)\hat{j} - 4\hat{k} = 5\hat{i} - 4\hat{j} - 4\hat{k}$
(c) The magnitudes of the two vectors are given by:

$$|\vec{a}| = \sqrt{4^2 + (-3)^2 + 0^2} = \sqrt{25} = 5$$

$$|\vec{b}| = \sqrt{(-1)^2 + 1^2 + 4^2} = \sqrt{18} = 4.24$$

The direction of $\vec{a}$ with respect to the x axis can be determined from the angle θ that the 2-D vector makes with the x axis, where $\tan\theta = a_y / a_x$:

$$\theta = \arctan\left(\frac{a_y}{a_x}\right) = \arctan\left(\frac{-3}{4}\right) = -36.9^\circ$$

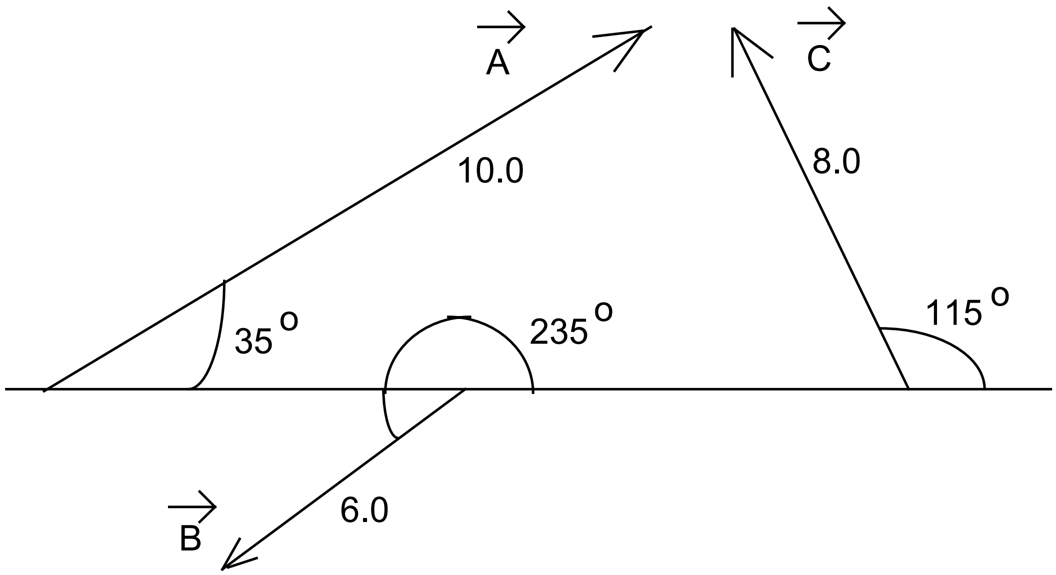


(d) Want $\vec{a} - 3\vec{b} + 2\vec{c} = 0$.

$$\begin{aligned} 2\vec{c} &= 3\vec{b} - \vec{a} \\ &= 3(-\hat{i} + \hat{j} + 4\hat{k}) - (4\hat{i} - 3\hat{j}) \\ &= (-3 - 4)\hat{i} + (3 + 3)\hat{j} + 12\hat{k} \\ &= -7\hat{i} + 6\hat{j} + 12\hat{k} \end{aligned}$$

and so $\vec{c} = -3.5\hat{i} + 3\hat{j} + 6k$

7. Given the vectors shown in the diagram below, calculate the x and y components, the magnitude, and the angle with respect to the x axis of the following combinations:
   (a) $\vec{A} + \vec{B}$
   (b) $\vec{B} - \vec{C}$
   (c) $1.5\vec{A} + 3.0\vec{B} - 1.7\vec{C}$.



Answer

First calculate the x and y components of each vector:

$\vec{A} = (10.0\cos 35^\circ)\hat{i} + (10.0\sin 35^\circ)\hat{j} = 8.19\hat{i} + 5.74\hat{j}$

$\vec{B} = (6.00\cos 235^\circ)\hat{i} + (6.00\sin 235^\circ)\hat{j} = -3.44\hat{i} - 4.91\hat{j}$

$\vec{C} = (8.00\cos 115^\circ)\hat{i} + (8.00\sin 115^\circ)\hat{j} = -3.38\hat{i} + 7.25\hat{j}$

(a) $\vec{A} + \vec{B} = (8.19 - 3.44)\hat{i} + (5.74 - 4.91)\hat{j} = 4.75\hat{i} + 0.83\hat{j}$

magnitude = $\left|\vec{A} + \vec{B}\right| = \sqrt{4.75^2 + 0.83^2} = 4.82$

$\theta = \arctan\left(\frac{0.83}{4.75}\right) = 9.91^\circ$

(b) $\vec{B} - \vec{C} = (-3.44 + 3.38)\hat{i} + (-4.91 - 7.25)\hat{j} = -0.06\hat{i} - 12.16\hat{j}$

magnitude = $\left|\vec{B} - \vec{C}\right| = \sqrt{(-0.06)^2 + (-12.16)^2} = 12.16$

$\theta = \arctan\left(\frac{-12.16}{-0.06}\right) = 89.7^\circ$ but must be in the –x/-y quadrant, so really = 269.7°

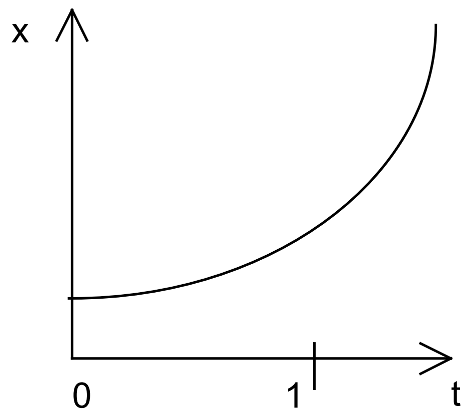
$$\begin{aligned}
&1.5\vec{A} + 3.0\vec{B} - 1.7\vec{C} \\
\text{(c)} &= [1.5(8.19) + 3.0(-3.44) - 1.7(-3.3)]\hat{i} + [1.5(5.74) + 3.0(-4.91) - 1.7(-7.25)]\hat{j} \\
&= 7.71\hat{i} - 18.4\hat{j}
\end{aligned}$$

magnitude = $\left|1.5\vec{A} + 3.0\vec{B} - 1.7\vec{C}\right| = \sqrt{(7.71)^2 + (-18.44)^2} = 20.0$

$\theta = \arctan\left(\frac{-18.4}{-7.71}\right) = 67.3^\circ$

Differentiation and Integration

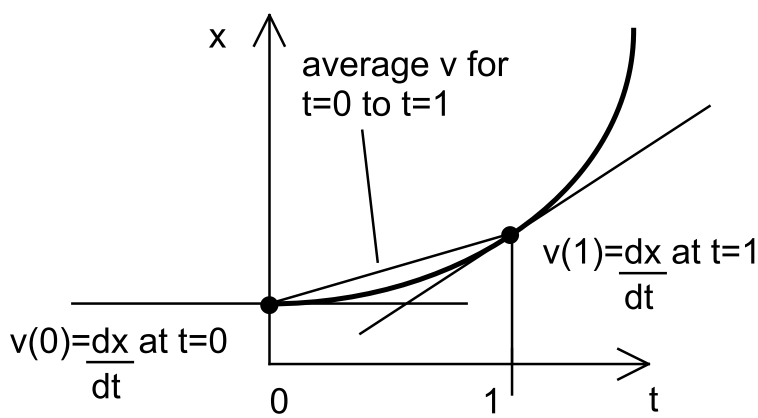
8. The position of an object is given as a function of time t: $x = a + bt + ct^2 + dt^3$ where a, b, c, and d are constants. What is the speed, v, of the object as a function of time? A portion of the curve x vs. t is shown below. Sketch the geometrical representation of the average velocity for the interval 0 to 1, v(0), and v(1).



Answer

The speed is the derivative of the position:

$$v = \frac{d}{dx}[x(t)] = \frac{d}{dx}\left[a + bt + ct^2 + dt^3\right] = 0 + b + 2ct + 3dt^2$$

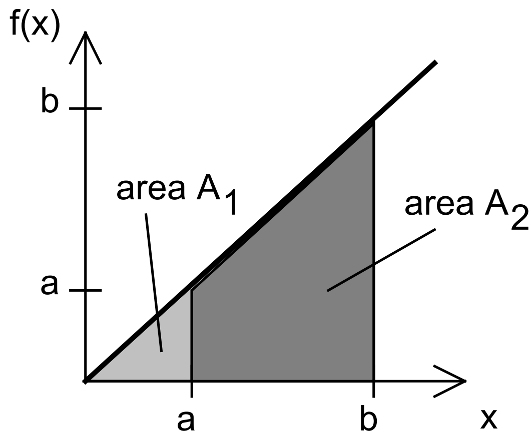


9. Given a function f(x) = x, what is the value of the definite integral $\int_a^b f(x)dx$?

Confirm this geometrically.

Answer

$$\int_a^b f(x)dx = \int_a^b x dx = \left.\frac{x}{2}\right|_a^b = \tfrac{1}{2}(b^2 - a^2)$$



Geometrically, the integral = area A_2 – area A_1 = $\frac{1}{2}bb - \frac{1}{2}aa = \frac{1}{2}(b^2 - a^2)$

10. Given a function $f(x) = 3x^2 + 7e^{4x}$. Derive the expression for the indefinite integral $g(x) = \int f(x)dx$. If g(0) = 1.5, then what is the expression for g(x)?

Answer

$$g(x) = \int f(x)dx = \int\left(3x^2 + 7e^{4x}\right)dx = 3\frac{x^3}{3} + 7\frac{e^{4x}}{4} + C$$

Given the initial condition g(0) = 1.5, we have:

$$g(0) = 3\frac{0^3}{3} + 7\frac{e^0}{4} + C = 1.5$$

$$C = 1.5 - \frac{7}{4} = -0.25$$

So $g(x) = 3\frac{x^3}{3} + \frac{7}{4}e^{4x} - 0.25$.

11. What are (a) the derivatives $\frac{df(x)}{dx}$, and (b) the indefinite integrals $\int f(x)dx$, of the following common functions (assume that "a" is a constant)?
$f(x)=a$, $f(x)=a\,x^n$, $f(x)=\sin(x)$, $f(x)=\cos(x)$, $f(x)=\tan(x)$, $f(x)=e^{ax}$, $f(x)=\ln(ax)$, $f(x)=1/x$, $f(x)=1/(x^2 + a^2)$, and $f(x)=x/(x^2 \pm a^2)^{0.5}$.

Answer
See Appendix 2 (pages A-6 to A-11) of Wolfson and Pasachoff for rules used in differentiating and integrating functions, and for short tables of derivatives and integrals. Longer tables can be found in calculus textbooks.

$f(x)$	$\dfrac{df(x)}{dx}$	$\int f(x)dx$
a	0	$a + C$
$a\,x^n$	$a\,n\,x^{n-1}$	$(a\,x^{n+1}) / (n+1) + C \; ; n \neq -1$
$\sin(x)$	$\cos(x)$	$-\cos(x) + C$
$\cos(ax)$	$-a\sin(x)$	$(1/a)\sin(ax) + C$
$\tan(x)$	$1 / \cos^2(x)$	$-\ln(\cos x) + C$
e^{ax}	$a\,e^{ax}$	$e^{ax} / a + C$
$\ln(ax)$	a/x	$x\ln(ax) - x + C$
$1/x$	$-1/x^2$	$\ln(x) + C$
$1 / (x^2 + a^2)$	$-2x / (x^2 + a^2)^2$	$(1/a)\tan^{-1}(x/a) + C$
$x / (x^2 \pm a^2)^{0.5}$	$1 / (x^2 \pm a^2)^{0.5} - x^2 / (x^2 \pm a^2)^{1.5}$	$(x^2 \pm a^2)^{0.5} + C$