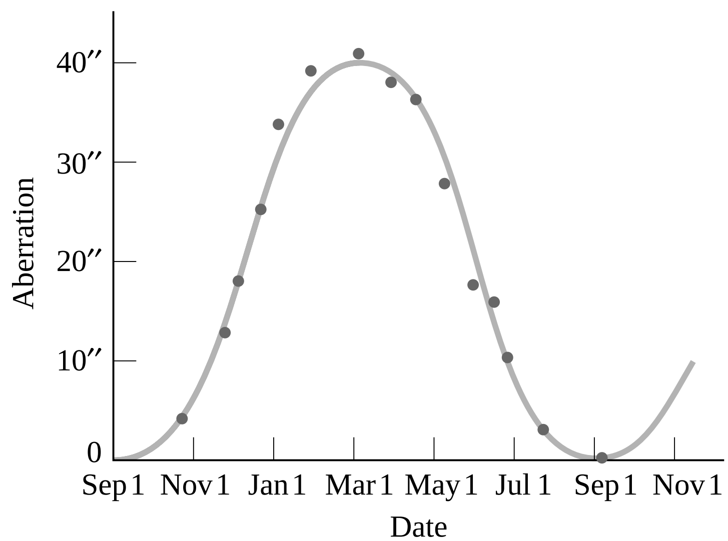
PHY 140Y – FOUNDATIONS OF PHYSICS
2001-2002
Tutorial Questions #11 – Solutions
November 26/27

Aberration of Starlight and the Michelson-Morley Experiment

1. Figure 38-30, below, shows a plot of James Bradley's data on the aberration of light from the star γ Draconis, recorded in 1727-1728.
(a) From the data, determine the magnitude of Earth's orbital velocity.
(b) The data very nearly fit a perfect sine curve. What does this say about the shape of Earth's orbit?



Solution:

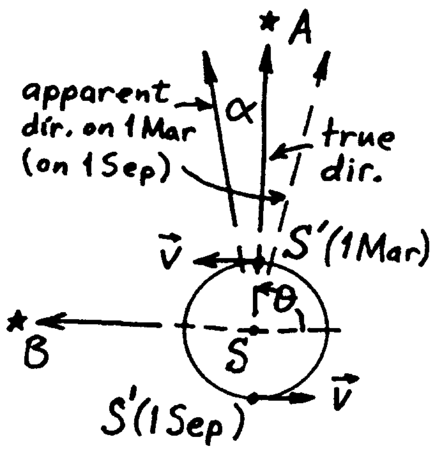
(a) The angle of aberration, α, is defined in the reference frame of the Sun, say S, whose motion we need not consider. Since v/c << 1 for the orbital speed of the Earth, we can use non-relativistic expressions, such as Equation 3-10, which says that the apparent velocity of starlight in the Earth's frame, S', is the vector difference of its velocity in S and the orbital velocity, or $\vec{c}' = \vec{c} - \vec{v}$.

The shift in the position of distant stars (with no measurable parallax) observed perpendicular to $\vec{v}$ (say on March 1), relative to stars seen parallel to $\vec{v}$, is α, which reverses after six months.

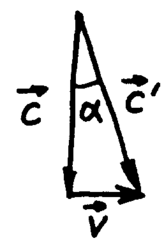
The maximum annual difference, $\alpha - (-\alpha) = 2\alpha$, is about 40" as shown in the figure below.
From the vector diagram: $v / c = \tan\alpha \approx \alpha = 20''$

Therefore: $v = c \times 20'' = (3.00 \times 10^5 \text{ km/s}) \times 20'' \times (1 \text{ rad} / 206{,}265'') = 29 \text{ km/s}$

(b) If the Earth's orbit were circular, the component of its velocity perpendicular to the direction of stars at A would vary as $v \sin \theta$ over a year ($\theta = 90°$ on March, as shown). Therefore, the aberration would vary as $\alpha = (v/c) \sin \theta$.



Problem 3 Solution. (1)



Problem 3 Solution. (2)

2. What would be the difference in light travel times in the two arms of the Michelson-Morley experiment if the ether existed and if Earth moved relative to it at:
(a) its orbital speed relative to the Sun (see Appendix E)?
(b) 0.01 c?
(c) 0.5 c?
(d) 0.99 c? (Not 0.9 as in hand-out.)
Assume that each light path is exactly 11 m in length, and that the paths are oriented parallel and perpendicular to the ether wind.

Solution:

The difference between the travel times in each arm of the interferogram is

$$\Delta t = t_{\parallel} - t_{\perp} = \frac{2cL}{c^2 - v^2} - \frac{2L}{\sqrt{c^2 - v^2}}$$

$$= \frac{2L}{c}\left(\frac{1}{1 - v^2/c^2} - \frac{1}{\sqrt{1 - v^2/c^2}}\right)$$

For the given interferometer: $$\frac{2L}{c} = \frac{2(11 \text{ m})}{3.00 \times 10^8 \text{ m/s}} = 73.3 \text{ ns}$$

If $v^2/c^2 << 1$, then we can expand the denominators (Appendix A) to obtain

$$\Delta t \approx \frac{2L}{c}\left(1 + \frac{v^2}{c^2} - \left[1 + \frac{v^2}{2c^2}\right]\right) = \frac{2L}{c}\frac{v^2}{2c^2}$$

(as was done in class).

(a) For the Earth's orbital speed (30 km/s), $v / c = 10^{-4}$, so that

$$\Delta t = (73.3 \text{ ns}) \times \frac{1}{2}(10^{-4})^2 = 3.67 \times 10^{-16} \text{ s}$$

(a fraction of the period of visible light).

(b) For $v / c = 10^{-2}$: $\Delta t = (73.3 \text{ ns}) \times \frac{1}{2}(10^{-2})^2 = 3.67 \times 10^{-12} \text{ s}$

(a few thousand periods of visible light).

(c) At relativistic speeds, we use the exact expression for Δt. For $v / c = 0.5$:

$$\gamma = \frac{1}{\sqrt{1 - v^2 / c^2}} = \frac{2}{\sqrt{3}} \quad \text{and} \quad \Delta t = (73.3 \text{ ns})\left(\frac{4}{3} - \frac{2}{\sqrt{3}}\right) = 13.1 \text{ ns}$$

(d) For $v / c = 0.99$: $\gamma = \frac{1}{\sqrt{1 - 0.99^2}} = 7.09$ and $\Delta t = \frac{2L}{c}\gamma(\gamma - 1) = 3.17 \ \mu\text{s}$

Lorentz Transformations

Answer Questions 3 and 4 using Lorentz transformations. Do not use short cuts associated with time dilation and/or length contraction. Once you have answered the questions, think about the implications for our concept of space and time.

3. How long would it take a spacecraft travelling at 65% of the speed of light to make the 5.8×10^9 km journey from Earth to Pluto according to clocks
   (a) on Earth, and
   (b) on the spacecraft?

Solution:

Define the reference frames: A = Earth, A' = spacecraft

(a) For a clock on Earth (in reference frame A):

distance from Earth to Pluto: $x = 5.8 \times 10^9 \text{ km} \times 1000 \text{ m/km} = 5.8 \times 10^{12} \text{ m}$

time taken for the spacecraft to travel distance x:

$$t = \frac{x}{v} = \frac{x}{0.65c} = \frac{5.8 \times 10^{12} \text{ m}}{0.65 \times 3.00 \times 10^8 \text{ m/s}} = 29{,}744 \text{ s} = 8.3 \text{ hours}$$

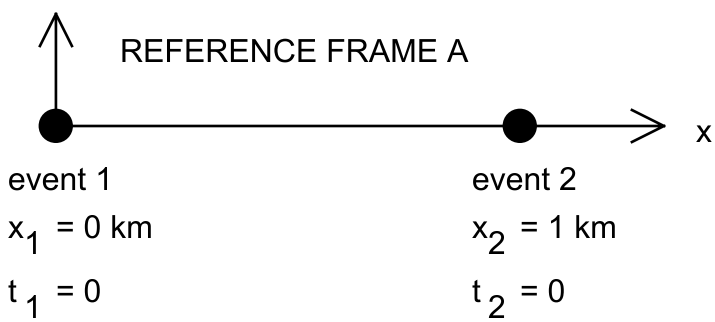
(b) For a clock on the spacecraft (in reference frame A'), we need to apply the Lorentz transformation for time:

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) = \frac{1}{\sqrt{1 - v^2/c^2}}\left(t - \frac{v}{c^2}x\right)$$
$$= \frac{1}{\sqrt{1 - 0.65^2}}\left(29{,}744\ \text{s} - \frac{0.65}{3.00 \times 10^8\ \text{m/s}}\left(5.8 \times 10^{12}\ \text{m}\right)\right)$$
$$= 22{,}604\ \text{s} = 6.3\ \text{hours}$$

4. An observer in a reference frame A observes two events to be simultaneous. The events are seen to occur at x = 0 km and x = 1 km in frame A. An observer moves at speed βc relative to reference frame A (where c is the speed of light). This observer sees the events occur with a time difference of 10^{-6} seconds. What is β?

Solution:

First consider the space-time coordinates of the two events.



Note that $t_1 = t_2$ because the events are simultaneous in reference frame A.

For the observer moving at speed βc relative to reference frame A (say in reference frame A'), apply the Lorentz transformation for time to each event:

$$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right) = 0$$
$$t_2' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) = 10^{-6}\ \text{s} \qquad \text{(given)}$$

Thus:

$$\frac{1}{\sqrt{1 - \beta^2}}\left(0 - \frac{\beta c}{c^2}x_2\right) = 10^{-6}\ \text{s}$$
$$-\frac{\beta}{c}x_2 = \left(10^{-6}\ \text{s}\right)\sqrt{1 - \beta^2}$$
$$\beta x_2 = 10^{-6}c\sqrt{1 - \beta^2}$$
$$\beta^2 x_2^{\,2} = 10^{-12}c^2\left(1 - \beta^2\right)$$

$$\beta^2\left(x_2^{\ 2} + 10^{-12}c^2\right) = 10^{-12}c^2$$

$$\beta^2 = \frac{10^{-12}c^2}{x_2^{\ 2} + 10^{-12}c^2}$$

$$\beta = \frac{10^{-6}c}{\sqrt{x_2^{\ 2} + 10^{-12}c^2}} = \frac{\left(10^{-6}\ \text{s}\right)\left(3.00\times 10^{8}\ \text{m/s}\right)}{\sqrt{(1000\ \text{m})^2 + (10^{-12}\ \text{s}^2)(3.00\times 10^{8}\ \text{m/s})^2}} = 0.287$$

So the observer travels at 28.7% of the speed of light.

5. Jessica embarks on a cosmic journey at a speed of $\frac{24}{25}$c relative to the Earth. Before leaving, she tells her twin brother, Tom, who stays on Earth, that she will travel outwards for 25 years of Earth time, then back for another 25 years of Earth time. Tom will thus be 50 years older when she returns. She promises to send a radio message on each of her birthdays. According to an Earth-based clock,
(a) when will these messages reach Tom, and
(b) how much older than the age of the clock at which she leaves will Jessica be when she returns to Earth?
Think about the implications of this answer for our concept of space and time!

Solution:

The distance from Earth to the turnaround point is

$$L = vt = \left(\frac{24}{25}c\right)\times 25\ \text{years} = 24\ \text{lightyears}$$

According to Tom, Jessica's clocks run slower, so the time interval between sending cards is

$$\Delta T' = \frac{\Delta T}{\sqrt{1 - v^2/c^2}} = \frac{1\ \text{year}}{\sqrt{1-(24/25)^2}} = \frac{25}{7}\ \text{years}$$

The distance that Jessica will be from Earth at the time that the first card is sent is

$$L_1 = vT_1' = \left(\frac{24}{25}c\right)\times\left(\frac{25}{7}\ \text{years}\right) = \frac{24}{7}\ \text{lightyears}$$

The time interval on Earth between takeoff and the arrival of the first card is

$$\Delta T_1 = T_1' + \frac{L_1}{c} = \left(\frac{25}{7}\ \text{years}\right) + \left(\frac{24}{7}\ \text{lightyears}\right)\Big/c = \frac{49}{7}\ \text{years} = 7\ \text{years}$$

This will be the interval between the reception of the cards until the card that is sent just before turning around is received. This card will be sent from the turnaround point at a time of 25 years, according to Tom, and will be received by Tom at a time

$$\left(25\ \text{years}\right) + \frac{24\ \text{lightyears}}{c} = 49\ \text{years after takeoff}$$

Thus Tom will receive cards every 7 years for 49 years, i.e., a total of 7 cards.

On the return journey, the eighth card will be sent at a time

$$T_8 = (25 \text{ years}) + \left(\frac{25}{7} \text{ years}\right) = \left(25 + \frac{25}{7}\right) \text{ years}$$

and a distance from Earth of

$$L_8 = (24 \text{ lightyears}) - \left(\frac{24}{25} c\right)\left(\frac{25}{7} \text{ years}\right) = \left(24 - \frac{24}{7}\right) \text{ lightyears}$$

This card will be received by Tom at a time

$$T_8' = T_8 + \frac{L_8}{c} = \left(25 + \frac{25}{7} \text{ years}\right) + \left(24 - \frac{24}{7} \text{ lightyears}\right) \Big/ c = 49 + \frac{1}{7} \text{ years}$$

Because Tom will receive card number 7 after 49 years, the time between receptions for cards sent on the return journey is

$$\Delta T_2 = \frac{1}{7} \text{ year}$$

Because the total trip takes 50 years, Tom will receive cards at this rate for only 1 year, or

$$\frac{1 \text{ year}}{1/7 \text{ year}} = 7 \text{ cards}$$

(a) Thus Tom receives a card every 7 years for 49 years and every 1/7 year for 1 year.

(b) Jessica will be 14 years older when she returns.

Note that this is consistent with the time dilation of Jessica's clocks:

$$\Delta T_{total}' = \frac{\Delta T_{total}}{\sqrt{1 - v^2/c^2}}$$

$$\begin{aligned}
\Delta T_{total} &= \Delta T_{total}' \sqrt{1 - v^2/c^2} \\
&= (50 \text{ years})\sqrt{1 - (24/25)^2} \\
&= 14 \text{ years}
\end{aligned}$$