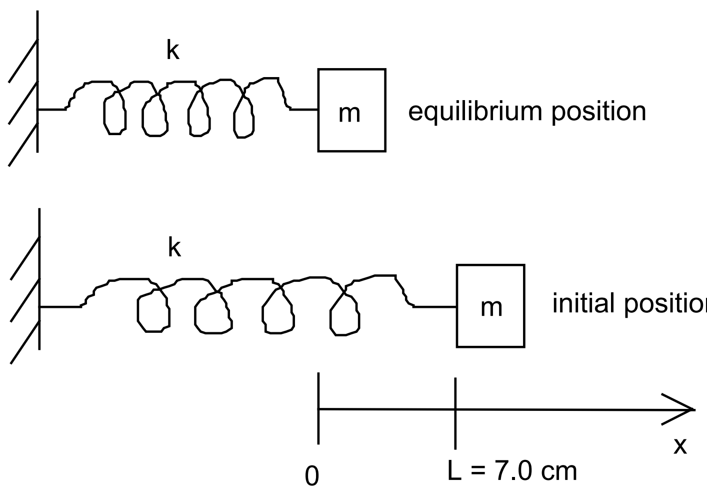
PHY 140Y – FOUNDATIONS OF PHYSICS
2001-2002
Tutorial Questions #10 – Solutions
November 19/20

Damped and Driven Harmonic Motion, Resonance

1. A mass of 220 g is connected to a light spring having force constant 5.4 N/m. It is free to oscillate on a horizontal frictionless surface.
(a) If the mass is displaced 7.0 cm from the equilibrium position and released from rest, what are the natural frequency and the period of its motion?
(b) What are the maximum speed and acceleration of the mass?
(c) What is the total energy of the system? What are the kinetic and potential energies of the system when the mass has a displacement of 2.0 cm from equilibrium?
(d) If the surface is no longer frictionless, but instead gives rise to a damping force with damping constant b = 1.7 kg/s, is the motion of the mass underdamped, critically damped, or overdamped? Justify your answer.

Solution:



(a) Given: $m = 220 \text{ g} = 0.22 \text{ kg}$
$L = 7.0 \text{ cm} = 0.070 \text{ m}$
$k = 5.4 \text{ N/m}$

natural angular frequency: $$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.4 \text{ N/m}}{0.22 \text{ kg}}} = 4.9(5) \text{ rad/s}$$

natural frequency: $$f_o = \frac{\omega}{2\pi} = \frac{4.95 \text{ rad/s}}{2\pi} = 0.79 \text{ Hz}$$

period: $$T = \frac{2\pi}{\omega} = \frac{2\pi}{4.95 \text{ rad/s}} = 1.3 \text{ seconds}$$

(b) The maximum speed occurs at x = 0, while the maximum acceleration is at x = L (the maximum extension).

Apply Conservation of Mechanical Energy to find the maximum speed:

$$K_0 + U_0 = K_L + U_L$$

$$\tfrac{1}{2} m v_{max}^2 + 0 = 0 + \tfrac{1}{2} k L^2$$

$$v_{max}^2 = \frac{kL^2}{m} = \omega_o^2 L^2$$

$$v_{max} = \omega_o L = 4.95 \text{ rad/s} \times 0.070 \text{ m} = 0.35 \text{ m/s}$$

Apply Newton's Second Law to find the maximum acceleration:

$$F = m a_{max} = -kL$$

$$|a_{max}| = \frac{kL}{m} = \omega_o^2 L$$

$$= (4.95 \text{ rad/s})^2 \times 0.070 \text{ m} = 1.7 \text{ m/s}^2$$

(c) The total energy of the system is:

$$E_m = \tfrac{1}{2} k L^2 = K_{max} = U_{max}$$

$$= \tfrac{1}{2}(5.4 \text{ kg/s}) \times (0.070 \text{ m})^2 = 0.013 \text{ J}$$

When the mass has a displacement of 2.0 cm from equilibrium,

$$U(0.020 \text{ cm}) = \tfrac{1}{2} k x^2 = \tfrac{1}{2}(5.4 \text{ kg/s}) \times (0.020 \text{ m})^2 = 0.0011 \text{ J}$$

$$K(0.020 \text{ cm}) = E_m - U(0.020 \text{ cm}) = \left[\tfrac{1}{2} m v^2\right] = 0.013 \text{ J} - 0.00108 \text{ J} = 0.012 \text{ J}$$

(d) In order to determine if the motion of the mass is underdamped, critically damped, or overdamped, need to compare b with $2m\omega_o$.

$$b = 1.7 \text{ kg/s}$$

$$2m\omega_o = 2 \times 0.22 \text{ kg} \times 4.95 \text{ rad/s} = 2.2 \text{ kg/s}$$

Thus: $b < 2m\omega_o$

and so the motion is underdamped, with the amplitude of the oscillation decreasing with time.

2. A 250-g mass is mounted on a spring with a spring constant of k = 3.3 N/m. The damping constant for this system is $b = 8.4 \times 10^{-3}$ kg/s. How many oscillations will the system undergo during the time it takes the amplitude to decay to 1/e of its original value?

Solution:

First, need to check whether the system is underdamped, critically damped, or overdamped, by comparing b with $2m\omega_o$.

$$2m\omega_o = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2\sqrt{0.25\ \text{kg} \times 3.3\ \text{N/m}} = 1.8\ \text{kg/s}$$

With $b = 8.4 \times 10^{-3}$ kg/s , $b \ll 2m\omega_o$, and so the motion is underdamped and will oscillate with:

$$x(t) = A \exp\left(-\frac{b}{2m} t\right) \cos(\omega t + \delta)$$

$$\omega = \sqrt{\omega_o^2 - \frac{b^2}{4m^2}}$$

Asked for the number of oscillations before the amplitude decays to 1/e of its original value

i.e., $A \exp\left(-\frac{b}{2m} t\right) = A \exp(-1)$

Solve for the time t at which this happens:

$$\frac{b}{2m} t = 1$$

$$t = \frac{2m}{b} = \frac{2(0.25\ \text{kg})}{8.4 \times 10^{-3}\ \text{kg/s}} = 59.5\ \text{s}$$

Also need to calculate the period for one oscillation:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_o^2 - \frac{b^2}{4m^2}}} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}} = \frac{2\pi}{\sqrt{\frac{3.3\ \text{N/m}}{0.25\ \text{kg}} - \frac{(8.4 \times 10^{-3}\ \text{kg/s})^2}{4(0.25\ \text{kg})^2}}} = 1.7(3)\ \text{seconds}$$

Thus, the number of oscillations is 59.5/1.73 = 34.

3. A pendulum of length 1.00 m is released from an initial angle of 15.0°
   (a) What is the resonant (angular) frequency of this pendulum?
   (b) After 1000. seconds, its amplitude is reduced by friction to 5.50°. What is the value of b/2m?

Solution:

(a) The resonant angular frequency is the same as the natural angular frequency, and so is:

$$\omega_o = \sqrt{\frac{g}{l}} = \sqrt{\frac{9.81\ \text{m/s}^2}{1.00\ \text{m}}} = 3.13\ \text{rad/s}$$

(b) The pendulum will oscillate following: $\theta(t) = \theta_o \exp\left(-\frac{b}{2m} t\right) \cos(\omega t + \delta)$

(assuming small angles θ; note that θ = 15.0° = 0.262 rad, while sin 15.0° = 0.259 rad)

At t = 0, the amplitude is: $\theta(0) = \theta_o \exp\left(-\frac{b}{2m}t\right) = \theta_o = 15.0^o$

At t = 1000. seconds, the amplitude is: $\theta(1000.) = \theta_o \exp\left(-\frac{b}{2m}t\right) = 5.50^o$

$$15.0^o \exp\left(-\frac{b}{2m}t\right) = 5.50^o$$

Thus: $\exp\left(-\frac{b}{2m}t\right) = \frac{5.50}{15.0}$

$$\frac{b}{2m} = -\frac{1}{t} \times \ln\left(\frac{5.50}{15.0}\right) = -\frac{1}{1000.\ s} \times \ln\left(\frac{5.50}{15.0}\right) = 1.00 \times 10^{-3}\ s^{-1}$$

4. A mass-spring system has $b/m = \omega_o/5$, where b is the damping constant and ω_o is the natural frequency. When this system is driven at frequencies 10% above and below ω_o, how does its amplitude compare with its amplitude at ω_o?

Solution:

In general, the amplitude is: $A = \frac{F_o}{\sqrt{m^2(\omega_d^2 - \omega_o^2)^2 + b^2\omega_d^2}}$

At resonance $\omega_d = \omega_o$, and so the amplitude is: $A_{res} = \frac{F_o}{b\omega_d}$

The ratio of the amplitude to the resonant amplitude is thus:

$$\frac{A}{A_{res}} = \frac{A}{F_o/b\omega_d} = \frac{F_o}{\sqrt{m^2(\omega_d^2 - \omega_o^2)^2 + b^2\omega_d^2}} \times \frac{b\omega_d}{F_o} = \frac{1}{\sqrt{\left(\frac{m\omega_o}{b}\right)^2\left(\frac{\omega_d^2}{\omega_o^2} - 1\right)^2 + \frac{\omega_d^2}{\omega_o^2}}}$$

If $\frac{m\omega_o}{b} = 5$ and $\frac{\omega_d}{\omega_o} = 1.1$ (10% above resonance), then

$$\frac{A}{A_{res}} = \frac{1}{\sqrt{5^2\left(1.1^2 - 1\right)^2 + 1.1^2}} = 0.66 = 66\%$$

If $\frac{\omega_d}{\omega_o} = 0.9$ (10% below resonance), then

$$\frac{A}{A_{res}} = \frac{1}{\sqrt{5^2\left(0.9^2 - 1\right)^2 + 0.9^2}} = 0.76 = 66\% = 76\%$$

5. A 2.00 kg mass attached to a spring is driven by an external force $F(t) = (3.00\text{N})\cos(2\pi t)$. If the force constant of the spring is 20.0 N/m and damping is negligible, determine:
(a) the period, and
(b) the amplitude of the motion.

Solution:

(a) natural angular frequency: $\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{20.0 \text{ N/m}}{2.00 \text{ kg}}} = 3.16 \text{ rad/s}$

angular frequency of motion: $\omega = \sqrt{\omega_o^2 - \frac{b^2}{4m^2}} = \omega_o$ (damping negligible)

angular frequency of driving force: $\omega_d = 2\pi \text{ rad/s}$

The period of the motion is calculated using the driving angular frequency, as the spring will oscillate at this frequency after the transients die out and it reaches a steady state motion.

period: $T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi \text{ rad/s}} = 1.00 \text{ seconds}$

(b) The amplitude of the motion is:

$$A = \frac{F_o}{\sqrt{m^2(\omega_d^2 - \omega_o^2)^2 + b^2\omega_d^2}} = \frac{F_o}{m(\omega_d^2 - \omega_o^2)} \quad \text{(damping negligible)}$$

Thus: $$A = \frac{F_o}{m(\omega_d^2 - \omega_o^2)} = \frac{3.00 \text{ N}}{(2.00 \text{ kg})\left[(2\pi \text{ rad/s})^2 - (3.16 \text{ rad/s})^2\right]} = 0.0509 \text{ m}$$