PHY 140Y – FOUNDATIONS OF PHYSICS
2001-2002
Problem Set #4 – Solutions

HANDED OUT: Friday, November 16, 2001 (in class).

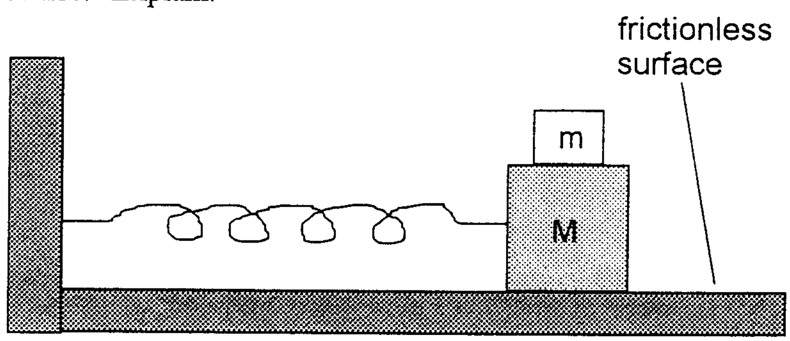
DUE: **5:00 PM, Thursday, November 29, 2001** in the appropriate box, labeled by tutorial group, in the basement at the bottom of the stairs leading down from MP202.

LATE PENALTY: 5 marks/day (which also applies to weekend days!) until 1:00 PM, Monday, December 3, after which it will not be accepted as solutions will then be available in tutorials and on the WWW.

NOTES: Answer all questions. A selected subset (3-4) will be marked out of 100%. Marks will be given for workings and units, as well as for final answers.

QUESTIONS:

1. A mass m rests on a block of mass M that is attached to a horizontal spring as indicated in the figure below. The block M rests on a frictionless surface. The coefficient of static friction for the contact surfaces of the two masses is μ_s and the corresponding coefficient of kinetic friction is μ_k. The system is set into simple harmonic oscillation with amplitude A. Identify the forces acting on mass m. What spring constant ensures that m is on the verge of slipping? If m is increased, will the mass slip or not? Explain.



Solution:

Choose the small mass m as the system. The forces on m are:

1. its weight $\vec{w}$, of magnitude mg and directed downward;
2. the normal force $\vec{N}$ of the surface of M on m, directed upward; and
3. the static force of friction $\vec{f}_s$, directed horizontally in the direction of acceleration so as to oppose slippage.

Choose a coordinate system with $\hat{i}$ pointed horizontally to the right in Figure , and with $\hat{j}$ pointing up (parallel to $\vec{N}$). Then

x direction.

$$F_{x\text{ total}} = ma_x \implies f_s = ma_x.$$

y direction.

$$F_{y\text{ total}} = ma_y \implies N - mg = m(0\text{ m/s}^2) \implies N = mg.$$

The force of static force of friction on m reaches its maximum magnitude when the acceleration component in the x direction reaches its maximum magnitude. Since the oscillation is described by $x(t) = A\cos(\omega t + \theta)$, the acceleration component is

$$a_x(t) = \frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \theta).$$

The maximum magnitude of the acceleration is therefore $a_{\text{max}} = A\omega^2$. Hence, if m is not slipping but is on the verge of slipping

$$f_{s\,\text{max}} = ma_{max} \implies \mu_s N = mA\omega^2 \implies \mu_s mg = mA\omega^2 \implies \mu_s g = A\omega^2.$$

The total mass on the end of the spring is $M + m$. The angular frequency of the oscillation is

$$\omega = \sqrt{\frac{k}{M+m}}.$$

Hence

$$\mu_s g = A\omega^2 = A\frac{k}{M+m} \implies k = \frac{\mu_s(M+m)g}{A}$$

If m is increased and k is held constant, the angular frequency of the oscillation will decrease, so the maximum magnitude of acceleration will also decrease, and the force of static friction needed to prevent slippage will be less than $f_{s\,\text{max}}$. Hence, increasing m will mean that m is no longer on the verge of slipping.

2. A 1.50-kg mass on a horizontal frictionless surface is attached to a horizontal spring with spring constant k = 200. N/m. The mass is in equilibrium at x = 0 m. The mass is released when t = 0 s at coordinate x = 0.100 m with a velocity (2.00 m/s) $\hat{i}$.
   (a) Find the constants A, ω, and φ (or δ as in class) in the equation: $x(t) = A\cos(\omega t + \phi)$.
   (b) Find the period of the oscillation.
   (c) Determine the maximum speed of the oscillation and the magnitude of the maximum acceleration.
   (d) Plot x(t) during the time interval from t = 0 s to t = 2T.

Solution:

a) The angular frequency is found from

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\text{ N/m}}{1.50\text{ kg}}} = 11.5\text{ rad/s}.$$

When $t = 0$ s, the mass is released at $x = 0.100$ m, so the general equation for x becomes

(1) $$0.100 \text{ m} = A\cos[\omega(0 \text{ s}) + \phi] = A\cos\phi.$$

Likewise, when $t = 0$ s, the velocity component is $v_x = 2.00$ m/s. The velocity at any instant is

$$v_x(t) = \frac{d}{dt}x(t) = -A\omega\sin(\omega t + \phi).$$

So

(2) $$2.00 \text{ m/s} = -A\omega\sin[\omega(0 \text{ s}) + \phi] = -A(11.5 \text{ rad/s})\sin\phi.$$

Divide equation (2) by equation (1).

$$\frac{2.00 \text{ m/s}}{0.100 \text{ m}} = \frac{-A(11.5 \text{ rad/s})\sin\phi}{A\cos\phi} \implies 20.0 \text{ s}^{-1} = -11.5 \text{ rad/s} \tan\phi$$
$$\implies \tan\phi = -1.74$$
$$\implies \phi = -1.05 \text{ rad}.$$

Now use this value for ϕ in equation (1) to find A:

$$0.100 \text{ m} = A\cos\phi = A\cos(-1.05 \text{ rad}) \implies A = 0.201 \text{ m}.$$

Hence

$$x(t) = (0.201 \text{ m})\cos[(11.5 \text{ rad/s})t - 1.05 \text{ rad}].$$

b) The period T of the oscillation is the inverse of the frequency ν:

$$T = \frac{1}{\nu} = \frac{2\pi}{\omega} = \frac{2\pi}{11.5 \text{ rad/s}} = 0.546 \text{ s}.$$

c) The velocity component at any time is

$$v_x(t) = \frac{dx(t)}{dt} = -A\omega\sin(\omega t + \phi).$$

Since the maximum magnitude of the sine is 1, the maximum speed is

$$v_{\text{max}} = \omega A = (11.5 \text{ rad/s})(0.201 \text{ m}) = 2.31 \text{ m/s}.$$

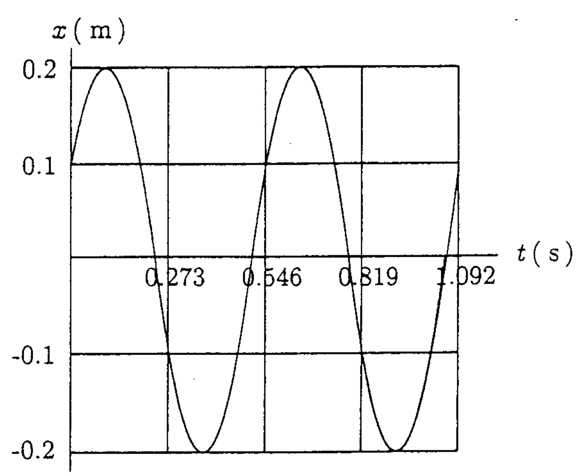
The acceleration component at any time is

$$a_x(t) = \frac{dv_x(t)}{dt} = -A\omega^2\cos(\omega t + \phi).$$

Since the maximum magnitude of the cosine is 1, the maximum magnitude of the acceleration is

$$a_{\text{max}} = \omega^2 A = (11.5 \text{ rad/s})^2(0.201 \text{ m}) = 26.6 \text{ m/s}^2.$$

d) A plot of $x(t)$ versus t for two periods is shown below.



3. The pendulum inside a grandfather clock has a half period of 1.0000 s at a location where the magnitude of the local acceleration of gravity is 9.800 m/s^2. The clock is carefully moved to another location at the same temperature and is found to run slow by 89.0 s per day. What is the magnitude of the local acceleration due to gravity at the new location?

Solution:

Since the clock runs slow, the period of its pendulum is longer at the new location, and since ℓ is unchanged we anticipate a smaller value for g. The period T of a simple pendulum of length ℓ is $T = 2\pi\sqrt{\frac{\ell}{g}}$. To see how the period is affected by changes in g, take the derivative of T with respect to g:

$$\frac{dT}{dg} = 2\pi\ell^{1/2}\left(-\frac{1}{2}g^{-3/2}\right) = -\left(2\pi\sqrt{\frac{\ell}{g}}\right)\frac{1}{2g} = -\frac{T}{2g}.$$

So, for a small change Δg in g and the resulting change ΔT in T we have approximately

$$\Delta T \approx -\frac{T}{2g}\Delta g \implies \Delta g \approx -2g\frac{\Delta T}{T}.$$

The fractional increase in the period is 89.0 s divided by the number of seconds per day:

$$\frac{\Delta T}{T} = \frac{89.0\text{ s}}{8.64\times 10^4\text{ s}} = 1.03\times 10^{-3} \implies \Delta g \approx -2g\frac{\Delta T}{T} = -2(9.800\text{ m/s}^2)(1.03\times 10^{-3}) = -2.02\times 10^{-2}\text{ m/s}^2.$$

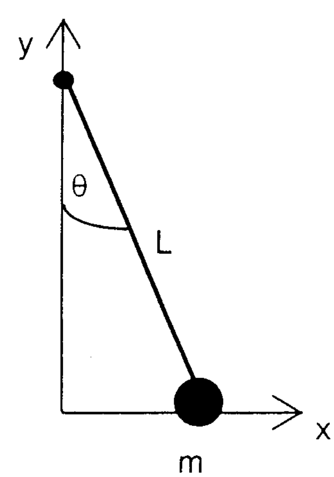
The magnitude of the acceleration due to gravity at the new location is therefore

$$g_{\text{new}} = g + \Delta g \approx 9.800\text{ m/s}^2 - 0.0202\text{ m/s}^2 = 9.780\text{ m/s}^2.$$

If the local value of g was originally 9.81 m/s^2, the value in the new location is 9.79 m/s^2

4. A plumb bob of length L is swung so that the mass on the end of the string moves in a horizontal circle at constant angular speed with the string making a constant small angle θ with the vertical direction, as shown in the adjacent figure. Show that if the motion is viewed sideways in the plane of the horizontal circle, then the motion is simple harmonic and of period

$$T = 2\pi\sqrt{\frac{L\cos\theta}{g}}.$$

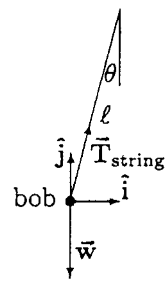


Solution:

The bob of the pendulum is swinging around at constant speed in a circle and so is in uniform circular motion. The projection of such uniform circular motion onto a line in the plane of the circle is simple harmonic motion with the same angular frequency as the angular speed of the circular motion. To find this angular speed, and from it the period of the motion, consider the forces acting on the pendulum bob:

1. the weight $\vec{w}$ of the bob; and
2. the force $\vec{T}_{\text{string}}$ of the cord on the bob.

The second law force diagram is shown below, along with an appropriate coordinate choice.



The unit vector $\hat{\imath}$ points directly from the bob towards the center of circular motion.

There is zero acceleration in the y direction, so the total force in that direction is zero. The only acceleration in the horizontal plane is the centripetal acceleration in the x direction toward the center of the circle. The radius of the circle is $\ell \sin\theta$, so the centripetal acceleration has magnitude $(\ell\sin\theta)\omega^2$.

y direction

$$F_{y\ \text{total}} = 0\ \text{N} \implies T_{\text{string}}\cos\theta - mg = 0\ \text{N} \implies T_{\text{string}}\cos\theta = mg.$$

x direction

$$F_{x\ \text{total}} = ma_x \implies T_{\text{string}}\sin\theta = m(\ell\sin\theta)\omega^2 \implies T_{\text{string}} = m\ell\omega^2.$$

Substitute the expression for T_{string} from the x equation into the y equation and solve for ω.

$$ml\omega^2\cos\theta = mg \implies \omega = \sqrt{\frac{g}{\ell\cos\theta}}$$

Hence the period T_{period} is

$$T_{\text{period}} = \frac{1}{\nu} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell\cos\theta}{g}}.$$

Notice that for small values of θ, $\cos\theta \approx 1$, so $T_{\text{period}} \approx 2\pi\sqrt{\frac{\ell}{g}}$, which is the same formula as the one for the period of a pendulum of length ℓ

5. (a) A 0.950-kg mass hangs vertically from a spring that has a spring constant of 8.50 N/m. The mass is set into vertical oscillation and after 600. s, you find that the amplitude of the oscillation is 1/10 that of the initial amplitude. What is the damping constant associated with this motion?

(b) Show that if the position of a damped oscillator is given by $x(t) = Ae^{-\alpha t}\cos(\omega t)$, where $\alpha = \beta/(2m)$, the time interval for the amplitude to decrease to half its initial value is $(2m/\beta)\ln 2$.

Solution:

(a) The amplitude is

$$A = A_0 e^{-\alpha t}.$$

Divide by A_0, take natural logarithms and solve for α.

$$\frac{A}{A_0} = e^{-\alpha t} \implies \ln\left(\frac{A}{A_0}\right) = -\alpha t \implies \alpha = -\frac{1}{t}\ln\left(\frac{A}{A_0}\right).$$

Since $\alpha = \frac{\beta}{2m}$, then $\beta = 2m\alpha$. Thus

$$\beta = -\frac{2m}{t}\ln\left(\frac{A}{A_0}\right).$$

Now substitute 0.950 kg for m, 600 s, for t, and 0.10 for $\frac{A}{A_0}$.

$$\beta = -\frac{2m}{t}\ln\left(\frac{A}{A_0}\right) = -\frac{2(0.950\text{ kg})}{600\text{ s}}\ln 0.10 = 7.3\times10^{-3}\text{ kg/s}.$$

(b) The amplitude is $A = A_0 e^{-\alpha t}$. Set $A = \frac{1}{2}A_0$ and solve for t.

$$\frac{1}{2}A_0 = A_0 e^{-\alpha t} \implies \frac{1}{2} = e^{-\alpha t} \implies -\ln 2 = -\alpha t \implies t = \frac{1}{\alpha}\ln 2 = \frac{1}{\frac{\beta}{2m}}\ln 2 = \frac{2m}{\beta}\ln 2.$$

6. (a) Show that as $v/c \to 1$, an approximate expression for the relativistic factor γ is

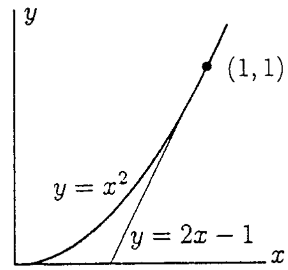
$$\gamma \cong \frac{1}{\sqrt{2\left(1-\frac{v}{c}\right)}}.$$

(b) Close to the San Andreas Fault, south of San Francisco, lies the Stanford Linear Accelerator (SLAC) which is used to project subatomic particles to great speed. The accelerator is 3.0 km long and passes under a major highway. The accelerator is capable of accelerating electrons to speeds that have a relativistic factor $\gamma = 1.00\times10^4$. Calculate the ratio of the speed of the electrons to the speed of light. Use the result of part (a) to find v/c. Can you easily find v/c on your calculator without the result of part (a)? Comment.

(c) In a reference frame in which the electrons are at rest, what is the length of the accelerator? Assume (incorrectly) that the speed of the electrons is constant along the length of the accelerator.

Solution:

(a) Use the calculus to linearly approximate $y = x^2$ near $x = 1$.



The approximation is

$$x^2 \approx 2x - 1.$$

Now use this expression to approximate $\frac{v^2}{c^2}$ in the formula for γ.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \approx \sqrt{\frac{1}{1 - \left(2\frac{v}{c} - 1\right)}} = \sqrt{\frac{1}{2\left(1 - \frac{v}{c}\right)}}.$$

For $\frac{v}{c} = 0.999\,999$, the exact expression for γ gives

$$\gamma = 707.1070$$

to 7 significant figures. The approximation gives

$$\gamma \approx 707.1068,$$

to 7 significant figures. So our approximation works well when $\frac{v}{c}$ is very close to 1.

Approximations such as this were commonly used before the advent of electronic calculators (pre 1970s).

(b) Use the result of part (a) to find $\frac{v}{c}$, since the ratio is very close to 1. Solve the "equation"

$$\gamma \approx \frac{1}{\sqrt{2\left(1 - \frac{v}{c}\right)}}$$

for $\frac{v}{c}$. Begin by squaring both sides

$$\gamma^2 \approx \frac{1}{2\left(1 - \frac{v}{c}\right)} \implies 1 - \frac{v}{c} \approx \frac{1}{2\gamma^2}$$

$$\implies \frac{v}{c} \approx 1 - \frac{1}{2\gamma^2} = 1 - \frac{1}{2(1.00 \times 10^4)^2} = 1 - 0.500 \times 10^{-8} = 0.999\,999\,995\,00.$$

Had we started with the equation

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and then solved it for $\frac{v}{c}$ we would end up with

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}. \quad (1)$$

Many calculators carry only 8 digits in their operations, so if we evaluated this last equation for $\gamma = 1.00 \times 10^4$ we would not have the right answer — or more precisely, the answer for $1 - \frac{v}{c}$ would not have any correct significant digits. If your calculator carries more than 8 digits (as many do), try doing this problem with Equation (1) when $\gamma = 10^6$ or 10^8.

(c) In the frame of the electrons, the lab is moving. Use length contraction:

$$\ell = \frac{\ell_0}{\gamma} = \frac{3.0 \times 10^3 \text{ m}}{1.00 \times 10^4} = 0.30 \text{ m}.$$

7. In an inertial reference frame S, the following observations are made: (i) a professor passes out a test at the origin and starts a stopwatch simultaneously, (ii) 10.0 s later, a student has a fit at x = 9.0 × 10^8 m. The dean (located at the origin in another inertial reference frame S' in the standard geometry) is cruising by at a speed of 0.98c.

(a) Calculate the relativistic factor γ.
(b) Specify the space and time coordinates of the two events in S.
(c) Use the Lorentz transformation equations to find the space and time coordinates of these events in the dean's reference frame.

Solution: a)

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - (0.98)^2}} = 5.0.$$

b)

(Event 1) $x = x_1 = 0 \text{ m}$ and $t = t_1 = 0 \text{ s}$.

(Event 2) $x = x_2 = 9.0 \times 10^8 \text{ m}$ and $t = t_2 = 10.0 \text{ s}$.

c) We'll apply the Lorentz transformation, to each event. In order to do this we need to know the speed v. Since $\frac{v}{c} = 0.98$, we have

$$v = 0.98c = 0.98(3.00 \times 10^8 \text{ m/s}) = 2.9 \times 10^8 \text{ m/s}.$$

Now apply the Lorentz transformation to each event.

(Event 1)

$$x' = x_1' = \gamma(x_1 - vt_1) = 5.0[0 \text{ m} - (2.9 \times 10^8 \text{ m/s})(0 \text{ s})] = 0 \text{ m}$$

$$t' = t_1' = \gamma(t_1 - \frac{v}{c^2}x_1) = \gamma[0 \text{ s} - \frac{2.9 \times 10^8 \text{ m/s}}{(3.00 \times 10^8 \text{ m/s})^2}(0 \text{ m})] = 0 \text{ s}.$$

(Event 2)

$$x' = x_2' = \gamma(x_2 - vt_2) = 5.0[9.0 \times 10^8 \text{ m} - (2.9 \times 10^8 \text{ m/s})(10.0 \text{ s})] = -1.0 \times 10^{10} \text{ m}$$

$$t' = t_2' = \gamma(t_2 - \frac{v}{c^2}x_2) = 5.0[10.0 \text{ s} - \frac{2.9 \times 10^8 \text{ m/s}}{(3.00 \times 10^8 \text{ m/s})^2}(9.0 \times 10^8 \text{ m})] = 36 \text{ s}.$$

8. You are in a spaceship of length 100. m moving at speed 0.900c to visit the star Arcturus. Back home, a friend notices that you forgot your toothbrush and so sends a light signal to the spaceship. Let A' be the reference frame of the spaceship and A be the reference frame of the friend back on Earth. The light signal arrives at the tail of the spaceship when t = t' = 0 sec.

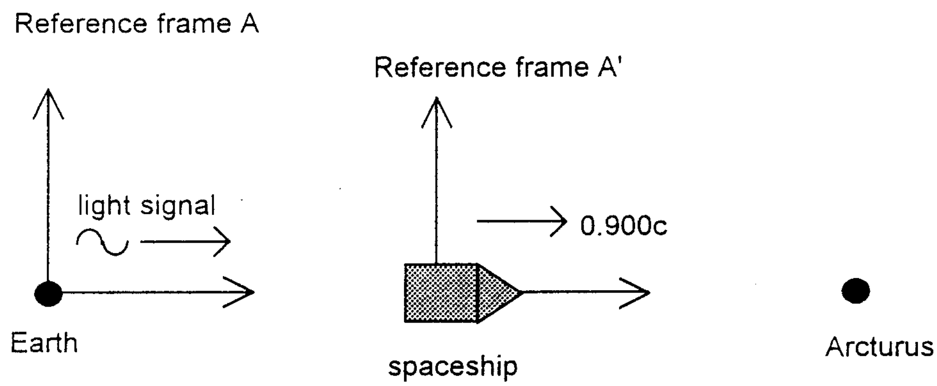
(a) When does the light signal reach the front of the spaceship according to the spaceship clock?

(b) When does the light signal reach the front of the spaceship according to your friend's clock?

(c) Are the answers to (a) and (b) related by the time dilation equation? Why or why not?

(d) The light signal is reflected back to the tail of the spaceship by a mirror in the nose of the spaceship. When does the light signal reach tail according to the spaceship clock? When does the light signal reach tail according to your friend's clock?

(e) Determine the total distance travelled by the light signal as it travels from the tail to the nose and then back to the tail according to (i) you, inside the spaceship, and (ii) your friend, back on Earth.



Solution:

a) In the frame of the space ship, the length of the spaceship is still 100 m, and the speed of light is c, so the time for the light to travel the 100 m length of the spaceship is

$$\text{time required} = \frac{100\text{ m}}{c} = \frac{100\text{ m}}{3.00 \times 10^8\text{ m/s}} = 3.33 \times 10^{-7}\text{ s}.$$

b) We'll use the inverse Lorentz transformation to transform the result of part a) back to Earth's reference frame.

Define two events. Event 1 is the light signal occuring at the tail of the spacecraft, and Event 2 is the light signal occuring at the front of the space craft. Let S be the frame of the Earth, and S' the frame of the spacecraft.

In S' let x_1' be the position of the tail of the spacecraft, and x_2' the position of its head. Let t_1' be the time that the light signal is at the tail of the craft, and t_2' be the time it is at the head. Then, using the result of part a), the two events are described in S' by

(Event 1) $$x' = x_1' \quad \text{and} \quad t' = t_1',$$

and

(Event 2) $$x' = x_2' = x_1' + 100\text{ m} \quad \text{and} \quad t' = t_2' = t_1' + 3.33 \times 10^{-7}\text{ s}.$$

Let $x_1, x_2, t_1,$ and t_2 be the corresponding positions and times in S. Then

(Event 1)
$$x = x_1 = \gamma(x_1' + vt_1')$$
$$t = t_1 = \gamma\left(t_1' + \frac{v}{c^2}x_1'\right),$$

and

(Event 2)
$$x = x_2 = \gamma(x_2' + vt_2') = \gamma[x_1' + 100\text{ m} + v(t_1' + 3.33 \times 10^{-7}\text{ s})]$$
$$t = t_2 = \gamma\left(t_2' + \frac{v}{c^2}x_2'\right) = \gamma\left[t_1' + 3.33 \times 10^{-7}\text{ s} + \frac{v}{c^2}(x_1' + 100\text{ m})\right],$$

To apply these equations, we first compute

$$v = 0.900c = 0.900(3.00 \times 10^8 \text{ m/s}) = 2.70 \times 10^8 \text{ m/s}$$
$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-(0.900)^2}} = 2.29.$$

Then, in the Earth's frame, the length of time it takes for the signal to go from the tail to the head of the space craft is

$$\Delta t = t_2 - t_1 = \gamma\left(t_1' + 3.33 \times 10^{-7}\text{ s} + \frac{v}{c^2}(x_1' + 100\text{ m})\right) - \gamma\left(t_1' + \frac{v}{c^2}x_1'\right) = \gamma\left(3.33 \times 10^{-7}\text{ s} + \frac{v}{c^2}(100\text{ m})\right)$$
$$= 2.29\left(3.33 \times 10^{-7}\text{ s} + \frac{2.70 \times 10^8 \text{ m/s}}{(3.00 \times 10^8 \text{ m/s})^2}(100\text{ m})\right) = 1.45 \times 10^{-6}\text{ s}.$$

c) The answers to a) and b) are not related by the time dilation equation, since the two events do not occur at the same place in either reference frame.

d) According to the spacecraft clocks, it takes another $\frac{100\text{ m}}{c}$ for the reflected signal to get back to the tail of the spacecraft, so the total elapsed time on spacecraft clocks is $\frac{200\text{ m}}{c} = 6.67 \times 10^{-7}\text{ s}$.

In order to find this time on Earth based clocks, we'll define Event 3 to be the arrival of the reflected light at the tail of spacecraft, and use the inverse Lorentz transformation to compute its coordinates in the Earth's frame. In S', the coordinates of Event 3 are

(Event 3) $\qquad x' = x_3' = x_1' \qquad$ and $\qquad t_3' = t_1' + 6.67 \times 10^{-7}\text{ s}.$

Apply the inverse Lorentz transformation to find

(Event 3)
$$x = x_3 = \gamma(x_3' + vt_3') = \gamma[x_1' + v(t_1' + 6.67 \times 10^{-7}\text{ s})]$$
$$t = t_3 = \gamma\left(t_3' + \frac{v}{c^2}x_3'\right) = \gamma\left[t_1' + 6.67 \times 10^{-7}\text{ s} + \frac{v}{c^2}x_1'\right],$$

The length of time on the Earth based clocks in S that it takes for the light signal to go from the tail to the head and then reflect back to the tail of the spacecraft is

$$\Delta t = t_3 - t_1 = \gamma\left(t_1' + 6.67 \times 10^{-7}\text{ s} + \frac{v}{c^2}x_1'\right) - \gamma\left(t_1' + \frac{v}{c^2}x_1'\right) = \gamma(6.67 \times 10^{-7}\text{ s})$$
$$= 2.29(6.67 \times 10^{-7}\text{ s}) = 1.53 \times 10^{-6}\text{ s}.$$

Notice that we did not need the full power of the Lorentz transformation to find this Δt. Because Events 1 and 3 both occur at the same position in S', it turns out that we could have just used "time dilation:"

$$\Delta t = \gamma \Delta t' = 2.29(6.67 \times 10^{-7}\text{ s}) = 1.53 \times 10^{-6}\text{ s}.$$

The extra work is not lost, however. We will need the position equations in part e) below.

e) The total distance traversed by the light according to you inside the spacecraft is 200 m.

Using the position equations from part b), the distance traveled by the light signal from the tail to the head of the space craft as viewed from S is

$$\Delta x = x_2 - x_1 = \gamma[x_1' + 100\text{ m} + v(t_1' + 3.33 \times 10^{-7}\text{ s})] - \gamma(x_1' + vt_1') = \gamma[100\text{ m} + v(3.33 \times 10^{-7}\text{ s})]$$
$$= 2.29[100\text{ m} + (2.70 \times 10^8\text{ m/s})(3.33 \times 10^{-7}\text{ s})] = 435\text{ m}.$$

Using the position equation from part d), the distance traveled by the reflected light signal from the head to the tail of the space craft as viewed from S is

$$\Delta x = |x_3 - x_2| = |\gamma[x_1' + v(t_1' + 6.67 \times 10^{-7}\text{ s})] - \gamma[x_1' + 100\text{ m} + v(t_1' + 3.33 \times 10^{-7}\text{ s})]|$$
$$= \gamma|v(3.34 \times 10^{-7}\text{ s}) - 100\text{ m}| = 2.29|(2.70 \times 10^8\text{ m/s})(3.34 \times 10^{-7}\text{ s}) - 100\text{ m}| = 23\text{ m}$$

Hence the total distance traveled is $435\text{ m} + 23\text{ m} = 458\text{ m}$.

9. A politician of mass 70.0 kg is ejected from office and sent at a speed of 0.980c to a planet located 20.0 LY from Earth.
(a) What is the kinetic energy of the politician according to us on Earth?
(b) What is the kinetic energy of the politician according to the politician?
(c) How many years (measured with clocks at rest on the Earth) will the one-way trip take?
(d) According to the politician, how far away will the Earth be when the trip is completed (but the politician is still moving at the same speed)?
(e) How many years will the trip take according to clocks at rest with respect to the politician?

Solution:

a) The kinetic energy is defined as

$$\mathrm{KE} = (\gamma - 1)mc^2.$$

So we'll start (as usual in these problems!) by computing the relativistic factor γ.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - (0.980)^2}} = 5.0.$$

Then

$$\mathrm{KE} = (\gamma - 1)mc^2 = (5.0 - 1)(70.0\ \mathrm{kg})(3.00 \times 10^8\ \mathrm{m/s})^2 = 2.5 \times 10^{19}\ \mathrm{J}.$$

b) The politician is at rest, in his own frame, so according to him his kinetic energy is 0 J.

c) For clocks on Earth the trip time is the distance divided by the speed. When light years are the unit of distance, it is most convenient to use c as the unit of speed, and years as the unit of time. In these units, $c = 1$ LY/y. Thus the time required is

$$\frac{\text{distance}}{\text{speed}} = \frac{20.0\ \text{LY}}{0.980c} = \frac{20.0}{0.980}\ \text{y} = 20.4\ \text{y}.$$

d) According to the politician, the distance between the Earth and the planet is contracted to

$$\ell = \frac{\ell_0}{\gamma} = \frac{20.0\ \text{LY}}{5.0} = 4.0\ \text{LY}.$$

Hence, on arrival at the planet, the Earth is only 4.0 LY away — according to the politician.

e) The politician measures the length of time for the trip to be

$$\frac{\text{distance}}{\text{speed}} = \frac{4.0\ \text{LY}}{0.980c} = 4.1\ \text{y}.$$

In summary, the politician concludes that he traveled 4.0 LY in 4.1 y, but Earth bound creatures believe the journey was 20.0 LY, and that it took the politician 20.4 y.

Suppose the politician quickly turns around and comes home at the same speed — perhaps by using his (completely elastic) head to bounce off the planet with a completely elastic collision, and then firing off a rocket or two to make up for any momentum imparted to the planet. Then when the politician arrives back on Earth, he will have aged only 8.2 y, while his Earth bound opposition will be 40.8 y older.

There are a number of subtleties involved here, which often are discussed as "the twin paradox." See, for example:

Gerald Holton, "AAPT Resource Letter SRT on Special Relativity Theory, *American Journal of Physics, 30*, #6, pages 462-469 (June 1962);

Richard A. Muller, "The Twin Paradox in Special Relativity," *American Journal of Physics, 40*, #7, pages 966-969 (July, 1972);

Margaret Stautberg Greenwood, "Use of Doppler-shifted light beams to measure time during acceleration," *American Journal of Physics, 44*, #3, pages 259-263 (March, 1976);

Donald E. Hall, "Intuition, Time Dilation and the Twin Paradox," *The Physics Teacher, 16*, #4, pages 209-215 (April, 1978); and

Robert H. Good, "Uniformly accelerated reference frames and [the] twin paradox," *American Journal of Physics, 50*, #3, pages 232-238 (March, 1982).

Also, note that there are many *experiments* confirming time dilation and the twin "paradox" (which, of course, isn't *really* a paradox). See, for example, the discussions in:

Vernon D. Barger and Martin G. Olsson, *Classical Mechanics: A Modern Perspective*, (McGraw-Hill, New York, 1995), pages 350-353;

John R. Taylor and Chris D. Zafiratos, *Modern Physics for Scientists and Engineers*, (Prentice-Hall, New York, 1991), pages 25-26 and 40-41 ; and

Ralph Baierlein, *Newton to Einstein: The Trail of Light* (Cambridge University Press, New York, 1992), pages 274-282.

10. A light source at point P, as shown in the figure below, is turned on at time $t = 0$ s for a very short interval and then extinguished.

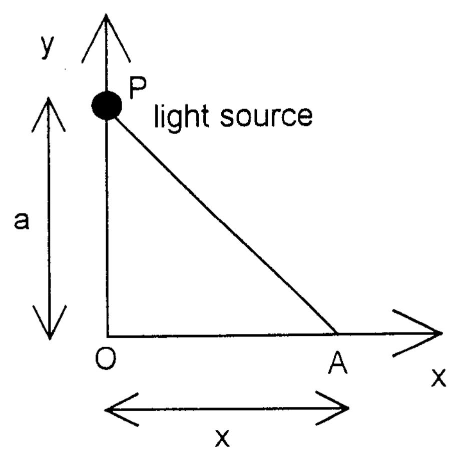
(a) At what time does the light arrive at the origin O?

(b) At what time does the light arrive at a point A located a distance x from the origin along the x-axis?

(c) The spot of light moves from point O to point A in a time given by the difference between the answer to part (b) and the answer to part (a). Show that the average speed $\langle v \rangle$ of the light spot in moving along the x-axis from O to A is

$$\langle v \rangle = c\frac{a + \sqrt{a^2 + x^2}}{x}.$$

(d) Show that when x is small, $\langle v \rangle \to \infty$, and when $x \to \infty$, $\langle v \rangle \to c$. The spot of light therefore, begins to move out along the x-axis at a speed *greater* than the speed of light and *slows down* to the speed of light as $x \to \infty$. Does this result contradict the special theory of relativity? Explain.



Solution:

a) The time required for light to travel from P to the origin is the distance divided by the speed, so $t = \dfrac{a}{c}$.

This is the time that light first appears at the origin.

b) The time required to travel from P to the point $(x, 0\text{ m})$ is the distance divided by the speed, so

$$t = \frac{\sqrt{x^2 + a^2}}{c}.$$

This is the time that light first appears at the point $(x, 0\text{ m})$.

c) The phenomenon of light appearing at a point on the x-axis first occured at time $\dfrac{a}{c}$ at the origin. It occurs at the point $(x, 0\text{ m})$ at time $\dfrac{\sqrt{x^2+a^2}}{c}$. The average speed $\langle v \rangle$ of this phenomenon is the distance traveled, $x - 0\text{ m}$, divided by the time it took to travel, $\dfrac{\sqrt{x^2+a^2}}{c} - \dfrac{a}{c}$, so

$$\begin{aligned}
\langle v \rangle &= \frac{x - 0\text{ m}}{\dfrac{\sqrt{x^2+a^2}}{c} - \dfrac{a}{c}} \\
&= \frac{xc}{\sqrt{x^2+a^2} - a} \\
&= \frac{xc}{\sqrt{x^2+a^2} - a}\left(\frac{\sqrt{x^2+a^2}+a}{\sqrt{x^2+a^2}+a}\right) \\
&= \frac{xc(\sqrt{x^2+a^2}+a)}{x^2 + a^2 - a^2} \\
&= c\frac{\sqrt{x^2+a^2}+a}{x}.
\end{aligned}$$

d) In part c) we found that the average speed $\langle v \rangle$ is

(1) $$\langle v \rangle = c\frac{\sqrt{x^2+a^2}+a}{x}.$$

Since $\sqrt{x^2+a^2} > \sqrt{x^2} = x$, this implies that

(2) $$\langle v \rangle > c\frac{x+a}{x} = c + \frac{ac}{x},$$

Thus $\langle v \rangle > c$ for *all* values of x, and $\lim_{x \to 0\text{ m}} \langle v \rangle = \infty\text{ m/s}$.

Here, apparently, is something whose speed is greater than the speed of light! But what precisely is it that has average speed $\langle v \rangle$? It is the phenomenon described by "light has arrived at the point $(x, 0\text{ m})$." This phenomenon has neither mass nor energy associated with it, so there is no violation of the special theory of relativity.

We were also asked to show that as $x \to \infty\text{ m}$, the average speed $\langle v \rangle$ approaches c. From Equation (1) we have,

$$\langle v \rangle = c\frac{\sqrt{x^2+a^2}+a}{x} = c\left(\sqrt{1 + \frac{a^2}{x^2}} + \frac{a}{x}\right).$$

As $x \to \infty\text{ m}$ the terms $\dfrac{a^2}{x^2}$ and $\dfrac{a}{x}$ both go to 0, so we have

$$\lim_{x\to\infty\text{ m}} \langle v \rangle = \lim_{x\to\infty\text{ m}} c\left(\sqrt{1 + \frac{a^2}{x^2}} + \frac{a}{x}\right) = c\left(\sqrt{1+0} + 0\right) = c,$$ as was to be shown.