PHY 140Y – FOUNDATIONS OF PHYSICS
2001-2002
Problem Set #3 – Solutions

HANDED OUT: Friday, November 2, 2001 (in class).

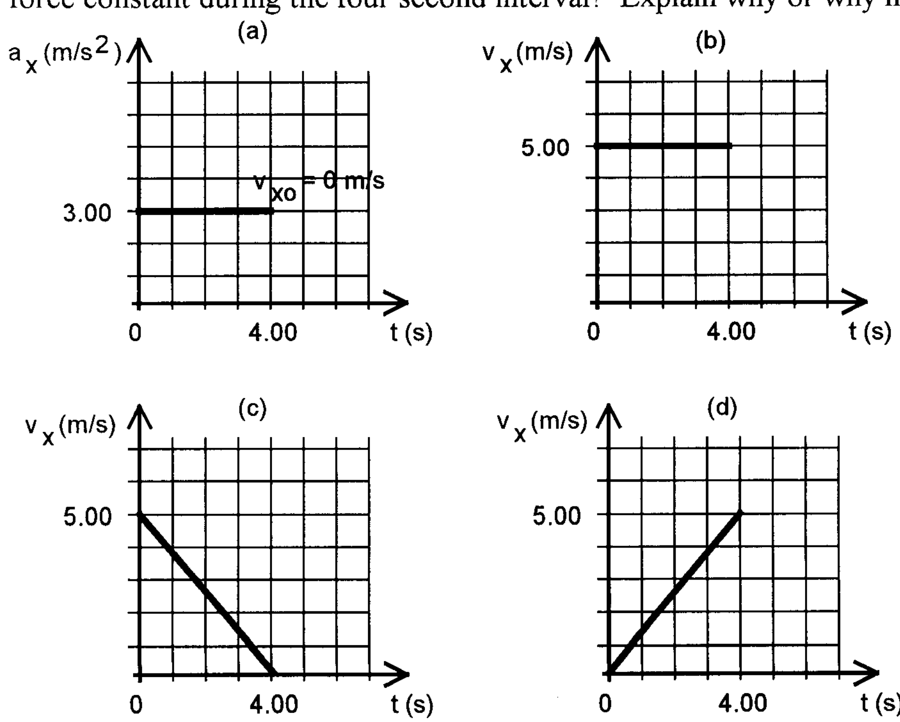
DUE: **5:00 PM, Thursday, November 15, 2001** in the appropriate box, labeled by tutorial group, in the basement at the bottom of the stairs leading down from MP202.

LATE PENALTY: 5 marks/day (which also applies to weekend days!) until 1:00 PM, Monday, November 19, after which it will not be accepted as solutions will then be available in tutorials and on the WWW.

NOTES: Answer all questions. A selected subset (3-4) will be marked out of 100%. Marks will be given for workings and units, as well as for final answers.

QUESTIONS:

1. Each graph in the figure below describes the one-dimensional motion of a 5.00 kg system during a 4.00 s interval. For each case, answer the following questions:
   (a) What is the work done on the system by the total force on the system during the interval?
   (b) Find an expression for the kinetic energy as a function of time that is valid during the four second interval.
   (c) What is the average power of the total force on the system?
   (d) Find P(t) at any instant during the four second interval. Is the instantaneous power of the total force constant during the four second interval? Explain why or why not.



Solution:

Case a.

a) The acceleration is constant during the interval. The position of the system is

$$x(t) = x_0 + v_{x0}t + a_x\frac{t^2}{2} = x_0 + (3.00\ \text{m/s}^2)\frac{t^2}{2}.$$

When $t = 4.00$ s

$$x = x_0 + (3.00\ \text{m/s}^2)\frac{(4.00)^2}{2} = x_0 + 24.0\ \text{m}.$$

So the change in the position vector of the system is

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_f - \vec{\mathbf{r}}_i = (x_0 + 24.0\ \text{m})\hat{\mathbf{i}} - x_0\hat{\mathbf{i}} = 24.0\ \text{m}\,\hat{\mathbf{i}}.$$

Since the acceleration component is constant, Newton's second law implies the total force component is constant, and

$$F_{x\ \text{total}} = ma_x = (5.00\ \text{kg})(3.00\ \text{m/s}^2) = 15.0\ \text{N}.$$

The work done by the constant force is

$$W = \vec{\mathbf{F}} \bullet \Delta\vec{\mathbf{r}} = (15.0\ \text{N})\hat{\mathbf{i}} \bullet (24.0\ \text{m})\hat{\mathbf{i}} = 360\ \text{J}.$$

b) The velocity component is

$$v_x(t) = v_{x0} + a_x t = (3.00\ \text{m/s}^2)t.$$

Since the motion is one-dimensional, $v = |v_x|$, so the kinetic energy is

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}(5.00\ \text{kg})[(3.00\ \text{m/s}^2)t]^2 = (22.5\ \text{J/s}^2)t^2.$$

c) In problem 8.13, we found the work done by the total force during the 4.00 s interval was 360 J. Hence the average power during this interval is

$$P_{\text{ave}} = \frac{W}{\Delta t} = \frac{360\ \text{J}}{4.00\ \text{s}} = 90.0\ \text{W}.$$

d) The instantaneous power of the total force is

$$P(t) = \frac{d}{dt}\text{KE}(t) = (22.5\ \text{J / s}^2)(2t) = (45.0\ \text{W/s})t.$$

The instantaneous power is not constant — even though the force is constant — because the velocity of the system changes with time.

Case b.

a) The velocity component is constant, which implies the acceleration component is zero. Newton's second law then implies the total force component also is zero. Therefore, the work done by the total force is 0 J.

b) The velocity component is constant during the interval. The kinetic energy is

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}(5.00\ \text{kg})(5.00\ \text{m/s})^2 = 62.5\ \text{J}.$$

c) The total force on the system is zero, since the velocity is constant (meaning zero acceleration). The average power of the total force is zero since no work is done.

d) The instantaneous power of the total force is the time rate of change of the kinetic energy. But the kinetic energy is constant, so

$$P(t) = \frac{d}{dt}\text{KE}(t) = 0 \text{ W}.$$

The power of the total force is constant and equal to zero during the 4.00 s interval.

Case c.

a) Since the velocity component has a constant slope, the acceleration component is constant. To find the acceleration component, use the one-dimensional kinematics equation for constant acceleration,

$$v_x(t) = v_{x0} + a_x t = 5.00 \text{ m/s} + a_x t.$$

When $t = 4.00$ s, the velocity component is zero, so

$$0 \text{ m/s} = 5.00 \text{ m/s} + a_x(4.00 \text{ s}) \implies a_x = -1.25 \text{ m/s}^2.$$

The position of the system at any time is

$$x(t) = x_0 + (5.00 \text{ m/s})t - (1.25 \text{ m/s}^2)\frac{t^2}{2}.$$

When $t = 4.00$ s, the x-coordinate is

$$x(t) = x_0 + (5.00 \text{ m/s})(4.00 \text{ s}) - (1.25 \text{ m/s}^2)\frac{(4.00 \text{ s})^2}{2} = x_0 + 10.0 \text{ m}.$$

Thus the change in the position vector of the system is

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_f - \vec{\mathbf{r}}_i = (x_0 + 10.0 \text{ m})\hat{\mathbf{i}} - x_0\hat{\mathbf{i}} = (10.0 \text{ m})\hat{\mathbf{i}}.$$

Since the acceleration component is constant, Newton's second law implies the total force component is constant, so

$$F_{x \text{ total}} = ma_x = (5.00 \text{ kg})(-1.25 \text{ m/s}^2) = -6.25 \text{ N}.$$

The work done by the force is

$$W = \vec{\mathbf{F}} \bullet \Delta\vec{\mathbf{r}} = (-6.25 \text{ N})\hat{\mathbf{i}} \bullet (10.0 \text{ m})\hat{\mathbf{i}} = -62.5 \text{ J}.$$

b) The velocity component at any instant was found in problem 8.13 to be

$$v_x(t) = 5.00 \text{ m/s} - (1.25 \text{ m/s}^2)t.$$

Since the motion is one-dimensional, $v = |v_x|$, so the kinetic energy is

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}(5.00 \text{ kg})[5.00 \text{ m/s} - (1.25 \text{ m/s}^2)t]^2.$$

c) The work done by the total force was found in problem 8.13 to be -62.5 J. Hence the average power of the total force is

$$P_{\text{ave}} = \frac{W}{\Delta t} = \frac{-62.5 \text{ J}}{4.00 \text{ s}} = -15.6 \text{ W}.$$

d) The instantaneous power of the total force is

$$P(t) = \frac{d}{dt}\text{KE}(t) = (5.00 \text{ kg})[5.00 \text{ m/s} - (1.25 \text{ m/s}^2)t](-1.25 \text{ m/s}^2) = -31.3 \text{ W} + (7.81 \text{ W/s})t.$$

Although the total force is constant, the instantaneous power is not constant because the velocity changes.

Case d.

a) Since the velocity component has a constant slope, the acceleration component is constant. To find the acceleration component, use the one dimensional kinematics equation for constant acceleration,

$$v_x(t) = v_0 + a_x t = 0 \text{ m/s} + a_x t = a_x t.$$

When $t = 4.00$ s, the velocity component is 5.00 m/s, so

$$5.00 \text{ m/s} = a_x(4.00 \text{ s}) \implies a_x = 1.25 \text{ m/s}^2.$$

The position of the system at any time is

$$x(t) = x_0 + (0 \text{ m/s})t + (1.25 \text{ m/s}^2)\frac{t^2}{2}.$$

So, when $t = 4.00$ s,

$$x = x_0 + (0 \text{ m/s})(4.00 \text{ s}) + (1.25 \text{ m/s}^2)\frac{(4.00 \text{ s})^2}{2} = x_0 + 10.0 \text{ m}.$$

Therefore, the change in the position vector of the system is

$$\Delta\vec{\text{r}} = \vec{\text{r}}_\text{f} - \vec{\text{r}}_\text{i} = (x_0 + 10.0 \text{ m})\hat{\text{i}} - x_0\hat{\text{i}} = (10.0 \text{ m})\hat{\text{i}}.$$

Since the acceleration component is constant, Newton's second law implies the total force component is constant, and

$$F_{x \text{ total}} = ma_x = (5.00 \text{ kg})(1.25 \text{ m/s}^2) = 6.25 \text{ N}.$$

The work done by the constant force is

$$W = \vec{\text{F}} \bullet \Delta\vec{\text{r}} = (6.25 \text{ N})\hat{\text{i}} \bullet (10.0 \text{ m})\hat{\text{i}} = 62.5 \text{ J}.$$

b) The velocity component at any instant was found in problem 8.13 to be

$$v_x(t) = v_{x0} + a_x t = (1.25 \text{ m/s}^2)t.$$

Since the motion is one-dimensional, $v = |v_x|$, so the kinetic energy is

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}(5.00 \text{ kg})[(1.25 \text{ m/s}^2)t]^2 = (3.91 \text{ J/s}^2)t^2.$$

c) The work done by the total force during the 4.00 s interval was found in problem 8.13 to be 62.5 J. Hence the average power of the total force is

$$P_\text{ave} = \frac{W}{\Delta t} = \frac{62.5 \text{ J}}{4.00 \text{ s}} = 15.6 \text{ W}.$$

d) The instantaneous power of the total force is

$$P(t) = \frac{d}{dt}\text{KE}(t) = 2(3.91 \text{ J / s}^2)t = (7.82 \text{ W/s})t.$$

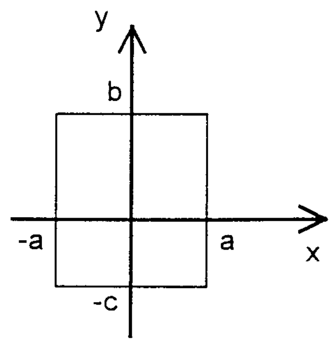
Although the total force is constant, the power of the total force is not constant because the velocity of the system changes.

2. A force $\vec{F} = (2.00\text{N})\hat{i} + (1.00\text{N})\hat{j}$ is one force on a system as it executes the rectangular path shown in the figure below.

(a) Find the work done on the system by this force if the path is traversed in the clockwise sense.

(b) Find the work done on the system by this force if the path is traversed in the counterclockwise sense.

(c) Is this force conservative or nonconservative? Explain.



Solution:

Notice that the force is a constant force. Therefore, we can calculate its work using $W = \vec{F} \bullet \Delta\vec{r}$.

a) and

b) After the system traverses the entire rectangular path, the change in the position vector of the system is $\Delta\vec{r} = 0$ m, regardless of whether the path is traversed in a clockwise or counterclockwise sense. Hence, the work done by the force is 0 J. Notice this is true for any *closed* path — it needn't be rectangular.

c) Since the work done by the force around any closed path is zero, the force is a conservative force — any constant force is a conservative force.

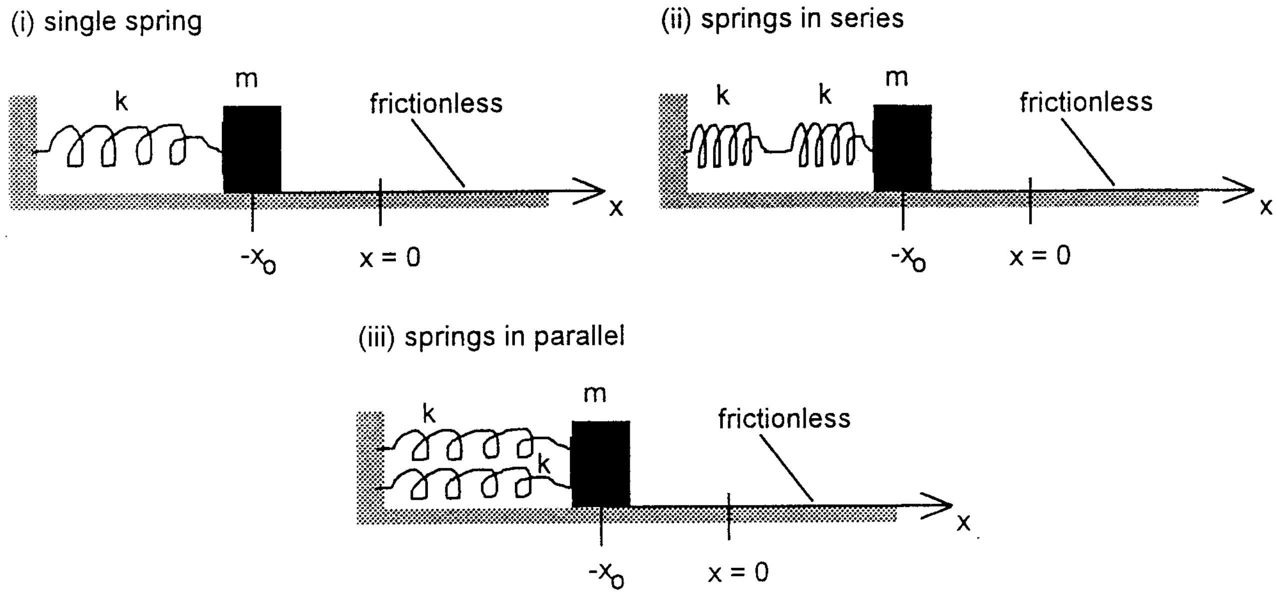
3. A system of two identical springs, each with spring constant k, is used to accelerate a mass m from rest to as great a speed as possible on a frictionless surface. Various arrangements of the springs are possible: (i) use just one spring; (ii) use two springs in series; or (iii) use two springs in parallel, as shown in the figure below. Regardless of which design is used, the spring system can be set the same distance x_o from the equilibrium position of mass m.

(a) Find the maximum potential energy of m in each spring system.

(b) Find an expression for the maximum speed of m as it leaves each system.

(c) To give m the greatest possible speed, which arrangement should be used (or does it not make any difference)?

Hint: The effective spring constant of two identical springs, each having spring constant k, is k/2 when they are attached in series, and is 2k when they are connected in parallel.



Solution:

The effective spring constant of two identical springs, each with spring constant k, when attached in series is

$$k_{\text{series}} = \frac{k}{2}.$$

If they are connected in parallel, the effective spring constant is

$$k_{\text{parallel}} = 2k.$$

a) For a single spring compressed to the point $-x_0$, the potential energy is

$$\text{PE}_{\text{single}} = \frac{1}{2}k(-x_0)^2 = \frac{1}{2}k(x_0)^2.$$

For two springs in series, compressed to the point $-x_0$,

$$\text{PE}_{\text{series}} = \frac{1}{2}k_{\text{series}}(-x_0)^2 = \frac{1}{2}\frac{k}{2}(-x_0)^2 = \frac{1}{4}kx_0^2.$$

For two springs in parallel, compressed to the point $-x_0$,

$$\text{PE}_{\text{parallel}} = \frac{1}{2}k_{\text{parallel}}(-x_0)^2 = \frac{1}{2}2kx_0^2 = kx_0^2.$$

b) To find the speed of the mass as it leaves the spring systems, use the CWE theorem. Take the initial position of m to be where it is at rest with the spring compressed and the final position where m leaves the spring. The forces on the mass are

1. its weight $\vec{\mathbf{w}}$, which here does zero work since the motion is perpendicular to $\vec{\mathbf{w}}$ — equivalently, there is no change in the gravitational potential energy of m since it does not change elevation;
2. the normal force $\vec{\mathbf{N}}$ of the surface on m — this force also does zero work since it always is perpendicular to the path of the mass; and
3. the force $\vec{\mathbf{F}}_{\text{spring}}$ of the spring on the mass — the work done by this force is accounted for by the change in the associated potential energy function.

There are no nonconservative forces, so there is no work done by them. Thus the CWE theorem becomes

$$0\text{ J} = W_{\text{nonconservative}} = \Delta(\text{KE}+\text{PE}) = (\text{KE}_f - \text{PE}_f) - (\text{KE}_i - \text{PE}_i) = \left(\frac{1}{2}mv^2 + 0\text{ J}\right) - (0\text{ J} + \text{PE}_{\text{spring}})$$

$$\Longrightarrow v = \sqrt{\frac{2\text{PE}_{\text{spring}}}{m}}$$

Now use the potential energies of found in part a) for each of the arrangements.

$$v_{\text{single}} = \sqrt{\frac{2\left(\frac{1}{2}kx_0^2\right)}{m}} = \sqrt{\frac{k}{m}}\,x_0$$

$$v_{\text{series}} = \sqrt{\frac{2\left(\frac{1}{4}kx_0^2\right)}{m}} = \sqrt{\frac{k}{2m}}\,x_0$$

$$v_{\text{parallel}} = \sqrt{\frac{2(2kx_0^2)}{m}} = \sqrt{\frac{4k}{m}}\,x_0$$

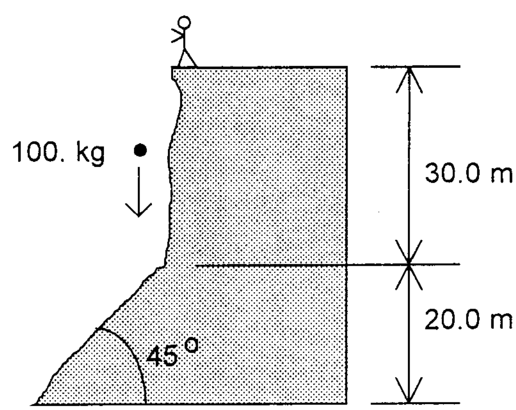
c) The expressions derived in part b) show that the greatest speed is secured with the parallel spring arrangement. In fact

$$v_{\text{parallel}} > v_{\text{single}} > v_{\text{series}}.$$

4. A hiker pushes a 100. kg boulder off a cliff as pictured in the figure below (not a smart idea!). The rock falls vertically for 30.0 m and then slides down a talus (gravel) slope. At the bottom of the slope and at a speed of 2.00 m/s, the rock passes some astonished park rangers who subsequently set off in hot pursuit of the hiker.

(a) How much work was done by the kinetic frictional force as the rock slid down the slope to the point where it passed the rangers?

(b) Assuming the kinetic frictional force is of constant magnitude, find the coefficient of kinetic friction for the boulder on the slope. In your answer, include a force diagram for the rock on the slope.



Solution:

a) Use the CWE theorem. Take the initial position to be the location of the rock as it begins its vertical descent and the final position where the rock passes the rangers. The work done by the nonconservative force of kinetic friction is equal to the change in the total mechanical energy of the rock.

$$W_{\text{nonconservative}} = \Delta(\text{KE} + \text{PE}) = (\text{KE}_f + \text{PE}_f) - (\text{KE}_i + \text{PE}_i)$$

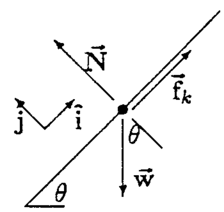
Let $\hat{j}$ point straight up and choose the origin at the height where the rock passes the rangers. The appropriate form for the potential energy function is mgy, since the entire motion occurs close to the Earth's surface. Hence,

$$W_{\text{nonconservative}} = \left(\frac{1}{2}mv_f^2 + mg(0\text{ m})\right) - \left(\frac{1}{2}m(0\text{ m/s}) + mgy_i\right)$$
$$= \frac{1}{2}(100\text{ kg})(2.00\text{ m/s})^2 - (100\text{ kg})(9.81\text{ m/s}^2)(50.0\text{ m}) = -4.89\times 10^4\text{ J}.$$

b) As the rock slides down the talus slope, the forces on the rock are:

1. its weight $\vec{w}$, of magnitude mg directed down;
2. the normal force $\vec{N}$ of the surface on the rock; directed perpendicularly out of the surface; and
3. the force of kinetic friction $\vec{f}_k$, directed opposite to the motion of the rock.

Here's a second law force diagram and a convenient coordinate system. (The rock is the little black dot in the middle of the picture.)



There is no acceleration in the $\hat{j}$ direction, so the total force in that direction is zero.

$$F_{y\text{ total}} = 0\text{ N} \implies N - mg\cos\theta = 0\text{ N} \implies N = mg\cos\theta.$$

Therefore the magnitude of the kinetic force of friction on the rock is

$$f_k = \mu_k N = \mu_k mg\cos\theta.$$

The kinetic force of friction is a constant force along the talus slope. The work done by the force is $W = \vec{f}_k \bullet \Delta\vec{r}$. The force $\vec{f}_k$ is directed opposite to the velocity of the rock and so is opposite to $\Delta\vec{r}$. Therefore

$$W = f_k\Delta r\cos 180° = -f_k\Delta r = (-\mu_k mg\cos\theta)\Delta r \implies \mu_k = -\frac{W}{(mg\cos\theta)\Delta r}.$$

The magnitude Δr is the length of the of the talus slope. From the geometry, we have

$$\Delta r = \frac{20\text{ m}}{\sin 45°} = 28\text{ m}.$$

Make the substitutions into the expression for μ_k, recognizing that W is the work done by the nonconservative force that we found from the CWE theorem.

$$\mu_k = -\frac{-4.89\times 10^4\text{ J}}{(100\text{ kg})(9.81\text{ m/s}^2)(\cos 45°)(28\text{ m})} = 2.5.$$

5. A popular recreation for the fearless is jumping from tall structures while attached to an elastic bungee cord to ensure that less than catastrophic consequences result from the fall. Model the bungee cord as an ideal spring. A student of mass m drops off a tower of height h, tethered to this bungee cord. The elasticity of the bungee cord is chosen so that its effective spring constant is k = 2mg/h. The jump is begun at zero speed at altitude h with the bungee cord fastened vertically above the jumper, barely taut (with no slack) but under no initial tension.

(a) Choose a suitable coordinate system to analyze this problem.
(b) What is the initial total mechanical energy of the bungee jumper?
(c) What is the kinetic energy of the jumper at the ground? What is the jumper's speed at this point?
(d) At what height above the ground is the speed of the jumper a maximum?
(e) Will the same bungee cord be safe to use for a jump from the same tower (of height h) for a person with mass m'<m? With mass m'>m? Explain your reasoning.

Solution:

a) Choose a coordinate system with $\hat{\jmath}$ pointing straight up and the origin on the ground.

b) The initial total mechanical energy is $E_i = \text{KE}_i + \text{PE}_{\text{gravity}} + \text{PE}_{\text{spring}}$. The student is initially at rest so the initial kinetic energy is zero; also, the spring is unstretched, so the initial potential energy of the spring is zero. The initial position of the student is at coordinate $y_i = h$. Hence, the initial total mechanical energy is

$$E_i = 0 \text{ J} + mgy_i + 0 \text{ J} = mgh.$$

c) There are no nonconservative forces acting. Hence according to the CWE theorem

$$0 \text{ J} = W_{\text{nonconservative}} = \Delta(\text{KE} + \text{PE}) = (\text{KE}_f + \text{PE}_f) - (\text{KE}_i + \text{PE}_i).$$

The quantity $(\text{KE}_i + \text{PE}_i)$ is the initial total mechanical energy, found in part b) above. When the student is on the ground, the gravitational potential energy is zero and the spring-like bungee cord is stretched a distance h from its unstretched position, Hence, the CWE theorem becomes

$$0 \text{ J} = \left(\text{KE}_f + 0 \text{ J} + \frac{1}{2}kh^2\right) - mgh \implies \text{KE}_f = mgh - \frac{1}{2}kh^2 = mgh - \frac{1}{2}\left(\frac{2mg}{h}\right)h^2 = 0 \text{ J}.$$

Therefore, the student arrives at the ground with zero speed. Whew!

d) Let y be the final height of the student above the ground, so the bungee cord is stretched a distance $h - y$. From the CWE theorem,

$$0 \text{ J} = (\text{KE}_f + \text{PE}_f) - (\text{KE}_i + \text{PE}_i) = \left(\text{KE}_f + mgy + \frac{1}{2}k(h-y)^2\right) - mgh$$

$$\implies \text{KE}_f = mg(h-y) - \frac{1}{2}k(h-y)^2.$$

The place where the student has maximum speed is also the place where the kinetic energy is a maximum. Hence, differentiate this expression for the kinetic energy with respect to y and set the result equal to zero.

$$\frac{d}{dy}\text{KE} = -mg + k(h-y) = 0 \text{ N} \implies y = h - \frac{mg}{k} = h - \frac{mg}{\left(\frac{2mg}{h}\right)} = \frac{h}{2}.$$

e) In part c) we found that

$$\mathrm{KE}_f = mgh - \frac{1}{2}kh^2$$

when the student reaches the ground. The spring constant k of the bungee cord was just the right value for this quantity to be zero at ground level. For a larger value of m, the equation implies that the kinetic energy would still be positive at ground level, so the student will still be moving when she hits the ground. Bump!

For smaller values of m, the kinetic energy of the student would be negative at ground level — which of course is impossible — so the student never actually reaches the ground. She must either let go of the cord at the bottom of her trip and fall the rest of the way (Bump!), or continue her simple harmonic motion (spring back up again).

6. (a) A system of mass m is traveling at a large initial speed v_o. A small amount of work ΔW_{total} is done by all the forces acting on the system. Begin with the Work-Energy Theorum in the form $W_{total} = \Delta$ KE and show that the change in the speed Δv of the system is approximately given by

$$\Delta v \approx \frac{\Delta W_{total}}{m v_o}.$$

Explain any assumptions made. Use this equation to estimate the change in the speed of a 10.0 kg mass initially travelling at 30.0 m/s when 1.00 J of work is done on it by the total force on the system.

(b) A system of mass m is undergoing vertical motion near the surface of the Earth with the gravitational force of the Earth the only force acting on the system. For such vertical motion near the surface of the Earth, the total mechanical energy of a system of mass m is

$$E = K + U = KE + PE = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + mgy$$

where y is postive upwards. The system has constant vertical (downward) acceleration and the position of the system at any time is

$$y(t) = y_o + v_{oy}t + \tfrac{1}{2}gt^2.$$

Use this in the expression for E to show that E is independent of time (that is, the total mechanical energy is conserved), and to find E in terms of y_o, v_{yo}, m, and g. The result should not be unexpected.

Solution:

a) Take the expression for kinetic energy and differentiate it with respect to v.

$$\frac{d}{dv}\mathrm{KE} = \frac{d}{dv}\left(\frac{mv^2}{2}\right) = mv.$$

Thus for small changes Δv in speed from an initial speed v_0, the corresponding change $\Delta \mathrm{KE}$ in kinetic energy is given approximately by

$$\Delta \mathrm{KE} \approx m v_0 \Delta v.$$

Solving for Δv and using the CWE theorem, $W_{\text{total}} = \Delta \mathrm{KE}$,

$$\Delta v \approx \frac{\Delta \mathrm{KE}}{m v_0} = \frac{W_{\text{total}}}{m v_0}.$$

With the given data

$$\Delta v \approx \frac{1.00 \text{ J}}{(10.0 \text{ kg})(30.0 \text{ m/s})} \approx 3.33 \times 10^{-3} \text{ m/s}.$$

b) Note that since $y = y_0 + v_{y0} t - \frac{1}{2} g t^2$,

$$v = \left| \frac{dy}{dt} \right| = |v_{0y} - gt|.$$

Hence the expression for E is

$$E = \mathrm{KE} + \mathrm{PE} = \frac{1}{2} m (v_{0y} - gt)^2 + mg \left(y_0 + v_{0y} t - g \frac{t^2}{2} \right)$$

When we expand the square and then simplify, this becomes

$$E = \frac{1}{2} m v_{0y}^2 + mgy_0 = \frac{1}{2} m v_{0y}^2 + mgy_0.$$

This is just the sum of the initial kinetic energy $\frac{1}{2} m v_{0y}^2$ and the initial potential energy mgy_0, and is independent of time.

7. A particle of mass m is dropped from rest from a height h in the uniform gravitational field near the surface of the Earth as shown in the figure below. For convenience, the indicated coordinate system has been chosen to analyze this problem.

(a) For review, starting with constant acceleration, find expressions for the y-component of velocity, $v_y(t)$, and the y coordinate of the particle, y(t). Also show that the time-of-flight of the particle is $T = \sqrt{\frac{2h}{g}}$.

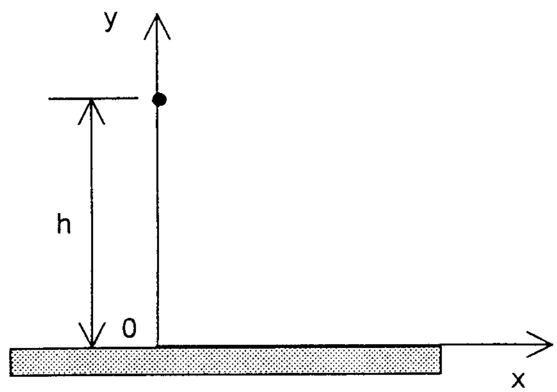
(b) The potential energy of the particle is time dependent. The following equation gives the average value of a time-dependent function f(t):

$$f_{ave} = \frac{1}{T} \int_0^T f(t)\, dt.$$

Use this equation to derive an equation for the time-average value of the potential energy during the time-of-flight T in terms of m, g, and h.

(c) The kinetic energy of the particle is also time dependent. Derive an equation for the time-average value of the kinetic energy of the particle during the time-of-flight T in terms of m, g, and h.

(d) How are PE_{ave} and K_{ave} related? *(This relationship is a consequence of a theorum in advanced mechanics known as the virial theorum.)*



Solution:

a) The acceleration is constant.

$$\vec{a} = a_y\hat{j} = -g\hat{j}.$$

Hence the velocity vector component is

$$v_y(t) = v_{y0} + a_y t = -gt,$$

and the position vector component is

$$y(t) = y_0 + v_{y0}t + a_y\frac{t^2}{2} = h - g\frac{t^2}{2}.$$

Impact occurs where $y = 0$ m, so at the time T of impact

$$0\text{ m} = h - g\frac{T^2}{2} \implies T = \sqrt{\frac{2h}{g}}.$$

b) The potential energy at any time t is $\quad \text{PE} = mgy(t) = mg\left(h - g\frac{t^2}{2}\right)$

Take the time average

$$\text{PE}_{\text{ave}} = \frac{1}{T}\int_{0\text{ s}}^{T} \text{PE}\,dt = \frac{1}{T}\int_{0\text{ s}}^{T} mg\left(h - g\frac{t^2}{2}\right)dt = \frac{mg}{T}\left(ht - g\frac{t^3}{6}\right)\Bigg|_{0\text{ s}}^{T} = mg\left(h - g\frac{T^2}{6}\right).$$

Therefore, substituting $\sqrt{\frac{2h}{g}}$ for T,

$$\text{PE}_{\text{ave}} = mg\left(h - g\frac{\left(\sqrt{\frac{2h}{g}}\right)^2}{6}\right) = \frac{2}{3}mgh.$$

c) The kinetic energy at any time is

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}m(-gt)^2 = \frac{1}{2}mg^2t^2.$$

Its time average is

$$\text{KE}_{\text{ave}} = \frac{1}{T}\int_{0\text{ s}}^{T} \text{KE}\,dt = \frac{1}{T}\int_{0\text{ s}}^{T} \frac{1}{2}mg^2t^2\,dt = \frac{m}{6T}g^2t^3\Bigg|_{0\text{ s}}^{T} = \frac{1}{6}mg^2T^2.$$

Therefore, substituting $\sqrt{\frac{2h}{g}}$ for T,

$$\text{KE}_{\text{ave}} = \frac{1}{6}mg^2\left(\sqrt{\frac{2h}{g}}\right)^2 = \frac{1}{3}mgh.$$

We've shown that $\text{KE}_{\text{ave}} = \frac{1}{2}\text{PE}_{\text{ave}}$. Although we've proved this just for the local gravitational force $\vec{F} = -mg\hat{j}$, it is true for any force with constant magnitude and direction — also known as a "uniform force."

8. A mass m is attached to a spring of spring constant k and is oscillating horizontally at angular frequency ω on a frictionless surface. The mass is released at rest when t = 0 s at the position x = A.

(a) What is the position x of the oscillator as a function of time?

(b) What is the velocity component $v = \frac{dx}{dt}$ as a function of time?

(c) What is the instantaneous power of the spring force when t = 0 s? Explain your answer.

(d) What is the instantaneous power of the spring force when the oscillator is at position x = 0 m? Explain your answer.

(The results of parts (c) and (d) imply that the oscillator reaches a peak power sometime during the first quarter of a complete oscillation.)

(e) Show that the instantaneous power P(t) of the oscillator force is $P(t) = \frac{k\omega A^2}{2}\sin(2\omega t)$.

(f) Show that the power is at a maximum for the first time when t is equal to one-eighth the period of oscillation.

(g) What is the average power of the oscillator force during one period of the oscillation?

(h) Make graphs of $F_{x,\,spring}(t)$, $v_x(t)$, and P(t) [the result of part (e)] versus time during one period of the oscillation if the mass is 1.00 kg, the spring constant is 49.0 N/m, and the amplitude of the oscillation is 0.100 m. What conclusions can be drawn from these graphs?

Solution:

Choose a coordinate system with $\hat{\mathbf{i}}$ pointing in the direction the spring is stretched, and with origin at the equilibrium position of the mass. So at any time, $x(t)$ is the position of the mass from its equilibrium position.

a) The position of the oscillator at any instant is $x(t) = A\cos(\omega t + \phi)$. When $t = 0$ s, the position is $x = A$, so when $t = 0$ s,

$$A = A\cos\phi \implies \cos\phi = 1 \implies \phi = 0 \text{ rad}.$$

Hence

$$x(t) = A\cos(\omega t).$$

b) The velocity component is

$$v_x(t) = \frac{d}{dt}x(t) = -A\omega\sin(\omega t).$$

c) When $t = 0$ s, the velocity $\vec{\mathbf{v}} = \mathbf{0}$ m/s, so at this time the instantaneous power of the force is

$$P = \vec{\mathbf{F}} \bullet \vec{\mathbf{v}} = \vec{\mathbf{F}} \bullet (\mathbf{0} \text{ m/s}) = 0 \text{ W}.$$

d) At the position $x = 0$ m, the force $\vec{\mathbf{F}} = \mathbf{0}$ N, so at this time the instantaneous power of the force is

$$P = \vec{\mathbf{F}} \bullet \vec{\mathbf{v}} = (\mathbf{0} \text{ N}) \bullet \vec{\mathbf{v}} = 0 \text{ W}.$$

e) The instantaneous power of the force is

$$P = \vec{\mathbf{F}} \bullet \vec{\mathbf{v}} = [-kx(t)]\hat{\mathbf{i}} \bullet \vec{\mathbf{v}}(t) = [-kA\cos(\omega t)]\hat{\mathbf{i}} \bullet [-A\omega\sin(\omega t)]\hat{\mathbf{i}} = k\omega A^2\cos(\omega t)\sin(\omega t) = \frac{k\omega A^2}{2}\sin(2\omega t).$$

f) The power is a maximum for the first time when the argument of the sine function is $\frac{\pi}{2}$ rad. That is, when

$$2\omega t = \frac{\pi}{2} \implies t = \frac{\pi}{4\omega}.$$

Since $\omega = 2\pi\nu$, and the period $T = \frac{1}{\nu}$, then

$$t = \frac{1}{8\nu} = \frac{T}{8}.$$

g) During one complete oscillation, the mass returns to its original position. The force on the oscillator is conservative, so the total work done by this force is zero over any closed path, in particular the path traversed during any period T. Hence, the average power of the total force over a whole period is 0 W.

h) With the given data, the angular frequency of the oscillation is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{49.0\ \text{N/m}}{1.00\ \text{kg}}} = 7.00\ \text{rad/s}.$$

The position at any time t is

$$x(t) = A\cos(\omega t) = (0.100\ \text{m})\cos[(7.00\ \text{rad/s})t].$$

The force component is

$$F_x(t) = -kx(t) = (-49.0\ \text{N/m})(0.100\ \text{m})\cos[(7.00\ \text{rad/s})t] = (-4.90\ \text{N})\cos[(7.00\ \text{rad/s})t].$$

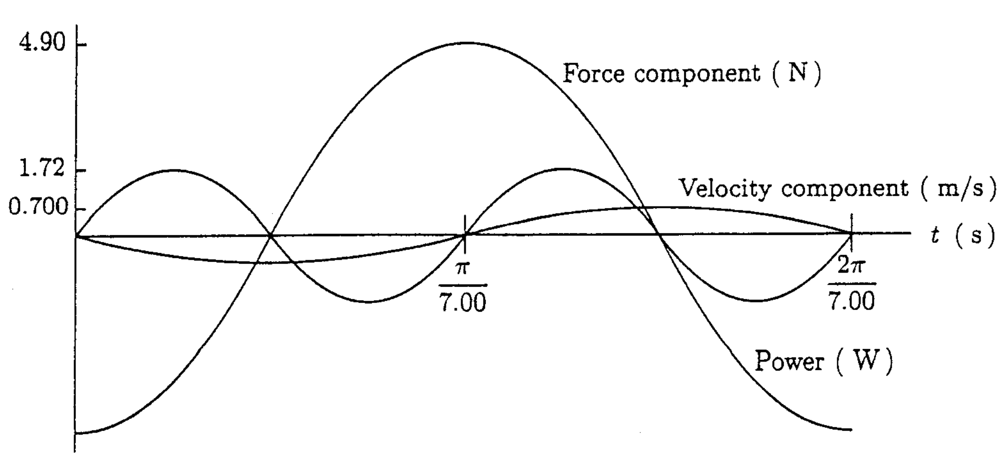
The velocity component is

$$v_x(t) = -A\omega\sin(\omega t) = (-0.100\ \text{m})(7.00\ \text{rad/s})\sin[(7.00\ \text{rad/s})t] = (-0.700\ \text{m/s})\sin[(7.00\ \text{rad/s})t].$$

The instantaneous power is

$$P(t) = \frac{k\omega A^2}{2}\sin(2\omega t) = \frac{(49.0\ \text{N/m})(7.00\ \text{rad/s})(0.100\ \text{m})^2}{2}\sin[2(7.00\ \text{rad/s})t]$$
$$= (1.72\ \text{W})\sin[(14.00\ \text{rad/s})t].$$

These functions, with the exception of $x(t)$, are plotted below. Note that the power oscillates at twice the frequency of the force and velocity components.



9. The potential energy function of a conservative force acting on a system moving in one dimension along the x-axis is described by U = PE = C |x|, where C is a positive constant. This conservative force is the only force on the system.

(a) What are the SI units of C?

(b) What is the x-component of the force on the system?

(c) Make a schematic graph of the potential energy as a function of x.

(d) Let the total mechanical energy E of the system be greater than zero. Sketch a representative value for E > 0 J on your graph of the potential energy function. Do turning points exist? If so, how many? Indicate prohibited values of x on the graph.

(e) Describe the motion of the system. Is the motion simple harmonic motion? In what ways is the motion similar to simple harmonic oscillation? In what ways is it different?

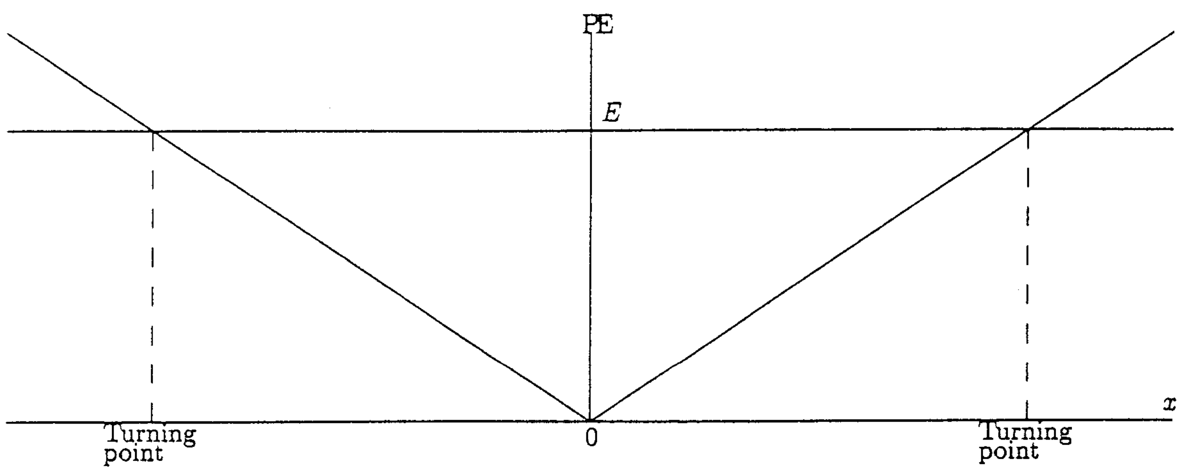
Solution:

a) Since the SI units of the potential energy are joules, and those of distance are meters, the SI units for the constant C must be J/m.

b) The x-component of the force acting on the system, here the only force component, is

$$F_x = -\frac{d}{dx}\text{PE} = \begin{cases} -C, & \text{for } x > 0\text{ m}, \\ C & \text{for } x < 0\text{ m}. \end{cases}$$

c) A graph of the potential energy as a function of x appears below. The absolute value of the slope of each line is C.



d) A particular value E of the total mechanical energy also is shown in the graph above. The potential energy cannot be greater than the total mechanical energy, since that would imply a negative kinetic energy (and the kinetic energy is intrinsically positive). Thus, there are two turning points, as indicated on the graph above. Values of $|x|$ greater than the turning points are prohibited, since for these coordinates the potential energy is greater than the total mechanical energy.

e) The motion of the particle is confined to coordinates x between the two turning points. The motion is oscillatory, but not simple harmonic oscillation since the potential energy function is not that of a simple harmonic oscillator — which has the form $\text{PE} = \frac{kx^2}{2}$.

For positive values of x, the force is directed toward $-\hat{\imath}$, and for negative values of x, the force is directed toward $\hat{\imath}$. In this respect the force is similar to a Hooke's law force. But unlike the Hooke's law force, the given force has a constant magnitude for all values of x.

10. A pendulum of mass m and length l makes an angle θ with the vertical direction.

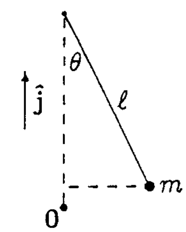
(a) Find an expression for the gravitational potential energy of m as a function of θ so that the potential energy is zero when θ = 0 rad.

(b) Make a sketch of this potential energy function in units of mgl as a function of θ over the range $-\pi/2$ rad $\leq \theta \leq \pi/2$ rad.

(c) Let PE_{max} be the maximum potential energy of m over this range of θ. Let the total mechanical energy E of m be constant and have a value of $E = PE_{max}/2$. What are the turning points of a pendulum with such a value for the total mechanical energy?

Solution:

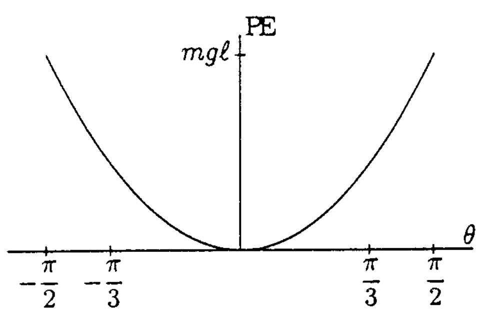
a) Here's the picture.



The gravitational potential energy is $\mathrm{PE} = mgy$. From the geometry, $y = \ell - \ell\cos\theta = \ell(1-\cos\theta)$. Therefore

$$\mathrm{PE} = mg\ell(1-\cos\theta).$$

b) Here's the graph of PE as a function of θ, for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.



c) The maximum value of the potential energy over the given range of θ occurs at $\theta = \pm\frac{\pi}{2}$. At these points $\cos\theta = 0$, so

$$\mathrm{PE}_{\text{max}} = mg\ell(1-0) = mg\ell.$$

From the statement of the problem, the total mechanical energy is

$$E = \frac{\mathrm{PE}_{\text{max}}}{2} = \frac{mg\ell}{2}.$$

The turning points are at those angles θ_{tp} where the potential energy is equal to the total mechanical energy. That is, where

$$E = \mathrm{PE} \implies \frac{mg\ell}{2} = mg(1-\cos\theta_{\text{tp}}) \implies \cos\theta_{\text{tp}} = \frac{1}{2} \implies \theta_{\text{tp}} = \pm\frac{\pi}{3}\text{ rad} = \pm 60^\circ.$$