PHY 140Y – FOUNDATIONS OF PHYSICS
2001-2002
Problem Set #2 – Solutions

HANDED OUT: Friday, October 5, 2001 (in class).

DUE: **5:00 PM, Thursday, October 18, 2001** in the appropriate box, labeled by tutorial group, in the basement at the bottom of the stairs leading down from MP202.

LATE PENALTY: 5 marks/day (which also applies to weekend days!) until 1:00 PM, Monday, October 22, after which it will not be accepted as solutions will then be available in tutorials and on the WWW.

NOTES: Answer all questions. A selected subset (3-4) will be marked out of 100%. Marks will be given for workings and units, as well as for final answers.

QUESTIONS:

1. A space station 120 m in diameter is set rotating in order to give its occupants "artificial gravity". Over a period of 5.0 minutes, small rockets bring the station steadily to its final rotation rate of 1 revolution every 20. seconds. What are the radial and tangential accelerations of a point on the rim of the station 2.0 minutes after the rockets start firing?

Solution:

If the rotation rate increases steadily from 0 to 1 revolution in 20 s, over a 5-minute interval, the rotation rate after 2 minutes is (2/5)(1 rev/20 s) = 1 rev/50 s, or the (instantaneous) period after 2 minutes is 50 s. Thus, by Equation 4-12, the centripetal (radial) acceleration is $a_c = 4\pi^2 r/T^2 = 4\pi^2(60\text{ m})/(50\text{ s})^2 = 0.947\text{ m/s}^2$. The tangential speed increases steadily from 0 to $2\pi(60\text{ m})/20\text{ s} = 6\pi\text{ m/s}$, in 5 minutes, so the tangential acceleration is $a_t = (6\pi\text{ m/s})/5\text{ min} = 6.28\times10^{-2}\text{ m/s}^2$.

2. You may have noticed that as a cassette tape is played, the radius of the tape left on the supply spool decreases slowly with time as the tape is drawn through the machine at a constant speed v_o. Because the radius of the tape remaining changes slowly compared with the speed of the tape through the machine, to a good approximation we can write:

$$v_o = r\frac{d\theta}{dt} = r\omega > 0$$

where ω is the angular speed of the tape reel. Notice that the tape speed v_o is constant, but radius r changes slowly with time. Therefore the angular speed of the reel also changes slowly with time. Assume that r represents the average radius of each coiled layer (since the tape is actually wound in a spiral). Let r_o be the initial radius of the tape when t = 0 s and $\theta_o = 0$ rad, where θ is the total angle of tape unwound from the spool. Represent the thickness of the tape by b. When the tape has unwound through a total angle θ, the radius of tape remaining is

$$r = r_o - b\frac{\theta}{2\pi}.$$

(a) Show that $\frac{dr}{dt} = -\frac{b}{2\pi}\frac{d\theta}{dt} = -\frac{b}{2\pi}\frac{v_o}{r}$.

(b) Integrate the result of part (a) to show that $\pi(r_o^2 - r^2) = bv_o t$.
What is the geometrical interpretation of the left-hand side of this equation? What is the physical interpretation of the right-hand side of this equation? Illustrate each with a sketch.

(c) Solve the result of part (b) for r and substitute this for r in Equation (1). Integrate the resulting expression to show that the total angle θ through which the tape has turned at time t is $$\theta = \frac{2\pi}{b}\left\{r_o - \sqrt{r_o^2 - \frac{bv_o t}{\pi}}\right\}.$$

Hint: The following integration formula may be useful: $\int \frac{-a}{\sqrt{b - at}}dt = 2\sqrt{b - at} + C$.

Solution:

a) Since r_0 and b are constants, the derivative is

$$\frac{dr}{dt} = \left(-\frac{b}{2\pi}\right)\left(\frac{d\theta}{dt}\right) = \left(-\frac{b}{2\pi}\right)\omega = \left(-\frac{b}{2\pi}\right)\frac{v_0}{r}.$$

b) For purposes of integration, rewrite

$$\frac{dr}{dt} = -\frac{b}{2\pi}\frac{v_0}{r}$$

as

$$r\,dr = -\frac{bv_0}{2\pi}dt.$$

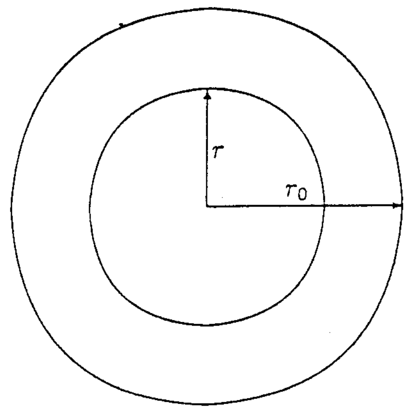
The limits of integration for t are from 0 s to t, and the corresponding limits for r are from r_0 to r.

$$\int_{r_0}^{r} r\,dr = -\int_{0\text{ s}}^{t}\frac{bv_0}{2\pi}dt \implies \frac{r^2 - r_0^2}{2} = -\frac{bv_0}{2\pi}t,$$

which can be rearranged as

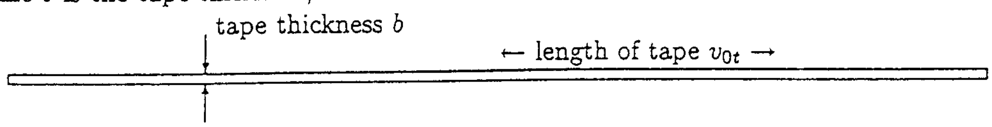
(1) $$\pi(r_0^2 - r^2) = bv_0 t.$$

The left-hand side of (1) is the area of the washer-shaped annulus of outer radius r_0 and inner radius r, as shown below.



It is the surface area of the edge of the tape as it was originally wound on the spool.

On the right-hand side of equation (1), the quantity v_0t is the total length of tape pulled through the machine while b is the tape thickness, as shown below.



Hence, the right-hand side of (1) is the total area of the edge of the tape pulled through the machine.

c) Solve equation (1) for r

$$r = \sqrt{r_0^2 - \frac{bv_0}{\pi}t}$$

Hence the equation $\frac{d\theta}{dt} = \frac{v_0}{r}$ becomes

$$\frac{d\theta}{dt} = \frac{v_0}{\sqrt{r_0^2 - \frac{bv_0}{\pi}t}}.$$

So

$$d\theta = \frac{v_0}{\sqrt{r_0^2 - \frac{bv_0}{\pi}t}}\,dt.$$

Integrate both sides

$$\int_0^\theta d\theta = \int_{0\,s}^t \frac{v_0}{\sqrt{r_0^2 - \frac{bv_0}{\pi}t}}\,dt.$$

The left side is simply θ. Hence,

$$\begin{aligned}
\theta &= \int_{0\,s}^t \frac{v_0}{\sqrt{r_0^2 - \frac{bv_0}{\pi}t}}\,dt \\
&= -\frac{\pi}{b}\int_{0\,s}^t \frac{\left(-\frac{bv_0}{\pi}\right)}{\sqrt{r_0^2 - \frac{bv_0}{\pi}t}}\,dt \\
&= -\frac{\pi}{b}\left(2\sqrt{r_0^2 - \frac{bv_0}{\pi}t}\right)\Bigg|_{0\,s}^t \\
&= \frac{2\pi}{b}\left(r_0 - \sqrt{r_0^2 - \frac{bv_0t}{\pi}}\right).
\end{aligned}$$

3. A kayaker attempts to cross a river in which the current has a uniform speed of 5.0 km/hr across the 50.0 m width of the river. The kayaker is paddling at the frenetic speed of 10.0 km/hr with respect to the water and wants to travel straight across the river to avoid a huge waterfall just downstream.

(a) What is the resulting speed of the kayak with respect to the ground?

(b) At what angle to the stream flow should the kayaker paddle so that the transit is made without moving upstream or downstream?

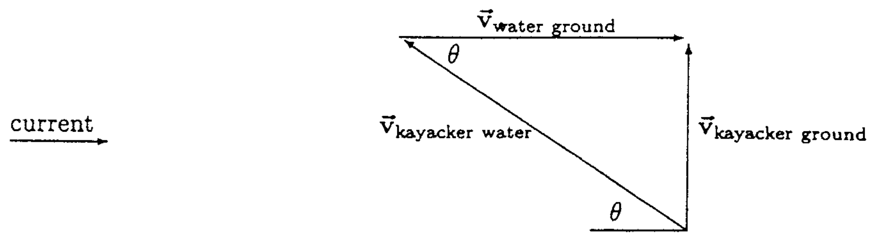
(c) How long does the trip take?

Solution:

The three velocities are related by the relative velocity addition equation:

$$\vec{v}_{\text{kayacker ground}} = \vec{v}_{\text{kayacker water}} + \vec{v}_{\text{water ground}}.$$

We want the velocity of the kayaker with respect to the ground to be directly across the river, so the three velocities must form a right triangle, as shown below:



(b) The cosine of the angle θ that $\vec{v}_{\text{kayacker water}}$ makes with the upstream direction is thus

$$\cos\theta = \frac{5.0 \text{ km/h}}{10.0 \text{ km/h}} \quad \Longrightarrow \quad \theta = 60°.$$

(a) The speed of the kayaker with respect to the ground is found from the Pythagorean theorem:

$$v_{\text{kayacker ground}} = \sqrt{(10.0 \text{ km/h})^2 - (5.0 \text{ km/h})^2} = 8.7 \text{ km/h}.$$

Convert this to m/s:

$$8.7 \text{ km/h} = (8.7 \text{ km/h})\left(\frac{10^3 \text{ m}}{\text{km}}\right)\left(\frac{\text{h}}{3600 \text{ s}}\right) = 2.4 \text{ m/s}.$$

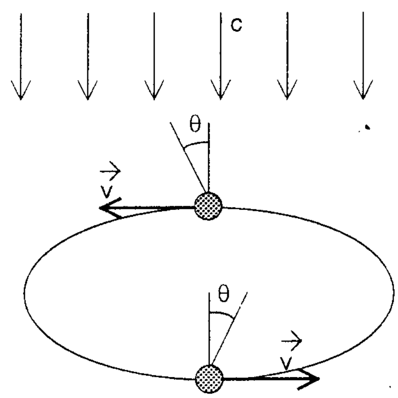
(c) The width of the river is 50 m. The time it takes the kayaker to cross the river is

$$t = \frac{\text{distance}}{\text{speed}} = \frac{50 \text{ m}}{2.4 \text{ m/s}} = 21 \text{ s}.$$

4. Imagine snow falling vertically downward at constant speed v_s. A car is driving horizontally in a straight line through the snow at constant speed v.

(a) Show that according to the driver, the snow appears to fall along a direction that makes an angle θ with the vertical direction, where $\tan\theta = v / v_s$. Include a sketch of the velocity vectors. If you tilt an umbrella at this angle while walking at speed v in snow (or rain) falling vertically at speed v_s, you keep driest.

(b) This problem also has implications in astronomy. Imagine the snow to be starlight moving at constant speed c towards the solar system, as shown in the figure below. The Earth, in its orbital motion, travels at speed v. Observers on the Earth see the light coming from a direction θ that is not in the direction of the true position of the star. The angle is found from $\tan\theta = v/c$. Thus a telescope must be tilted in the direction of the orbital velocity of the Earth by an angle θ to see the star. Over the course of a year, the star executes a small circle of angular radius θ about its true position perpendicular to the plane of the orbit of the Earth. This effect is known as *stellar aberration*. The orbital speed of the Earth is about 30 km/s. Evaluate the angle θ. If the star is in the plane of the orbit, then the aberrational path of the star is a line centered on the true position of the star. If the star is between the plane of the orbit and the perpendicular to the orbit, then the aberrational path of the star is an ellipse.



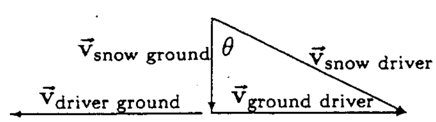
Earth in approximately circular orbit, in perspective but not to scale

Solution:

a) The velocity of the snow with respect to the driver is

$$\vec{v}_{\text{snow driver}} = \vec{v}_{\text{snow ground}} + \vec{v}_{\text{ground driver}}.$$

Since $\vec{v}_{\text{snow ground}}$ is vertically down and $\vec{v}_{\text{ground driver}} = -\vec{v}_{\text{driver ground}}$ is horizontal, the three vectors form a right triangle as shown below.



From the trigonometry,

$$\tan\theta = \frac{v_{\text{ground driver}}}{v_{\text{snow ground}}} = \frac{v}{v_s}.$$

b)

$$\tan\theta = \frac{30 \text{ km/s}}{3.00 \times 10^5 \text{ km/s}} = 1.0 \times 10^{-4}.$$

Hence, using the small angle approximation, and then converting from radians to arc seconds,

$$\theta = 1.0 \times 10^{-4} \text{ rad} = (1.0 \times 10^{-4} \text{ rad})\left(\frac{180°}{\pi \text{ rad}}\right)\left(\frac{3600 \text{ arc second}}{1°}\right) = 21 \text{ arc second}.$$

5. When the starting gun fires, a 60.0 kg sprinter accelerates with a constant acceleration for the first 10.0 m of the track and then runs with constant speed for the duration of the race, completing the 100. m sprint in 10.0 s. What is the magnitude of the total force causing the acceleration of the sprinter?

Solution:

Choose $\hat{\mathbf{i}}$ to point in the direction the sprinter runs, and let the origin be at the starting line. Then

$$x_0 = 0\ \text{m} \qquad \text{and} \qquad v_{x0} = 0\ \text{m/s},$$

so during the interval of constant acceleration,

$$x(t) = a_x \frac{t^2}{2}.$$

Now let t be the time that the acceleration ceases. At this time, the sprinter has covered 10.0 m, so $10.0\ \text{m} = a_x \frac{t^2}{2}$, hence

$$\text{(1)} \qquad a_x = \frac{20.0\ \text{m}}{t^2}.$$

Since $v_{x0} = 0$ m/s, the velocity component at time t is

$$v_x = a_x t = \frac{20.0\ \text{m}}{t^2} t = \frac{20.0\ \text{m}}{t}.$$

The remaining 90.0 m of the track are run with this constant velocity component, over the remaining time $10.0\ \text{s} - t$. Hence

$$90.0\ \text{m} = \frac{20.0\ \text{m}}{t}(10.0\ \text{s} - t) = \frac{(20.0\ \text{m})(10.0\ \text{s})}{t} - 20.0\ \text{m}.$$

So, solving for t,

$$t = \frac{(20.0\ \text{m})(10.0\ \text{s})}{90.0\ \text{m} + 20.0\ \text{m}} = 1.82\ \text{s}.$$

Use this value of t in equation (1) to find a_x.

$$\frac{20.0\ \text{m}}{(1.82\ \text{s})^2} = 6.04\ \text{m/s}^2.$$

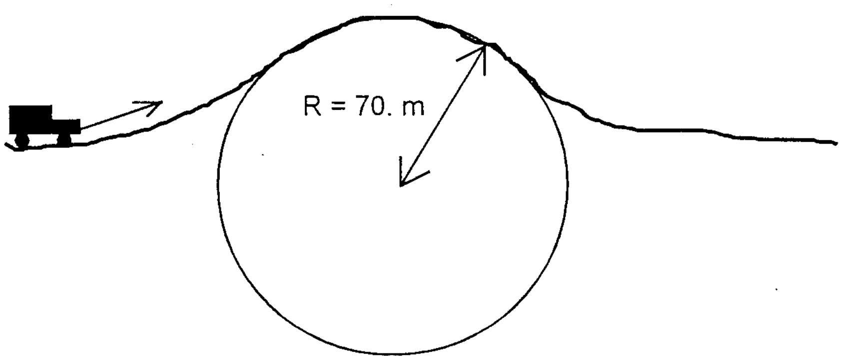
Therefore, the magnitude of the total force on the sprinter during the interval of constant acceleration is

$$F_{\text{total}} = ma = (60.0\ \text{kg})(6.04\ \text{m/s}^2) = 362\ \text{N}.$$

6. A car travelling at constant speed reaches the crest of a huge hill that is a cylindrical cross section of radius 70. m as shown in the figure below.

(a) At the instant when the car is at the crest of the hill, make a Second Law force diagram indicating schematically all forces acting on the car.

(b) What is the maximum speed that the car can travel without becoming airborne at the crest of the hill? Express your result in km/hr.



Solution:

a) Take the system to be the car. When the car is at the crest of the hill, there are two forces acting on it:

1. the gravitational force of the Earth on the car, i.e., the weight of the car $\vec{w}$, directed downward; and
2. the normal force $\vec{N}$ of the road surface on the car, directed upward.

b) Choose $\hat{j}$ to point up. Since the car is moving in a circular path, its acceleration is the centripetal acceleration $-\frac{v^2}{r}\hat{j}$ toward the center of the circle, so Newton's second law implies

$$-mg\hat{j} + N\hat{j} = m\left(-\frac{v^2}{r}\hat{j}\right).$$

If the speed of the car is just sufficient for it to become airborne, then the normal force $\vec{N}$ of the road surface on the car is zero. Hence

$$\begin{aligned}
-mg\hat{j} + (0\text{ N})\hat{j} &= m\left(-\frac{v^2}{r}\hat{j}\right)\\
&\Longrightarrow -mg\hat{j} = m\left(-\frac{v^2}{r}\hat{j}\right)\\
&\Longrightarrow g = \frac{v^2}{r}\\
&\Longrightarrow v = \sqrt{rg} = \sqrt{(70\text{ m})(9.81\text{ m/s}^2)} = 26\text{ m/s}.
\end{aligned}$$

Converting to km/h, this is

$$v = (26\text{ m/s})\left(\frac{\text{km}}{10^3\text{ m}}\right)\left(\frac{3600\text{ s}}{\text{h}}\right) = 94\text{ km/h}.$$

7. A large Ferris wheel of radius 15.0 m is rotating wildly out of control at 4.00 rev/min. A student of mass 80.0 kg is becoming increasingly uncomfortable with the ride. On the other hand, you (of mass 50.0 kg) are rather enjoying the wild ride. To calm your companion's panic, you decide to explain soothingly the physics of the effects that you are both experiencing on the Ferris wheel. You first point out that during the entire ride, there are only two forces acting on each of you: (1) your weight, and (2) the normal force of the bench seat on your backside.

(a) When the chair is at the top of the circle, which of the two forces has the greater magnitude? What is the magnitude of the difference between your weight and the normal force of the seat on your backside?

(b) When the chair is at the bottom of the circle, which of the two forces has the greater magnitude? What is the difference in the magnitudes of the forces on you at this location?

(c) When the chair is not at the top or bottom of the circle, the two forces are no longer antiparallel to each other, since the chair swivels. At what two places in the motion is the deviation of the direction of the normal force from the local vertical a maximum? At these locations, what is the angle θ between the direction of this force and the local vertical direction?

(d) At what angular speed must the wheel rotate so that you are both apparently "weightless" at the top of the spin? Give the result in rev/min. Are you truly weightless at the top of the spin? Explain why or why not.

Hints: You may need to use the Law of Cosines. Also, an extremum of function f(x) can be found using df(x)/dx=0.

Solution:

Throughout this problem let $\hat{\jmath}$ point straight up.

a) The angular speed of the Ferris wheel is

$$\omega = 4.00 \text{ rev/min} = (4.00 \text{ rev/min})\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{\text{min}}{60 \text{ s}}\right) = 0.419 \text{ rad/s}.$$

When at the top of the ride, the normal force $\vec{\mathbf{N}}_{\text{top}}$ points upwards and your weight $\vec{\mathbf{w}}$ points downwards, and these are the only forces on you. Since your motion is circular, you are accelerating towards the center of the circle, which at this time is downwards. Thus the total force must be downwards, which means that the magnitude of your weight exceeds N_{top}.

The acceleration is the centripetal acceleration toward the center of the circle, which at this time is in the direction $-\hat{\jmath}$ and has magnitude $r\omega^2$, where r is the radius of the Ferris wheel. Thus

$$\vec{\mathbf{F}}_{\text{total}} = m\vec{a} \implies N_{\text{top}}\hat{\jmath} - mg\hat{\jmath} = m(-r\omega^2\hat{\jmath}).$$

The magnitude of the difference between your weight and the normal force is the magnitude of the right-hand side of this equation,

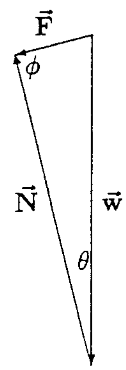
$$mr\omega^2 = (50.0 \text{ kg})(15.0 \text{ m})(0.419 \text{ rad/s})^2 = 132 \text{ N}.$$

b) At the bottom of the ride, the forces on you are your weight $-mg\hat{\jmath}$ pointing down, and the normal force $\vec{\mathbf{N}}_{\text{bottom}}$ pointing up. But now the centripetal acceleration points up, so the normal force must have *greater* magnitude than the weight. We now have

$$\vec{\mathbf{F}}_{\text{total}} = m\vec{a} \implies N_{\text{bottom}}\hat{\jmath} - mg\hat{\jmath} = mr\omega^2\hat{\jmath} = (50.0 \text{ kg})(15.0 \text{ m})(0.419 \text{ rad/s})^2\hat{\jmath} = (132 \text{ N})\hat{\jmath}$$

The magnitude of the difference between the normal force and weight is again 132 N, but now it is directed upwards.

c) We'll assume that $F_{\text{total}} < w$. Here is a force diagram



For simplicity, we have written $\vec{\mathbf{F}}$ instead of $\vec{\mathbf{F}}_{\text{total}}$. We want to know at which points in the ride the angle θ will be a maximum. We'll first show that in order for θ to be a maximum, the angle ϕ must be 90°.

The magnitudes F and w are constant at $mr\omega^2$ and mg respectively. The magnitude N varies throughout the ride. We will consider the value of N for which θ is a maximum. The law of cosines gives us

$$F^2 = N^2 - 2Nw\cos\theta + w^2.$$

Differentiate this equation with respect to N, remembering that both F and w are constant (we omit the units on zeros for clarity):

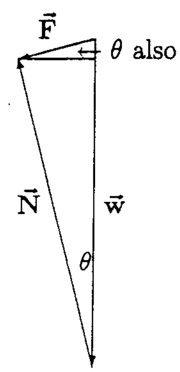
$$0 = 2N - 2w\cos\theta + 2Nw\sin\theta\frac{d\theta}{dN} + 0.$$

In order for θ to be a maximum, we must have $\dfrac{d\theta}{dN} = 0$. Substituting this into the last equation and simplifying,

$$\cos\theta = \frac{N}{w}.$$

This equation holds only if the triangle is a right triangle, i.e., $\phi = 90°$.

Now draw a horizontal line through the 90° angle ϕ (see the figure below). Since $\phi = 90°$, then both the angle that $\vec{\mathbf{F}}$ makes with the horizontal, and θ, have mutually perpendicular sides and so are equal.



This is the angle that a line drawn from the center of the wheel to the chair makes with the horizontal. The chair will be at this angle twice in each revolution, once on the way up and once on the way down.

To evaluate θ, note that from either diagram,

$$\sin\theta = \frac{F}{w} = \frac{mr\omega^2}{mg} = \frac{r\omega^2}{g} = \frac{(15.0\text{ m})(0.419\text{ rad/s})^2}{9.81\text{ m/s}^2} = 0.268 \implies \theta = 15.5°.$$

The value of N at this point may be found from the Pythagorean theorem.

$$N = \sqrt{w^2 - F^2} = \sqrt{\left((50.0\text{ kg})(9.81\text{ m/s}^2)\right)^2 - \left((15.0\text{ m})(0.419\text{ rad/s})^2\right)^2} = 490\text{ N}.$$

This is not appreciably less than the weight of your 50.0 kg mass.

d) You are not truly weightless, since the gravitational force of the Earth on you never ceases its action, even for an instant. You are normal-force-less. When the normal force is zero, Newton's second law becomes

$$-mg\hat{\jmath} = m(-r\omega^2\hat{\jmath}) \implies$$

$$\omega = \sqrt{\frac{g}{r}} = \sqrt{\frac{9.81 \text{ m/s}^2}{15.0 \text{ m}}} = 0.809 \text{ rad/s} = (0.809 \text{ rad/s})\left(\frac{\text{rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{\text{min}}\right) = 7.73 \text{ rev/min}.$$

8. A string made from braided dental floss has a maximum permissible tension of 500. N and is used by a 60.0 kg inmate to escape prison (apparently this actually happened!). The inmate starts at rest and slides vertically down the string for 8.0 m.

(a) Draw a force diagram indicating all forces acting on the inmate in the descent.

(b) Determine the magnitude of the acceleration of the inmate so that the string barely does not break. Is this a minimum or a maximum value? Explain.

(c) What is the maximum speed of the former inmate at the end of the 8.0 m descent if the string does not break?

Solution:

a) Consider the inmate to be the system. During the descent, the forces acting on the inmate are:

1. the weight $\vec{\mathbf{w}}$ of the inmate (the gravitational force of the Earth on the inmate) with magnitude mg, and directed straight down;
2. the force $\vec{\mathbf{T}}$ of the rope on the inmate, directed straight up.

b) Let $\hat{\jmath}$ point down. Newton's second law is:

$$F_{y \text{ total}} = ma_y \implies mg - T = ma_y.$$

When the string has its maximum tension, $T = 500$ N. Hence

$$(60.0 \text{ kg})(9.81 \text{ m/s}^2) - 500 \text{ N} = (60.0 \text{ kg})a_y \implies a_y = (1.48 \text{ m/s}^2).$$

The magnitude of the acceleration is the absolute value of this single component: 1.48 m/s^2. From Newton's second law, we see that for the magnitude of the acceleration of the inmate to be smaller, the tension would have to be larger. However, 500 N is the largest tension the string can support. Hence, this is the *minimum* value for the acceleration if the string is not to break.

c) Let the inmate begin at rest at the y-axis origin. Then the y-coordinate of the inmate is

$$y(t) = y_0 + v_{y0}t + a_y\frac{t^2}{2} = 0 \text{ m} + (0 \text{ m/s})t + a_y\frac{t^2}{2} = (1.48 \text{ m/s}^2)\frac{t^2}{2}.$$

When the inmate reaches the ground, the y-coordinate is 8.0 m. Hence

$$8.0 \text{ m} = (1.48 \text{ m/s}^2)\frac{t^2}{2}.$$

Solve for t, taking the positive root (since the journey began when $t = 0$ s): $t = 3.29$ s.

The velocity component of the inmate is

$$v_y(t) = v_{y0} + a_y t = 0 \text{ m/s} + (1.48 \text{ m/s}^2)t = (1.48 \text{ m/s}^2)t.$$

When $t = 3.29$ s, the velocity component is $v_y = (1.48 \text{ m/s}^2)(3.29 \text{ s}) = 4.87$ m/s. The speed of the inmate is the absolute value of this single velocity component, 4.87 m/s.

9. Amusement parks occasionally have a large cylindrical tube mounted to spin about the vertical axis of the cylinder as shown in the figure below. Patrons stand against the wall of the cylinder and it is set into rotation. At a certain critical angular velocity, the floor on which the patrons are standing is lowered and the patrons are left hanging on the wall (but on the verge of slipping) due to the force of static friction on them. Let R be the radius of the cylinder, m the mass of one patron, and μ_s the coefficient of static friction for this patron on the wall.

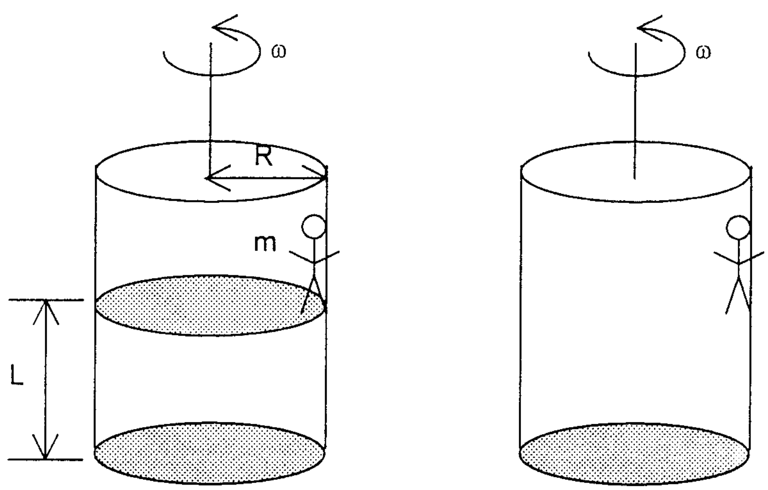
(a) Draw the force diagram for the patron, indicating all forces acting on the patron.

(b) Is there an acceleration of the patron in the vertical direction? What does this imply about the relationship between the static frictionless force on the patron and the weight of the patron? Are these forces a Newton's Third Law pair?

(c) Is there an acceleration of the patron in the horizontal direction? How is this acceleration related to the speed of the patron?

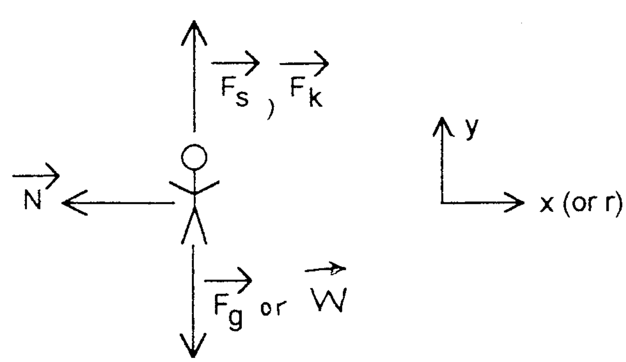
(d) How is the angular velocity ω of the cylinder related to the tangential speed of the patron? If the patron is ready to slip, find the angular velocity ω of the cylinder in terms of R, μ_s, and g. Express your answer in rev/s.

(e) If the floor of the cylinder drops a distance L, how long does it take for the patron to slide down the distance L? Consider the coefficient of kinetic friction to be μ_k.



Solution:

(a) Force diagram



The forces acting on the patron are:

1. the weight $\vec{w}$ of the patron, directed down;
2. the normal force $\vec{N}$ of the surface on the patron, directed perpendicularly from the surface toward the center of circular motion; and
3. the force of static friction $\vec{f}_s$ on the patron, directed upward to resist slippage.

Choose $\hat{\imath}$ to point horizontally from the patron toward the center of circular motion (parallel to $\vec{N}$), and choose $\hat{\jmath}$ to point up (parallel to $\vec{f}_s$).

b) There is zero acceleration in the vertical direction, so the total force in this direction is zero.

$$F_{y\ \text{total}} = 0\ \text{N} \implies f_s - mg = 0\ \text{N} \implies f_s = mg$$

Thus, the magnitude of the static force of friction is equal to that of the weight.

Note that although these forces are equal in magnitude and opposite in direction, they are not a Newton's third law force pair since they both act on the same system (the patron).

c) There is a nonzero centripetal acceleration in the horizontal direction because the patron is moving in circular motion. This acceleration has magnitude

$$a_{\text{centripetal}} = \frac{v^2}{R}$$

where R is the radius of the tube, and v is the speed of the patron.

d) The tangential speed v is related to the angular speed ω, by

$$v = R\omega.$$

If the patron is ready to slip, then the static force of friction on the patron is of magnitude

$$f_s = f_{s\,\max} = \mu_s N.$$

From part b) $f_s = mg$, so

$$mg = \mu_s N \implies N = \frac{mg}{\mu_s}.$$

Apply Newton's second law to the horizontal direction.

$$F_x = ma_x \implies N = mR\omega^2 \implies \frac{mg}{\mu_s} = mR\omega^2 \implies \omega = \sqrt{\frac{g}{\mu_s R}}.$$

This result has units of radians per unit time. To convert to revolutions per unit time, multiply by $\frac{\text{rev}}{2\pi\ \text{rad}}$.

$$\nu = \left(\frac{\text{rev}}{2\pi\ \text{rad}}\right)\sqrt{\frac{g}{\mu_s R}}.$$

If you ever actually go on this ride, don't wear a silk shirt, or any clothes whose μ_s with the inside of the tube is small. The manufacturer of the ride assumes that everyone will wear clothes whose coefficient of static friction with the inside of the tube is greater than some threshold value. Those who don't, will fall!

(e) Apply Newton's Second Law for the situation where the patron is sliding downward. Because the patron is sliding, kinetic friction is used.

x direction: $F_{\text{net},x} = -N = ma_x = -m\frac{v^2}{R}$ $\quad\therefore\ N = m\frac{v^2}{R} = m\frac{(\omega R)^2}{R} = m\omega^2 R$

y direction: $F_{\text{net},y} = F_k - F_g = ma_y$

$$\mu_k N - mg = ma_y$$

$$\therefore\ \mu_k(m\omega^2 R) - mg = ma_y$$

$$a_y = \mu_k \omega^2 R - g$$

This acceleration is constant and can be used in the equations for v_y and y.

$$v_y = v_{yo} + a_y(t - t_o) = a_y t = (\mu_k \omega^2 R - g)t \qquad \text{with } v_{yo} = 0 \text{ at } t_o = 0$$

$$y = y_o + v_{yo}(t - t_o) + \tfrac{1}{2} a_y (t - t_o)^2 = \tfrac{1}{2} a_y t^2 = \tfrac{1}{2}(\mu_k \omega^2 R - g)t^2 \qquad \text{with } y_o = 0 \text{ at } t_o = 0$$

Thus $t^2 = \dfrac{2y}{\mu_k \omega^2 R - g}$

When the patron slides down a distance L: $t^2 = \dfrac{2(-L)}{\mu_k \omega^2 R - g}$

Therefore the time required is: $t = \sqrt{\dfrac{2L}{g - \mu_k \omega^2 R}}$

10. Two masses, m_1 and m_2, are connected by an ideal cord as shown in the figure below. The cord passes over an ideal pulley. A student determines that when the surface is inclined at an angle θ to the horizontal, m_1 is just on the verge of slipping to the right up the incline.

(a) Draw a force diagram showing all forces acting on mass m_2.

(b) Draw a force diagram showing all forces acting on mass m_1.

(c) Introduce appropriate coordinate system(s) and show that the coefficient of static frictions for m_1 on the inclined plane is

$$\mu_s = \frac{|m_2 - m_1 \sin\theta|}{m_1 \cos\theta}.$$

Now the inclination angle of the inclined plane is increased to angle ϕ so that the mass m_1 is just on the verge of slipping to the left down the incline.

(d) Draw a force diagram showing all forces acting on mass m_2.

(e) Draw a force diagram showing all forces acting on mass m_1.

(f) Show that the coefficient of static frictions for m_1 on the inclined plane is now

$$\mu_s = \frac{|m_1 \sin\phi - m_2|}{m_1 \cos\phi}.$$

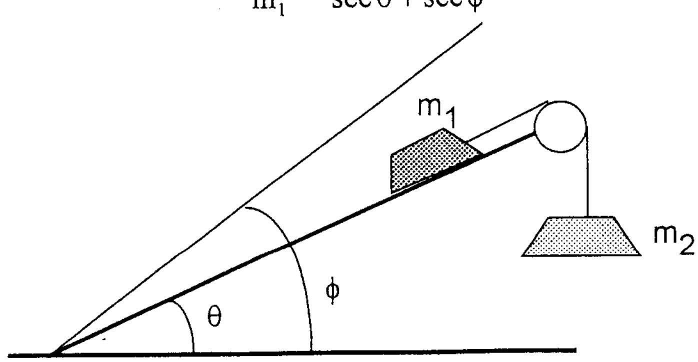
Notice that if the suspended mass $m_2 = 0$ kg, then $\phi = 0$ and this equation reduces to

$$\mu_s = \tan\phi = \tan\theta.$$

(g) Is this the same coefficient of friction as that determined in part (c)? Why or why not?

(h) Show that the ratio of the two masses is given by the rather elegant expression

$$\frac{m_2}{m_1} = \frac{\tan\theta + \tan\phi}{\sec\theta + \sec\phi}.$$



Solution:

a) The forces acting on m_2 are:

1. its weight $\vec{\mathbf{w}}_2$, directed down; and

2. the force $\vec{\mathbf{T}}_2$ of the cord on m_2,directed up.

b) The forces on m_1 when it is ready to slip to the right are:

1. its weight $\vec{\mathbf{w}}_1$, directed straight down;

2. the force $\vec{\mathbf{T}}_1$ of the cord on m_1, directed up the plane toward the pulley;

3. the normal force $\vec{\mathbf{N}}$ of the surface on m_1, directed perpendicularly out of the surface; and

4. the maximum magnitude force of static friction $\vec{\mathbf{f}}_{s\,\text{max}}$, directed down the plane opposite to $\vec{\mathbf{T}}_1$ — since the mass is ready to slip up the plane.

c) Although $\vec{\mathbf{T}}_1$ and $\vec{\mathbf{T}}_2$ point in different directions, their magnitudes are the same. Call their common magnitude T, so $T_1 = T_2 = T$.

To analyze m_2, let $\hat{\jmath}$ point straight down, parallel to $\vec{\mathbf{w}}_2$. Then, since it is not accelerating, the total force on it is zero.

(1) $$F_{y\ \text{total}} = 0\ \text{N} \implies m_2 g - T = 0\ \text{N} \implies T = m_2 g.$$

Now, in order to analyze m_1, choose a coordinate system with $\hat{\imath}$ parallel to T_1 (opposite $\vec{\mathbf{f}}_{s\,\text{max}}$), and with $\hat{\jmath}$ parallel to $\vec{\mathbf{N}}$. Since m_1 is not accelerating, the total force on it is zero. Thus,

x direction

$$T - m_1 g \sin\theta - f_{s\,\text{max}} = 0\ \text{N} \implies T - m_1 g \sin\theta - \mu_s N = 0\ \text{N}$$

y direction

$$F_{y\ \text{total}} = 0\ \text{N} \implies N - m_1 g\cos\theta = 0\ \text{N} \implies N = m_1 g\cos\theta.$$

Use equation (1) and the y direction equation to substitute for T and N in the x direction equation.

$$m_2 g - m_1 g\sin\theta - \mu_s m_1 g\cos\theta = 0\ \text{N} \implies \mu_s = \frac{m_2 - m_1\sin\theta}{m_1\cos\theta}.$$

Since m_2 must be greater than $m_1 \sin\theta$ for m_1 to slide up the plane, one can (or not) put in absolute value signs on the right hand side to reinforce the fact that $\mu_s > 0$.

d) See part a)

e) The forces on m_1 when it is ready to slip to the left are:

1. its weight $\vec{\mathbf{w}}_1$, directed straight down;

2. the force $\vec{\mathbf{T}}_1$ of the cord on m_1, directed up the plane toward the pulley;

3. the normal force $\vec{\mathbf{N}}$ of the surface on m_1, directed perpendicularly out of the surface; and

4. the maximum magnitude force of static friction $\vec{\mathbf{f}}_{s\,\text{max}}$, directed up the plane parallel to $\vec{\mathbf{T}}_1$ — since the mass is ready to slip down the plane.

These forces are the same as in part b), except that $\vec{\mathbf{f}}_{s\,\text{max}}$ is now in the opposite direction.

f) Again, although $\vec{\mathbf{T}}_1$ and $\vec{\mathbf{T}}_2$ point in different directions, their magnitudes are the same. Call their common magnitude T, so $T_1 = T_2 = T$.

Just as in part c), we find that

$$T = m_2 g. \tag{2}$$

Now analyze m_1 just as we did in part c) with $\hat{\imath}$ parallel to $\vec{\mathbf{T}}_1$ and with $\hat{\jmath}$ parallel to $\vec{\mathbf{N}}$. The only difference is that $\vec{\mathbf{f}}_{s\,\max}$ points in the opposite direction, and the angle is ϕ in place of θ.

x direction

$$T - m_1 g \sin\phi + f_{s\,\max} = 0\ \mathrm{N} \implies T - m_1 g \sin\phi + \mu_s N = 0\ \mathrm{N}$$

y direction

$$F_{y\ \mathrm{total}} = 0\ \mathrm{N} \implies N - m_1 g \cos\phi = 0\ \mathrm{N} \implies N = m_1 g \cos\phi.$$

Use equation (2) and the y direction equation to substitute for T and N in the x direction equation.

$$m_2 g - m_1 g \sin\phi + \mu_s m_1 g \cos\phi = 0\ \mathrm{N} \implies \mu_s = \frac{m_1 \sin\phi - m_2}{m_1 \cos\phi}.$$

Here, $m_1 \sin\phi$ must be greater than m_2 for m_1 to slide down the plane. We can (or not) put in absolute value signs on the right hand side to emphasize that $\mu_s > 0$.

g) This is the same coefficient of static friction as in part c) since the same two surfaces are in contact.

h) Equate the two expressions from parts c) and f) (without absolute value signs, since they are, in a sense, superfluous) for the static coefficient of friction.

$$\frac{m_2 - m_1 \sin\theta}{m_1 \cos\theta} = \frac{m_1 \sin\phi - m_2}{m_1 \cos\phi} \implies \frac{m_2}{m_1}\sec\theta - \tan\theta = \tan\phi - \frac{m_2}{m_1}\sec\phi.$$

Now solve for the ratio of the masses.

$$\frac{m_2}{m_1} = \frac{\tan\theta + \tan\phi}{\sec\theta + \sec\phi}$$

In this expression, for the case where $m_2 = 0$ kg, we must have $\tan\theta = -\tan\phi$ or $\theta = -\phi$. In other words, if ϕ is the angle where m_1 (by itself) slides to the left down the incline, the angle θ where m_1 (by itself) slides to the right is negative, meaning the board must be inclined downward to the right.