LECTURE #32 – SUMMARY

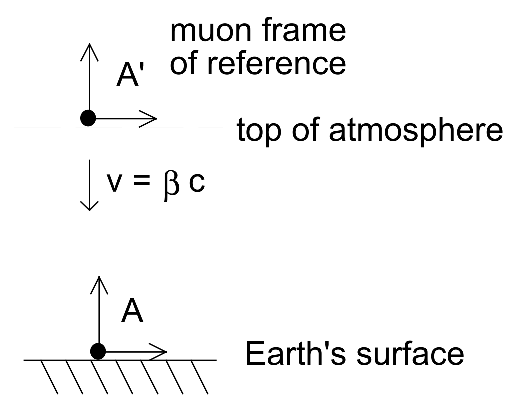
Muons - An Example of Time Dilation and Length Contraction

Muons are unstable particles created when cosmic rays interact with the upper atmosphere. They move at very high velocities ($\beta \sim 0.9999$) and have very short lifetimes, $\tau = 2 \times 10^{-6}$ s, as measured in the lab.

Do muons reach the ground, given an atmospheric "thickness" of about 10 km?
"Classical" answer: distance = velocity $\times$ time = $0.9999c \times 2 \times 10^{-6}$ s $\cong$ 0.6 km
$\therefore$ conclude that muons will never reach the ground.
However, they do! What is wrong? Because muons move so quickly, relativistic effects are important. The classical answer mixes up reference frames.

- τ - refers to lifetime in the muon reference frame (lab where τ is measured)
- atmospheric thickness - refers to length in the Earth's reference frame

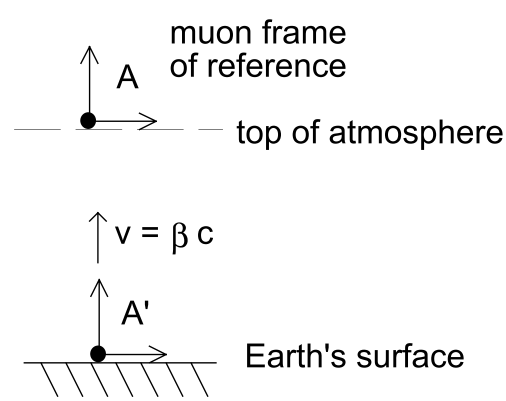
(1) Time dilation approach – from Earth frame



$\rightarrow$ treat the muons as a clock in A'
t' = 0 muons are created
t' = τ muons are destroyed
$\Delta t' = \tau$

The creation and destruction of muons occurs at the same place in A'.
What is the time interval as measured in A?
$\Delta t = \gamma \Delta t' = \gamma \tau = 1.4 \times 10^{-4}\,\mathrm{s}$ i.e., it takes 70 times longer
$\therefore$ distance = $0.9999c \times 1.4 \times 10^{-4}$ s $\cong$ 42 km
Since 42 km > 10 km, the muons will reach the ground.

(2) Length contraction approach – from muon frame



In reference frame A (now with the muons)
lifetime = $\tau = 2 \times 10^{-6}$ s
velocity of ground = 0.9999c

So the distance that the ground travels before the muon decays is 0.6 km. But what is the thickness of the atmosphere that the muon sees?
proper length of atmosphere = 10 km
length of atmosphere in muon frame is

$$\Delta x = \Delta x' \sqrt{1 - v^2/c^2} = 10\sqrt{1 - 0.9999^2} = 0.14 \text{ km}$$

i.e., the atmosphere that the muon sees is 70 times thinner
0.6 km > 0.14 km and so the ground will reach the muon.

Thus, length contraction and time dilation are real!

Section V.5 Energy in STR

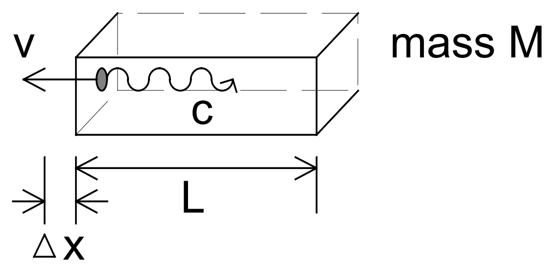
Derivation of $E = mc^2$

Einstein conceived of this equation using a thought experiment, which we will "reproduce". Imagine a closed box of mass m and length L, initially at rest. A flash of light is emitted from one end of the box.

We know from mechanics that moving objects carry both energy and momentum. This is also true for electromagnetic waves. Maxwell showed that the energy E and momentum p of a wave are related by $p = E/c$.

Because the emitted light wave carries momentum E/c, the box must recoil in order to conserve momentum. The recoil speed is v, which we assume is much less than c, since the box is massive.

Momentum conservation: $\frac{E}{c} = Mv$ where M = mass of the box.



The light will move down the length of the box in time: $\Delta t = \frac{L}{c}$ (again assumes v << c, so that distance travelled by light is L).

During Δt, the box moves a very small distance Δx, given by: $\Delta x = v\Delta t = \left(\frac{E}{Mc}\right)\left(\frac{L}{c}\right) = \frac{EL}{Mc^2}$.

Once the light hits the end of the box, it transfers its momentum to the box and brings it to a stop. The box is now in a new position. Its centre of mass appears to have moved, but the box is an isolated system whose centre of mass cannot move! To deal with this contradiction, Einstein assumed that light carries mass (as well as energy and momentum).

If m = mass carried by the light, then: $mL = M\Delta x$

in order that the centre of mass of (the box + light) does not move.

Using the expression for Δx and solving for m gives: $m = \frac{M\Delta x}{L} = \frac{M}{L}\frac{EL}{Mc^2} = \frac{E}{c^2}$

$$\boxed{\therefore E = mc^2}$$

where

E = energy of the light, usually written as E_o to denote that it is the rest energy

m = equivalent mass of the light

Although this expression has been derived for light energy, it is universal statement of the equivalence of mass and energy. It can also be derived from the definition of kinetic energy of a moving body. Because mass and energy are not independent quantities, the separate conservation principles of energy and mass are properly combined in the <u>principle of conservation of mass-energy</u>.

Energy and Momentum in Special Relativity

The energy term in $E_o = mc^2$ is the rest energy of a particle, not the total energy.

Newtonian mechanics: $\vec{p} = m\vec{u}$ where $\vec{u}$ is the particle velocity

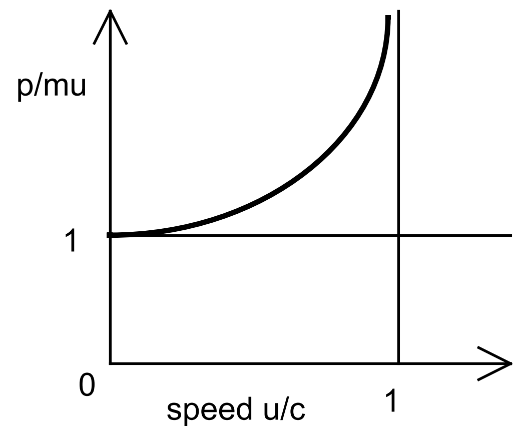
This suggests that if momentum is conserved in one reference frame, then it will not be conserved in another because of Lorentz velocity addition.

In fact, $\vec{p} = m\vec{u}$ is only an approximation for momentum for u << c.

General equation for momentum, valid at all speeds u: $\boxed{\vec{p} = \gamma m\vec{u} = \dfrac{m\vec{u}}{\sqrt{1-u^2/c^2}}}$

This momentum is conserved in all inertial reference frames and reduces to $\vec{p} = m\vec{u}$ for small u.

Also note that as $u \to c$, then $\gamma \to \infty$ and so $p \to \infty$. Since force = dp/dt, this means that a very large force is needed to produce even a small change in the velocity of a rapidly moving particle. This helps to explain why it is impossible to accelerate an object to the speed of light: the object's momentum would $\to\infty$ and no matter how close u is to c, an ∞ force would be needed to give the object the "last bit of speed" needed to reach c.



Now, calculate kinetic energy gained as a particle accelerates from rest to speed u.

$$F = \frac{dp}{dt} = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}}\right) = \frac{m}{\left(1-u^2/c^2\right)^{3/2}}\frac{du}{dt}$$

So: $$K = \int F dx = \int \left[\frac{m}{\left(1-u^2/c^2\right)^{3/2}}\frac{du}{dt}\right] u dt = ... = \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2 = \gamma mc^2 - mc^2$$

= kinetic energy of a particle in a reference frame in which it has speed u

= total energy – rest energy = E - E_o

For u<<c, it can be shown (problem 43) that: $K = \frac{1}{2}mu^2$

For a particle at rest, u = 0 and then: $E = E_0 = mc^2$

This is an alternative derivation of the equivalence of mass and energy.

Let's combine the equations $p = \gamma mu$ and $E = \gamma mc^2$: $\boxed{p = u/c^2 E}$

We can also write: $E^2 - c^2p^2 = (\gamma mc^2) - c^2(\gamma mu)^2$ to get: $\boxed{E^2 = c^2p^2 + (mc^2)^2}$

This is the energy-momentum relation. For particles at rest, p=0 and $E = mc^2$. For particles at high speed, rest energy is negligible and $E \approx pc$. For photons (bundles of EM energy, no mass), $E = pc$, as we used before.

<u>Section V.6 General Relativity</u> (Einstein, 1914-16)

Recall that the Special Theory of Relativity is "special" because it is restricted to reference frames in uniform motion. Einstein later tried to develop a theory that expressed the laws of physics in the same form all frames of reference, including those in accelerated motion. However, he recognized that it is impossible to distinguish the effects of uniform acceleration from those of a uniform gravitational field, and so his theory became a theory of gravity.

Einstein built on the idea of 4-D spacetime, introducing geometrical curvature of spacetime to account for gravity $\rightarrow$ <u>General Theory of Relativity</u>

Two postulates of General Theory of Relativity:

(1) All the laws of physics have the same form for observers in any frame of reference, whether accelerated or not.

(2) In the vicinity of a given point, a gravitational field is equivalent to an accelerated frame of reference in the absence of gravitational effects (this is the <u>principle of equivalence</u>).

The predictions of the GTR differ from those of Newton's theory of gravity only in regions of very strong gravitational fields or when the overall structure of the universe is considered. "strong fields" = those of objects whose escape speed is comparable to that of light

We have no direct lab experience of such strong fields, and so the GTR is not as solidly established as the STR. However, GTR is an important theory for astrophysics, helping to explain the physics of neutron stars and black holes.

Space-Time Diagrams

These are graphical depictions of the space and time coordinates of a reference frame. A complete diagram is 4D, but we usually suppress one or more spatial dimensions. An event is represented by a point on the diagram and the path followed by a system on the diagram is called a <u>world-line</u>.

The speed of a particle is the inverse of the slope of the world-line. Nothing can have a world-line with a slope whose absolute magnitude is less than that of the world-line for light (slope = $\pm 1/c$).

A point at the origin is "here and now". The future is t>0 and the past is t<0. Travel to the past requires the slope of a particle's world-line to equal and then become less than 1/c – this is prohibited and so travel into the past is impossible under Special Relativity.