LECTURE #31 – SUMMARY

Lorentz (or Length) Contraction

Let's return to our example, and consider just the one-way travel of the spaceship from Earth to the star. In the Earth-star reference frame A, the spaceship travels 10 ly in 12.5 years at speed 0.8c. So the distance between Earth and the star, as seen in A is:

$$\Delta x_{one-way} = v\Delta t_{one-way} = v\frac{L}{\beta} = cL\ \ (m) = L\ \ (lightyears) = 10\ \ ly$$

In the spaceship reference frame A', we already found that:

$$\Delta t'_{one-way} = \frac{\Delta t}{1\big/\sqrt{1-v^2/c^2}} = 7.5\ \ years$$

Because the Earth and star move past the spaceship at speed v = 0.8c, the distance between Earth and the star, as seen in A', is

$$\Delta x'_{one-way} = v\Delta t'_{one-way} = v\frac{\Delta t}{1\big/\sqrt{1-v^2/c^2}} = \Delta x\sqrt{1-v^2/c^2} = (10\ \ ly)(\sqrt{1-0.8^2}) = 6.0\ \ ly$$

This indicates that the distance between any two points is always greatest in a frame (the Earth-star frame A in our example) fixed with respect to those points. In any other reference frame is smaller. This is called the Lorentz contraction.

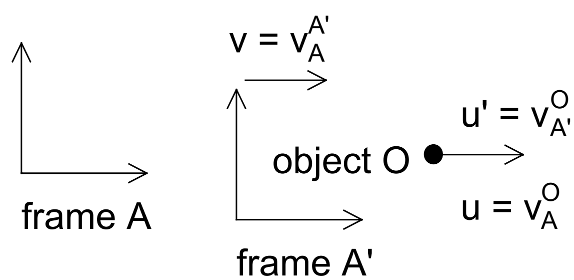
In general: $\boxed{\Delta x' = \Delta x\sqrt{1-\frac{v^2}{c^2}} = \frac{\Delta x}{\gamma}}$ where A' is moving w.r.t. the two points.

Note: an object's length in a frame in which it is at rest is called its proper length.

Lorentz Velocity Addition

Let's say that object O has velocity u' in reference frame A'.

What will the velocity of the object be as measured by an observer in A?



Galilean Approach: $v_A^O = v_{A'}^O + v_A^{A'}$ $\boxed{\therefore u = u' + v}$

Does this still hold? No! e.g., if v = 0.75 c and u' = 0.80 c, then u = 1.55 c > c !

STR Approach: Apply the Lorentz transformations and take the differentials.

$$x = \gamma(x' + vt') \quad \rightarrow \quad dx = \gamma(dx' + vdt')$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \quad \rightarrow \quad dt = \gamma\left(dt' + \frac{v}{c^2}dx'\right)$$

$$u = \frac{dx}{dt} = \frac{dx' + v dt'}{dt' + \frac{v}{c^2} dx'} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} \quad \text{Thus} \quad \boxed{u = \frac{u' + v}{1 + \frac{v}{c^2} u'}} \quad \text{and} \quad \boxed{u' = \frac{u - v}{1 - \frac{v}{c^2} u}}$$

Special cases:

(1) $u' \ll c$ and $v \ll c$: $\boxed{u \cong u' + v}$ and $\boxed{u' \cong u - v}$

i.e., we get the Galilean velocity transforms

(2) $u' \approx c$ and $v \approx c$:

Let's say:

$$u' = (1-\delta_1)c : \quad \delta_1 \approx 0, \quad \delta_1 > 0$$
$$v = (1-\delta_2)c \quad \delta_2 \approx 0, \quad \delta_2 > 0$$

Then:

$$u = \frac{c(2-\delta_1-\delta_2)}{1+(1-\delta_1)(1-\delta_2)} = \frac{c(2-\delta_1-\delta_2)}{2-\delta_1-\delta_2+\delta_1\delta_2} < c!!$$

So u is still less than c. Note: it would be 2c using the Galilean velocity transform.

Now, let's make the x component of u and u' explicit: $u_x = \frac{u'_x + v}{1 + \frac{v}{c^2} u'_x}$

where v is the relative velocity of A' to A along the x axis.

What happens to the velocity perpendicular to the direction of motion of the two reference frames?

Galilean Approach: $u_y = u'_y$

With STR, we still have y = y', but $t \neq t'$. Again apply the Lorentz transformations and take the differentials.

$$y = y' \quad \rightarrow \quad dy = dy'$$
$$t = \gamma\left(t' + \frac{v}{c^2} x'\right) \quad \rightarrow \quad dt = \gamma\left(dt' + \frac{v}{c^2} dx'\right)$$

Thus: $$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{v}{c^2} dx'\right)} = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} = \frac{u'_y}{\gamma\left(1 + \frac{v}{c^2} u'_x\right)}$$ v in x,x' direction

So $u_y \neq u'_y$ Note: even if $u'_x = 0$, so that the object only has a y' velocity component in reference frame A', we still have $u_y \neq u'_y$. In this case: $u_y = u'_y / \gamma$.

We can derive a similar equation for the z component: $u_z = \frac{u'_z}{\gamma\left(1 + \frac{v}{c^2} u'_x\right)}$