LECTURE #25 – SUMMARY

(1) Grouping together the "sin(ωt+δ)" terms gives:

$$\left(m\omega\alpha + m\alpha\omega - b\omega\right)\ Ae^{-\alpha t}\sin(\omega t+\delta) = 0 \qquad \text{so} \qquad \boxed{\alpha = \frac{b}{2m}}$$

(2) Grouping together the "cos(ωt+δ)" terms gives:

$$\left(-m\omega^2 + m\alpha^2 - b\alpha + k\right)\ Ae^{-\alpha t}\cos(\omega t+\delta) = 0 \qquad \text{so} \qquad \omega^2 = \alpha^2 + \frac{k}{m} - \frac{b\alpha}{m}$$

where $\omega_o = \sqrt{\frac{k}{m}}$ = natural angular frequency of oscillation

and $\boxed{\omega = \sqrt{\omega_o^{\ 2} - \frac{b^2}{4m^2}}}$ = damped angular frequency

Note: $\omega < \omega_o$ (damping reduces frequency)

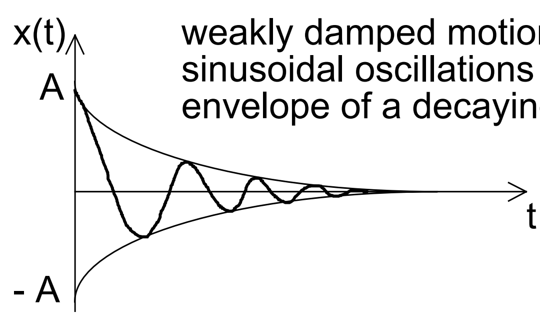
Damped displacement: $$x = Ae^{-\left(\frac{b}{2m}\right)t}\cos\left[\left(\sqrt{\omega_o^{\ 2} - \frac{b^2}{4m^2}}\right)t + \delta\right]$$

Special case: no damping, b = 0: $x = A\cos(\omega t + \delta)$ ⇒ SHM !

As b (damping) increases:

→ ω decreases and T increases, slower (more sluggish) oscillation

→ α increases, amplitude decreases more rapidly



If ω = 0, then $\omega_o^{\ 2} = \frac{b^2}{4m^2}$

⇒ $\boxed{b = 2m\omega_o}$

If $b < 2m\omega_o$, then the motion is underdamped.

→ oscillation occurs with an amplitude that decreases with time (as shown)

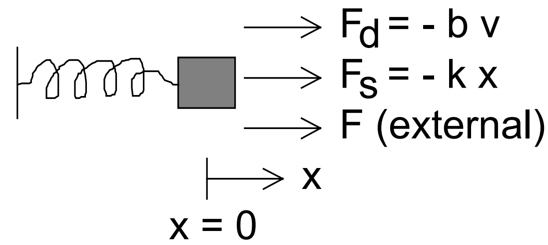
If $b = 2m\omega_o$, then the motion is critically damped.

→ the damping force is the same as the spring force and the system returns to its equilibrium state with no oscillations

If $b > 2m\omega_o$, then the motion is overdamped.

→ the damping force is so large that the equation for x(t) is no longer a valid solution of the equation of motion, and the system slowly returns to its equilibrium state with no oscillations

Driven Harmonic Motion
Now add an external harmonic forcing to the system.



Driving force: $F = F_o \cos \omega_d t$
where ω_d = driving angular frequency

The equation of motion is now $F_s + F_d + F = ma$

$$\boxed{m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx - F_o \cos \omega_d t = 0}$$

What is the solution to this equation?
→ on short time scales - complicated transient solution
→ on longer time scales - not complicated

If $F_o \cos \omega_d t$ is applied for long enough, then the response will eventually have the frequency of the driving force, ω_d. So, we expect a solution of the form
$x = A\cos(\omega_d t + \delta)$ … after the transients die off.

Calculate v and a, and substitute in the terms:

$$-m\omega_d^2 A\cos(\omega_d t+\delta) - b\omega_d A\sin(\omega_d t+\delta) + kA\cos(\omega_d t+\delta) - F_o\cos\omega_d t = 0$$

Apply the trigonometric relations:

$$\cos(\omega_d t+\delta) = \cos(\omega_d t)\cos(\delta) - \sin(\omega_d t)\sin(\delta)$$
$$\sin(\omega_d t+\delta) = \sin(\omega_d t)\cos(\delta) + \cos(\omega_d t)\sin(\delta)$$

First, group together the $\cos(\omega_d t)$ terms:

$$-m\omega_d^2 A\cos(\delta) - b\omega_d A\sin(\delta) + kA\cos(\delta) - F_o = 0$$

Next, group together the $\sin(\omega_d t)$ terms:

$$m\omega_d^2 A\sin(\delta) - b\omega_d A\cos(\delta) + kA\sin(\delta) = 0$$

Solve for A and δ - messy but not hard!

The result: $A = \dfrac{F_o}{\sqrt{m^2(\omega_d^2 - \omega_o^2)^2 + b^2\omega_d^2}}$

This has a general form - a resonance curve.