LECTURE #24 – SUMMARY

Section IV.3 Energy in Simple Harmonic Motion

No dissipation in the system, so use $E = K + U(\vec{r}) = \frac{1}{2}mv^2 + U(\vec{r}_s) - \int_{\vec{r}_s}^{\vec{r}} \vec{F} \bullet d\vec{r}$

Energy of a Spring

$$U(x) = U(x_s) - \int_{x_s}^{x} F dx = -\int_0^x (-kx)dx = \int_0^x kx dx = \tfrac{1}{2}kx^2 \quad \text{using } x_s = 0 \text{ and } U(x_s) = 0$$

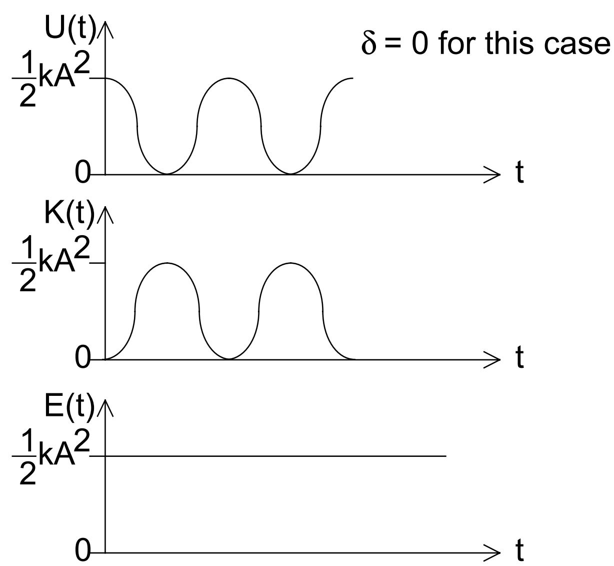
$$\therefore \quad U(x) = \tfrac{1}{2}kx^2 = U(t) = \tfrac{1}{2}kA^2 \cos^2(\omega t + \delta)$$

$$K(x) = \tfrac{1}{2}mv(x)^2 = K(t) = \tfrac{1}{2}mA^2\omega^2 \sin^2(\omega t + \delta) = \tfrac{1}{2}kA^2 \sin^2(\omega t + \delta)$$

using $x(t) = A\cos(\omega t + \delta)$ and $v(t) = -A\omega \sin(\omega t + \delta)$

Total mechanical energy: $E = K + U = \frac{1}{2}kA^2 \sin^2(\omega t + \delta) + \frac{1}{2}kA^2 \cos^2(\omega t + \delta)$

$$\therefore \quad \boxed{E = \tfrac{1}{2}kA^2}$$



Notice that U(t) and K(t) are 90° out of phase.

For maximum displacement x:
U(x) = max, v(x) = min, K(x) = min
(spring extended or compressed)

For minimum displacement x:
U(x) = min, v(x) = max, K(x) = max
(spring at equilibrium)

Energy flows back and forth between U and K.

Energy of a Pendulum

$$U(s) = -\int_0^s F_{\hat{t}} ds = -L\int_0^\theta F_{\hat{t}} d\theta = mgL \int_0^\theta \sin\theta d\theta = mgL(1 - \cos\theta) \quad \text{using } U(s = 0) = 0$$

$$U \cong mgL\frac{\theta^2}{2} = \frac{mg}{2L}s^2 = \frac{mg}{2L}s_o^2 \cos^2(\omega t + \delta) \qquad \text{using } \cos\theta \cong 1 - \frac{\theta^2}{2} \text{ for } \theta \approx 0.$$

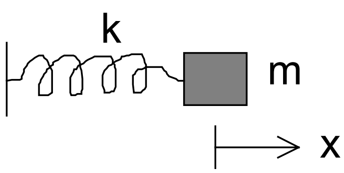
$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{mg}{2L}s_o^2 \sin^2(\omega t + \delta)$$

Total mechanical energy: $E = K + U = \frac{mg}{2L}s_o^2 \cos^2(\omega t + \delta) + \frac{mg}{2L}s_o^2 \sin^2(\omega t + \delta)$

$$\therefore \quad \boxed{E = \frac{mg}{2L}s_o^2 = \frac{mgL}{2}\theta_o^2}$$

Simple Harmonic Motion – A Summary

SPRING

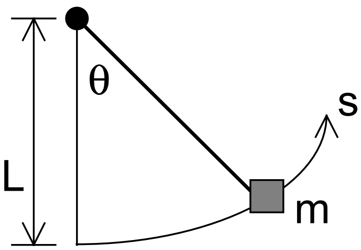


$F = ma$

$\Downarrow$

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

PENDULUM



$F = ma$

$\Downarrow$

$$\boxed{\frac{d^2s}{dt^2} = -\frac{g}{L}s}$$ for small angles

Both equations have the same form.

SOLUTION:

$$x = A\cos(\omega t + \delta)$$

$$\omega = \sqrt{\frac{k}{m}}$$

SOLUTION:

$$s(t) = s_o \cos(\omega t + \delta)$$

$$\omega = \sqrt{\frac{g}{L}}$$

A and s_o are the maximum displacements.

ENERGY

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2$$

$$= \tfrac{1}{2}kA^2 \sin^2(\omega t + \delta)$$

$$U = \tfrac{1}{2}kx^2$$

$$= \tfrac{1}{2}kA^2 \cos^2(\omega t + \delta)$$

$$\boxed{E_m = U + K = \tfrac{1}{2}kA^2}$$

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\frac{ds}{dt}\right)^2$$

$$= \tfrac{1}{2}\frac{mg}{L}s_o^2 \sin^2(\omega t + \delta)$$

$$U = \tfrac{1}{2}\frac{mg}{L}s^2$$

$$= \tfrac{1}{2}\frac{mg}{L}s_o^2 \cos^2(\omega t + \delta)$$

(we showed for small angles)

$$\boxed{E_m = U + K = \tfrac{1}{2}\frac{mg}{L}s_o^2}$$

Mechanical energy is conserved.

Section IV.4 Damped and Driven Harmonic Motion, Resonance

Damped Harmonic Motion

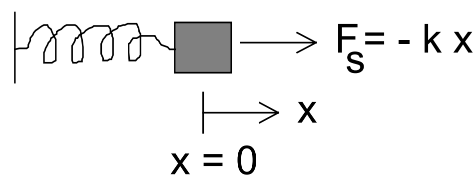
Simple harmonic motion includes no forces which can dissipate energy so mechanical energy is conserved. However, in real oscillating systems, the energy IS usually dissipated by forces like friction. The result is NOT simple harmonic motion. Such motion is called damped harmonic motion.

In many systems, the damping force is approximately proportional to velocity and is in the opposite direction:

$$F_d \propto \frac{dx}{dt} = -b\frac{dx}{dt}$$

where
b = constant that describes the strength of the damping
"– " sign indicates that the damping opposes the motion
If $v > 0$, then $F_d < 0$. If $v < 0$, then $F_d > 0$.

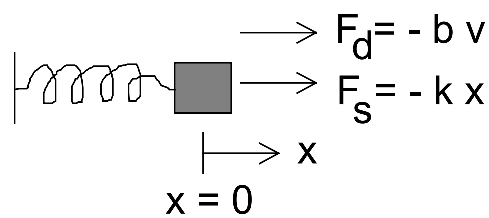
Reconsider the case of a spring with no damping:



$$F_s = -kx = ma \quad \text{so} \quad \boxed{-kx = m\frac{d^2x}{dt^2}}$$

$\Rightarrow$ this is the eqn of motion for SHM, with $\omega = \sqrt{\frac{k}{m}}$

Now, what if damping is active? The equation of motion becomes:



$$F_{net} = -kx - bv = ma \quad \text{so} \quad \boxed{m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0}$$

This is a second order ordinary differential equation. How do we solve it?

b = 0 : $x = A\cos(\omega t + \delta)$ SHM

b ≠ 0 : try a solution of the form $x = Ae^{-\alpha t}\cos(\omega t + \delta)$
where $e^{-\alpha t}$ takes the damping into account

So

$$\frac{dx}{dt} = -\omega Ae^{-\alpha t}\sin(\omega t + \delta) - \alpha Ae^{-\alpha t}\cos(\omega t + \delta)$$

$$\frac{d^2x}{dt^2} = -\omega^2 Ae^{-\alpha t}\cos(\omega t + \delta) + \omega\alpha Ae^{-\alpha t}\sin(\omega t + \delta)$$

$$+ \alpha\omega Ae^{-\alpha t}\sin(\omega t + \delta) + \alpha^2 Ae^{-\alpha t}\cos(\omega t + \delta)$$

Substitute these expressions back into the equation of motion…