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The Chaotic Motion of the Solar System: A Numerical Estimate of the Size of the Chaotic Zones

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In a previous paper (J. Laskar, *Nature* 338, 237–238), the chaotic nature of the Solar System excluding Pluto was established by the numerical computation of the maximum Lyapunov exponent of its secular system over 200 myr. In the present paper an explanation is given for the exponential divergence of the orbits: it is due to the transition from libration to circulation of the critical argument of the secular resonance $2(g_4 - g_3) - (s_4 - s_3)$ related to the motions of perihelions and nodes of Earth and Mars. Another important secular resonance is identified: $(g_1 - g_5) - (s_1 - s_2)$. Its critical argument stays in libration over 200 myr with a period of about 10 myr and amplitudes from 85 to 135°. The main features of the solutions of the inner planets are now identified when taking these resonances into account. Estimates of the size of the chaotic regions are determined by a new numerical method using the evolution with time of the fundamental frequencies. The chaotic regions in the inner Solar System are large and correspond to variations of about 0.2 arcsec/year in the fundamental frequencies. The chaotic nature of the inner Solar System can thus be considered as robust against small variations in the initial conditions or in the model. The chaotic regions related to the outer planet frequencies are very thin except for those of g_6 which present variations sufficiently large to be significant over the age of the Solar System.

1. INTRODUCTION

In the last few years, our vision of the dynamics of the Solar System has notably changed and the picture of the planets moving around the Sun in a regular quasi-periodic motion has suffered many outrages. In particular, Sussman and Wisdom (1988) have shown by direct numerical integration of the outer planets over nearly 10^9 years with the Digital Orrery that the motion of Pluto presents a positive Lyapounov exponent of 1/20 myr^{-1}. On the other hand, in a recent paper (Laskar 1989a), I established the chaotic nature of the solutions of the secular system for the Solar System excluding Pluto (the mass of Pluto is only 1/130,000,000 of the Solar mass) with a maximum Lyapunov exponent reaching the value of 1/5 myr^{-1}.

From a rapid analysis of the resonances in the secular system, it was possible to forecast that this high value of the Lyapunov exponent was mainly due to the existence of secular resonances among the inner planets, but this needed more explanation. One of the major questions, faced with the appearance of chaos in the Solar System, was to know where this chaos comes from, and what the sizes of the chaotic zones are. Do very small changes in the initial conditions or in the approximation modify the behavior of the solutions from chaotic to quasi-periodic, or do we stay in a large chaotic zone? I believe that I give here the answer to some of these questions. In the first part (Sections 2–5), I describe further the construction of the secular system and the accuracy of this system. In particular, I use a little trick to improve the

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length of validity of the solution of the outer planets which becomes very close to the results of direct numerical integrations. In the second part (Sections 6–8), I describe the secular resonances which are present in the secular system and analyze their effects on the appearance of the chaos. In the last section (Section 9), I give a new method for a numerical estimation of the size of the chaotic zones, based on the variation with time of the main frequencies of the system. This analysis confirms that the chaos comes mainly from the inner Solar System and in particular from the resonance $2(g_3 - g_4) - (s_3 - s_4)$ between Mars and Earth. The method used for the analysis of the size of the chaotic zones is well suited for the dynamical system of many degrees of freedom when the classical surface of section method is no longer possible. It can be used only when the chaos is not too large (i.e., when the Lyapunov exponent is small with respect to the main frequencies of the system).

2. THE LAGRANGE EQUATIONS AND THE SECULAR SYSTEM

The construction and integration of the secular system for the eight major planets of the Solar System were described in Laskar (1984, 1985, 1986, 1988). Its construction is based on the earlier works of Brumberg (1970), Brumberg and Chapront (1973), Abu El Ata and Chapront (1974), and Duriez (1977, 1979). The main improvement with respect to these previous works consists in the extension to the eight main planets of the Solar System and special dedicated techniques for the manipulation of the series (Laskar 1985), which allowed an extensive computation of the secular system of order 2 with respect to the masses, up to degree 5 in eccentricities and inclinations.

The disturbing function is developed in noncanonical heliocentric coordinates (Abu El Ata and Chapront 1974) with the notations:

a, semimajor axis;
e, eccentricity;
i, inclination;
Ω, longitude of the ascending node;
ϖ, longitude of perihelion;
n, mean motion;
λ, mean longitude;
N, mean mean motion;
p, deviation from the mean mean motion, defined by $n = N(1 + p)$;
q, defined by $\lambda = Nt - \sqrt{-1}q$;
$z = e \exp\sqrt{-1}\varpi$;
$\zeta = \sin(i/2) \exp\sqrt{-1}\Omega$;
g_i, s_i, fundamental frequencies of the secular system. (1)

If we gather the variables $p_i, q_i, z_i, \bar{z}_i, \zeta_i, \bar{\zeta}_i$ $(i = 1, 8)$ in a unique vector V, the Lagrange equations giving the variations of the elliptical elements of the planets can be written in the condensed form

$$\frac{dV}{dt} = \Lambda(V, t). \tag{2}$$

The right member can be expanded in Fourier series with generic term of the form

$$T = Cp_i^{n_1}\bar{z}_i^{n_2}\bar{\zeta}_i^{n_3}z_i^{n_4}\zeta_i^{n_5}p_j^{n_6}\bar{z}_j^{n_7}\bar{\zeta}_j^{n_8}z_j^{n_9}\zeta_j^{n_{10}} \times \exp(k_iq_i + k_jq_j + \sqrt{-1}(k_iN_i + k_jN_j)t), \tag{3}$$

where C is a numerical coefficient depending only on the masses of the planets, and $k_i, k_j, n_1, n_2, \ldots, n_{10}$ are integers. In order to integrate this system, we can split V into a secular part V_0 and a short period part $\Delta V(V_0, t)$, which is supposed to be small compared to the secular part.

$$V = V_0 + \Delta V(V_0, t). \tag{4}$$

Formally expanding Eq. (2) in Taylor expansion around V_0 we obtain

$$\frac{dV_0}{dt} + \frac{\partial \Delta V}{\partial V_0}(V_0, t)\frac{dV_0}{dt} + \frac{\partial \Delta V}{\partial t} = \Lambda(V_0, t) + \frac{\partial \Lambda}{\partial V_0}(V_0, t)\Delta V(V_0, t) + \frac{1}{2}\frac{\partial^2 \Lambda}{\partial V_0^2}(V_0, t)(\Delta V(V_0, t))^2 + \cdots. \tag{5}$$

V_0 is defined as the secular part of Eq. (4), that is the solution of

$$\frac{dV_0}{dt} = \langle \Lambda(V_0, t)\rangle + \left\langle \frac{\partial \Lambda}{\partial V_0}(V_0, t)\Delta V(V_0, t)\right\rangle + \frac{1}{2}\left\langle \frac{\partial^2 \Lambda}{\partial V_0^2}(V_0, t)(\Delta V(V_0, t))^2\right\rangle + \ldots, \tag{6}$$

where $\langle x\rangle$ is the average value of x with respect to the time t, and $\Delta V(V_0, t)$ is a short period solution given by

$$\frac{\partial \Delta V}{\partial t} = -\frac{\partial \Delta V}{\partial V_0}(V_0, t)\frac{dV_0}{dt} + \{\Lambda(V_0, t)\} + \left\{\frac{\partial \Lambda}{\partial V_0}(V_0, t)\Delta V(V_0, t)\right\} + \ldots. \tag{7}$$

The solutions of Eqs. (6) and (7) are then obtained order by order with respect to the masses (Duriez 1977, 1979, Laskar 1984, 1985). At the second order with respect to the masses, we obtain the secular system of order 2

$$\frac{dV_0}{dt} = \langle \Lambda(V_0, t)\rangle + \left\langle \frac{\partial \Lambda}{\partial V_0}(V_0, t)\Delta_1 V(V_0, t)\right\rangle, \tag{8}$$

where $\Delta_1 V(V_0, t)$ is a short period solution of order 1, that is a solution of

$$\frac{\partial \Delta V}{\partial t} = \{\Lambda(V_0, t)\}. \tag{9}$$

Poisson has shown that there are no secular variation up to the second order in the semimajor axes at the second order with respect to the masses. (It is no longer the case at third order as is shown by the studies of Simon and Francou (1981) and Milani *et al.* (1987)). In the secular system of second order (Eq. (5)), the semimajor axes are therefore constants; that is, the secular part of the variables p_i is constant up to order 2 with respect to the masses. The secular system (Eq. (8)) has thus only (!!) 16 degrees of freedom (the phase space is of dimension 8 $\times$ 4 = 32). It has a polynomial form and can be written in the form

$$\frac{d\alpha}{dt} = \sqrt{-1}(A\alpha + \Phi_3(\alpha, \bar{\alpha}) + \Phi_5(\alpha, \bar{\alpha})) \tag{10}$$

with

$$\alpha = (z_1, \ldots, z_8, \zeta_1, \ldots, \zeta_8) = \begin{cases} z = e \exp\sqrt{-1}\varpi \\ \zeta = \sin(i/2)\exp\sqrt{-1}\Omega \end{cases}. \tag{11}$$

A is a real matrix with constant coefficients; $\Phi_3(\alpha, \bar{\alpha})$ gathers the terms of degree 3 and $\Phi_5(\alpha, \bar{\alpha})$ the terms of degree 5 in the variables $(z_1, \ldots, z_8, \zeta_1, \ldots, \zeta_8)$. Of course, we are not here in the nice Hamiltonian formalism which people like to see today and which can be applied using the canonical heliocentric coordinates of Poincaré in the planetary case (Laskar 1989b). This implies some complications in the formulation of the equations, but these options were taken in order to make a more easy comparison with previous works and allow one to take directly the initial conditions given in Bretagnon (1982). Up to order 2 with respect to the masses, this does not change much in the computations, and let us remember the words of Poincaré (1893, p. 37): "*Les équations où s'introduisent les crochets de Lagrange prennent ainsi une forme en apparence plus compliquée. Mais cette différence n'a rien d'essentiel.*"

3. PROPER MODES OF THE SECULAR SYSTEM

We are in the planetary case. The eccentricities and inclinations are always supposed to be small. The terms of degree 3 and 5 are thus small in a first approximation and the differential system is close to its linear part which is integrable. In the same way as in the resolution of linear differential systems, the first thing to do is to diagonalize the linear part of Eq. (10). Let S be the matrix of the eigenvectors of A and C the diagonal matrix of its eigenvalues (c_i). We can diagonalize the system (Eq. (10)) by the linear change of variables

$$\alpha = S\beta \tag{12}$$

which leads to the new system

$$\frac{d\beta}{dt} = \sqrt{-1}(C\beta + \Psi_3(\beta, \bar{\beta}) + \Psi_5(\beta, \bar{\beta})). \tag{13}$$

With analogy to the elliptical elements, the new variables β will be denoted

$$\beta = (z_1^*, \ldots, z_8^*, \zeta_1^*, \ldots, \zeta_8^*) = \begin{cases} z^* = e^* \exp\sqrt{-1}\varpi^* \\ \zeta^* = \sin(i^*/2) \exp\sqrt{-1}\Omega^* \end{cases}. \tag{14}$$

As in the linear case, the variables β_i will be called the proper modes of the secular system. This terminology was introduced in the case of the secular system by Laskar (1987) in order to analyze the effects of the resonances in the Solar System. It is important to note that due to the coupling between the planets, the secular motions of the planets are not, in first approximation, independent uniform rotations of the perihelion and nodes, but rotations of the proper modes.

4. BIRKHOFF NORMALIZATION OF THE SECULAR SYSTEM

The formal resolution of the complete system (Eq. (13)) was made by Poincaré, or more explicitly in Birkhoff (1927). An exposition of this method adapted to the present case can be found in Brumberg (1980). One searches for a change of variables close to the identify

$$B = U + \Gamma(U, \bar{U}) \tag{15}$$

which transforms the differential system (Eq. (10)) into a new system gathering only the resonant terms

$$\frac{dU}{dt} = \sqrt{-1}CU + \sqrt{-1}F(U, \bar{U}), \tag{16}$$

where $F(U, \bar{U})$ denotes the resonant terms, that is for the equation i

$$F_i = u_i \sum^{*} \Psi_{(n\bar{n})}^{(i)} \prod_{j=1}^{16} (u_j\bar{u}_j)^{\bar{n}_j}. \tag{17}$$

$(n\bar{n})$ denotes the multi-index $(n_1, n_2, \ldots, n_{16}, \bar{n}_1, \bar{n}_2, \ldots, \bar{n}_{16})$. The coefficients $\Psi_{(n\bar{n})}^{(i)}$ are real numbers and the Σ^* denotes the sum over the reasonant terms, that is, $n_j = \bar{n}_j + \delta_{ij}$ for $(j = 1, 16)$. The moduli $|u_i|$ are then integrals of Eq. (16). The differential system (Eq. (16)) is thus integrable and

$$\frac{dU}{dt} = \sqrt{-1}(C + \delta C)U \tag{18}$$

with

$$\delta C = \sum^{*} \Psi_{(n\bar{n})}^{(i)} \prod_{j=1}^{16} |u_j|^{2n_j} \tag{19}$$

which gives

$$u_i = u_i(0) \exp\sqrt{-1}(c_i + \delta c_i)t. \tag{20}$$

The moduli $|u_j|$ are constant along the orbits of Eq. (16). The linear system with constant coefficients (Eq. (18)) is thus a linear approximation of the solution of Eq. (16) in the vicinity of the given initial conditions. The solutions (Eq. (20)) of Eq. (18) can thus be referred to as the proper modes of the solution of the secular system $\mathcal{S}$ with given initial conditions $u_i(0)$, in contrast with β_i which are the proper modes of the secular system $\mathcal{S}$ itself.

Unfortunately, Birkhoff's normalization method is only valid formally. If the differential system (Eq. (13)) is not integrable, which is the general case, it is not possible to transform this system into an integrable system (Eq. (16)) by an analytical change of variables (Eq. (15)). The expression (15) is a formal series which is divergent, in general. It is nevertheless possible to obtain in this way a good approximation of the solution valid over a limited time, if the first terms of Eq. (15) *converge rapidly enough*. Practically, the development of Eq. (15) is truncated at the same degree as the secular system (Eq. (10)). In the first approximation of degree 3, for example, the coefficient of a monomial $U^{(n\bar{n})}$ is given by

$$\Gamma_{(n\bar{n})}^{(i)} = \Psi_{(n\bar{n})}^{(i)} \Big/ \left(\sum_{j=1}^{16} (n_j - \bar{n}_j)c_j - c_i\right). \tag{21}$$

The denominator is not zero because we have excluded the resonant terms, but it

can become very small. We call it a small divisor. The presence of these small divisors in the expansions of $\Gamma(U, \overline{U})$ prevents the convergence of this formal series. The earlier these small divisors will appear and the larger their number in the expansions, the less the expansion (Eq. (15)) will approximate the true solution of the differential system (Eq. (13)).

Practically, there is no method to know in advance if the small divisors will prevent the construction of a solution of sufficient accuracy. Only the computation itself and the comparison with numerical integrations or with analytical or semianalytical solutions of higher order can allow one to evaluate the precision of the computations. From one planetary system to an other, the small divisors can behave in very different manners.

During the construction of an analytical solution for the Solar System, one needs to distinguish between two very different steps. The first one is the averaging of the short periods which leads to the construction of the differential secular system (Eq. (10)). The second one is the integration of this system, for example, with Birkhoff normalization. It is important to distinguish between these two steps since the small divisors which appear in the two steps are different. In the first step, they are combinations of the eight mean mean motions, and in the second step they are combinations of the 16 secular frequencies g_i, s_i (i = 1, 8) of the system (Eq. (10)). In fact only 15 frequencies are present since, due to the degeneracy of the system, $s_5 = 0$. In Laskar (1984), I realized that the analytical averaging of second order which leads to the construction of the secular system behaves much better than the Birkhoff normalization of this system, due to numerous small divisors present in the secular system. For this reason, the Birkhoff normalization of the secular system was abandoned and replaced by a numerical integration of the secular system for which small divisors are not an immediate problem.

5. ACCURACY OF THE SECULAR SYSTEM

The secular system of order 2 (Eq. (10)) was obtained in an extensive manner up to the degree 5 in eccentricity and inclination. It contains about 150,000 polynomial terms. The secular effects of general relativity and the Moon represent a few terms which are then added to the secular system (Laskar 1985, 1986). The difficult question is then to evaluate the precision of this secular system $\mathcal{S}$ (Laskar 1986). The secular system $\mathcal{S}$ is derived in the same variables as the semianalytical theory VSOP82 (Bretagnon 1982). In VSOP82, the secular terms are expanded in polynomial of the time t. VSOP82 is developed up to order 3 with respect to the masses for the inner planets, and even more by an iterative way for the outer planets. It has also been compared with ephemerides resulting from direct numerical integrations of the planets (DE200 and DE102) (Newhall *et al.* 1983). The expansion in power of t of the secular terms of VSOP82 can thus be used as a reference in order to check the accuracy of the secular system $\mathcal{S}$. In this expansion, the coefficient of t is obtained with a precision of order 3 (at least) with respect to the masses, the coefficient of t^2 at order 2 and the coefficient of t^3 at order 1. Moreover, as VSOP82 is semianalytical, there is no truncation in the powers of eccentricity and inclination. The similar polynomial expansions are obtained for the solutions of $\mathcal{S}$ by direct numerical integration of $\mathcal{S}$ with the initial conditions of VSOP82. Basically, the only comparison we can make is the comparison of the derivatives of the secular terms at the origin, which reflect the behavior of the solutions over the first few thousand years. The comparison of the coefficient of t in the two solutions is then an estimate of the precision of the secular system $\mathcal{S}$ at the origin. This estimate is given in Table I. In this table the quantities $\delta z^1 = |\Delta z^1|/|z^1|$ and $\delta\zeta^1 = |\Delta\zeta^1|/|\zeta^1|$ are given, where z^1 and ζ^1 denotes the coefficient of t in the polynomial expansion of the secular variables z and ζ

TABLE I

PRECISION OF THE SECULAR SYSTEM

i	$\lvert\Delta z_i^1\rvert$ (yr^{-1})	$\lvert\Delta\zeta_i^1\rvert$ (yr^{-1})	$\frac{\lvert\Delta z_i^1\rvert}{\lvert z_i^1\rvert}$	$\frac{\lvert\Delta\zeta_i^1\rvert}{\lvert\zeta_i^1\rvert}$
		before correction		
1	964 10^{-12}	366 10^{-12}	0.000169	0.000255
2	44 10^{-12}	62 10^{-12}	0.000092	0.000043
3	71 10^{-12}	75 10^{-12}	0.000069	0.000066
4	549 10^{-12}	246 10^{-12}	0.000075	0.000225
5	14126 10^{-12}	149 10^{-12}	0.005759	0.000381
6	66055 10^{-12}	213 10^{-12}	0.010182	0.000213
7	3606 10^{-12}	931 10^{-12}	0.004693	0.005442
8	2706 10^{-12}	74 10^{-12}	0.035447	0.002869
		after correction		
1	964 10^{-12}	366 10^{-12}	0.000169	0.000255
2	44 10^{-12}	62 10^{-12}	0.000092	0.000043
3	71 10^{-12}	75 10^{-12}	0.000069	0.000066
4	549 10^{-12}	246 10^{-12}	0.000075	0.000225
5	5635 10^{-12}	102 10^{-12}	0.002297	0.000261
6	5935 10^{-12}	234 10^{-12}	0.000915	0.000234
7	3964 10^{-12}	150 10^{-12}	0.005159	0.000877
8	2390 10^{-12}	30 10^{-12}	0.031307	0.001163

Note. The precision of the secular system is evaluated by comparison of the evaluation of the secular system at the origin with the coefficient of t in the polynomial expansions of the corresponding secular variable in VSOP82. Δ denotes the absolute difference with VSOP82. These evaluations are given before and after the correction applied to the secular system in order to partially take into account the third-order contribution. The relative precision of the secular system is also an estimate of the relative precision obtained on the main frequencies of the system.

for the different planets. Δ denotes the absolutes difference with VSOP82, used as a reference.

If we assume that the secular solution is limited to a single circular term $z = A \exp ict$, we have $z^1 = icA$. If we assume that all the uncertainty comes from the frequency c, the relative measure δz^1 is thus a measure of the error in the frequency c of the differential system $\mathcal{S}$. From Table I we can see that the relative precision of the system for the inner planets is very good and is about 1 to 2×10^{-4} for the inner planets, while it reaches nearly 10^{-2} for the variable z of Saturn. This is not surprising. The 1% error in the solution of Saturn comes from the third order with respect to the masses which has been neglected. This lead to a fundamental frequency of 29.96 arcsec/year for the frequency g_6 while an adjustment by least squares with VSOP82 leads to 28.23 arcsec/year (Laskar 1988), according to the results of long term numerical integrations of Applegate *et al.* (1986) or Carpino *et al.* (1987).

In Laskar (1988), I made an adjustment of the frequency g_6 after the integration of the secular system $\mathcal{S}$; this may give a good value for g_6 but does not give the effect of this change of frequency on the whole solution. In order to obtain a more reliable solution for the outer planets over 100 myr, I did the same thing here in a slightly different manner. The solution for the outer planets (Laskar 1988) is mostly contained in the leading terms which give a quasi-periodic representation of the solution over 100 myr. A small change in the initial conditions will only be reflected in a small change of the frequencies but will not lead to any strong resonance which can destroy the macroscopic aspect of the solution (this is not the case for the inner planets). We can thus adjust slightly the differential system $\mathcal{S}$ in order to force the values of the frequencies of the outer planet's solutions. Of course, this should be done with great care.

The change on the secular system is made only on the outer planets part. Moreover, it should correspond to the change of frequencies expected from the contribution of the third order with respect to the masses. The change will then be made in the diagonalized system (Eq. (13)). This change is obtained by least-squares adjustment with the coefficients of t from VSOP82 which gives 32 data for the whole system (in eccentricity and inclination). I kept only the changes which were very stable under a change of the set of "observations" on which the diagonal terms were fitted. Finally, I modify only three diagonal terms of C in Eq. (13) which correspond to a change of 0.26 arcsec/year in g_6, 0.02 arcsec/year in g_7, and -0.02 arcsec/year in s_7. These changes are made in the matrix C of Eq. (13) and transmitted to A by applying the 4×4 matrix S_4 formed by the part of S concerning only the outer planets. (I do not

touch on the inner planet equations.) The comparison with VSOP82 can be made again and leads to the second part of Table I, after the modification. We can see that we have an improvement for the outer planets by more than a factor 10 in eccentricity (variable z). This will thus allow us to obtain a frequency g_6 with a precision of about 0.02 arcsec/year. This will be more than sufficient to obtain a good orbit for the outer planets over 100 myr. (I do not pretend here to obtain an ephemeris of the outer planets over 100 myr, and I am interested essentially here in a good modeling of the effect of the outer planets on the inner planets.)

Another check of the accuracy of the new solutions for the outer planets will be obtained by direct comparison of the solutions as described in the next section.

6. THE MFT ANALYSIS OF THE SOLUTION OVER 20 MILLION YEARS

I have shown (Laskar 1989a) that the solutions of $\mathcal{S}$ diverge exponentially with a Lyapunov exponent of about 1/5 myr^{-1} (Fig. 1). This implies that it is not possible to give a quasi-periodic solution for the whole Solar System over 100 myr. Nevertheless, it is possible to derive by special Fourier analysis a quasi-periodic approximation of the solution over 20 myr. This was made in Laskar (1988) for 10 myr in the past and in the future with the noncorrected system $\mathcal{S}$. Here I will not try to obtain a very precise representation of the solutions in quasi-periodic form, but I will only search for the first 20 leading terms of the solutions. The Fourier analysis used here is nearly the same as in Laskar (1988); it will be called MFT (modified Fourier transform) from now on and is briefly described below.

Let us assume that we have the tabulated values of a quasi-periodic function $\alpha(t) = \sum_{j=1}^{M} A_j e^{\sqrt{-1}\gamma_j t}$ (where A_j are complex amplitudes) over a time span $[0, T]$. The problem is to make the reconstruction of the function $\alpha(t)$ on a quasi-periodic form,

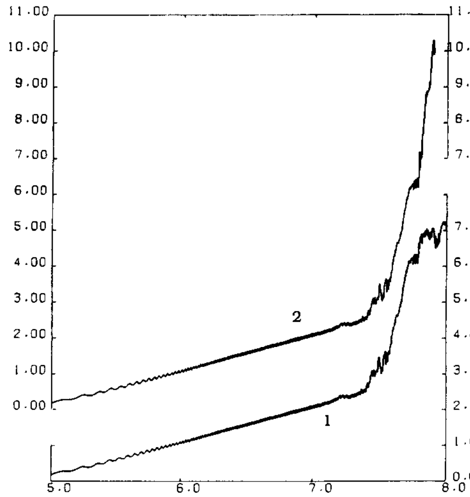


FIG. 1. Divergence of two nearby orbits. Log10(d/d_0) versus Log10(T) (T in years) over 100 myr. At the beginning, the distance of two nearby orbits increases with a nearly linear rate. After about 30 myr, the exponential divergence of the orbits dominates. The initial relative distance in the phase space is $d_0 = 10^{-7}$. In the absence of renormalization, the distance is bounded after about 60 myr (curve 1). With a renormalization every 40 myr, the second orbit stays close to the first one, and the cumulated distance is not bounded (curve 2).

knowing only the tabulated values of the function. The stepsize of the table of values of $\alpha(t)$ is supposed to be much smaller than the smallest period of the function, so that we do not have here any problem of aliasing. The first thing one can do is a FFT (fast Fourier transform) of the function $\alpha(t)$ which gives the projection $\tilde{\alpha}(t) = \sum_{j=0}^{N} \tilde{A}_j e^{\sqrt{-1}j2\pi t/T}$ of the function $\alpha(t)$ on the vector space generated by the functions $(h_j(t) = e^{\sqrt{-1}j2\pi t/T})_{j=1,N}$ which are orthogonal for the usual scalar product

$$\langle f, g\rangle = \int_0^T f(t)\overline{g(t)}dt.$$

It gives a quasi-periodic representation of $\alpha(t)$, but this representation can be very different from the original one. In particular, the determination of the main frequencies γ_j is only made with a precision of $\eta_0 = 2\pi/T$ which is not always sufficient here (it corresponds to a precision of about 0.05 arc-

sec/year for $T = 20$ myr). The MFT analysis will make the orthogonal projection of the function $\alpha(t)$ on the vector space generated by a nonorthogonal basis of function $(e_i(t) = \exp\sqrt{-1}(\nu_i t))_{i=1,N}$ where the values of the frequencies $(\nu_i,\ i = 1, N)$ are to be determined step by step. This will give a quasi-periodic approximation of the function much closer to the original one. The description of the first two steps is sufficient to understand the process:

I first make a FFT of the solution $\alpha(t)$ and search for the term of maximum amplitude in the FFT. Around this term, I search for the value of the frequency ν_1 which gives the maximum of the power spectrum

$$\phi(\nu) = \int_0^T \alpha(t) \exp\sqrt{-1}(-\nu t)dt. \tag{22}$$

This fine tuning of the frequency is made with a given precision of 0.000001 arcsec/year for the outer planets and 0.00001 arcsec/year for the inner planets where the determination is more difficult. This determination of the frequency is critical in this problem, and I improved the method by using a Hanning filter (Blackman and Tukey 1958) in the determination of the frequency in order to reduce the disturbing effect of the other terms (experiment shows a typical improvement by a factor 10 using this method). Once the frequency is found, the power spectrum gives the amplitude of the first term which is then substracted from the solution. The process is started again to search for the following frequency ν_2. The computation of the amplitude of the different terms in the MFT analysis is more difficult than in the regular Fourier analysis since the functions $e_i(t) = \exp\sqrt{-1}(\nu_i t)$ are not orthogonal. On the second step, we thus have to orthogonalize our basis of functions by the Gauss algorithm, and then compute the amplitude of each term by computing the inner products

$$\int_0^T \alpha(t)\overline{f_i(t)}dt, \tag{23}$$

where the $(f_i(t))_{i=1,2}$ are the functions of the orthogonal basis derived from the functions $(e_i(t))_{i=1,2}$ (the $f_i(t)$ are linear combinations of the $e_i(t)$). We thus obtain the decomposition of the orthogonal projection of the solution on the orthogonal basis $(f_i(t))_{i=1,2}$ which gives the decomposition on the nonorthogonal basis $(e_i(t) = \exp\sqrt{-1}(\nu_i t))_{i=1,2}$. The process is then repeated. It is important to notice that in each step the amplitudes of the terms previously determined are modified, but no changes are made in the frequencies.

In this process, I also use a termination test. If T is the length of the time interval of data, the fundamental frequency of the FFT is $\eta_0 = 2\pi/T$. This is also the width of the peak in the periodogram of the unfiltered MFT. Using the Hanning filter, this width is widened to $2\eta_0$. Therefore, the identification of the frequencies is stopped if the value of the new determined frequency is within $1.5\eta_0$ of any of the previously determined frequencies. This explains why solutions are sometimes given with fewer terms than others, when the termination test was reached before the end of the analysis. This does not affect very much the present results as we are here principally interested in the determination of the fundamental frequencies.

Actually, during all of the present analysis, I considered only the first 20 leading terms of the solutions. In order to check the accuracy of the determination of the frequencies, a new solution was constructed out of the results of the MFT and analyzed in the same way again. An offset in the initial conditions is made before doing this new analysis in order to prevent any spurious effect. In Tables II and III, I give the solutions for the variables z_i and ζ_i of the eight planets including only the 10 first terms although 20 terms were computed each time in the solution and in the test solution (for a more complete solution, refer to Laskar, 1988).

The identification of a term as a combination of the fundamental frequencies g_i and s_i

TABLES II and III

SOLUTIONS La90 OBTAINED OVER THE FIRST 20 MYR OF THE PRESENT WORK FOR THE VARIABLES $z = e \exp i\varpi$ AND $\zeta = \sin(i/2) \exp i\Omega$ OF THE EIGHT MAIN PLANETS OF THE SOLAR SYSTEM

var.	i		ν_i (″/yr)	$A_i \times 10^6$	ϕ_i (°)
z_1	1	g_1	5.5964355	185444	111.179
	2	$g_1 - \sigma$	5.4744873	66814	274.217
	3	$g_1 + \sigma$	5.7167139	52366	121.597
	4	g_5	4.2472969	36353	26.449
	5	$g_1 - 2\sigma$	5.3581992	28415	94.683
	6	g_2	7.4559375	27700	20.295
	7	$g_5 + \sigma$	4.3707920	13105	224.261
	8	$g_2 - s_1 + s_2$	5.9923652	10123	114.094
	9		6.9339902	9344	166.112
	10		5.6548916	8548	48.431
z_2	1	g_2	7.4559199	20733	200.618
	2	g_5	4.2487822	19636	30.571
	3	g_4	17.9155986	13464	335.715
	4	g_3	17.3646299	11671	124.965
	5	g_1	5.5964531	6668	112.487
	6	$g_3 - \theta$	17.0827998	2848	175.821
	7	$g_3 - 2\theta$ $2g_3 - g_4$ $g_4 + s_3 - s_4$	16.8131865	2350	274.361
	8	$g_1 - \sigma$	5.4740830	2242	272.955
	9		6.9349658	2184	170.325
	10	$g_1 + \sigma$	5.7163008	1893	123.764
z_3	1	g_5	4.2487910	18913	30.597
	2	g_2	7.4559199	16047	200.630
	3	g_4	17.9155811	13159	155.568
	4	g_3	17.3646123	9406	304.902
	5	g_1	5.5964883	4248	112.565
	6	$g_3 - \theta$	17.0830020	2238	351.000
	7	$g_3 + \theta$	17.6287061	1956	7.770
	8	$g_3 - 2\theta$ $2g_3 - g_4$ $g_4 + s_3 - s_4$	16.8132568	1722	94.866
	9		6.9349219	1598	169.028
	10	g_6	28.2206602	1506	128.008
z_4	1	g_4	17.9154053	49032	334.852
	2	g_3	17.3647617	40133	305.507
	3	g_5	4.2488174	20300	30.679
	4	$g_3 - 2\theta$ $2g_3 - g_4$ $g_4 + s_3 - s_4$	16.8127207	18587	91.894
	5	$g_4 + 2\theta$ $2g_4 - g_3$ $g_3 - s_3 + s_4$	18.4679297	13583	9.651
	6	$g_3 - \theta$	17.0825625	12109	356.776
	7	$g_3 + \theta$	17.6322393	11559	204.018
	8	g_6	28.2206777	7030	128.063
	9		17.8093564	5094	50.822
	10	$g_4 + 4\theta$ $g_4 - s_3 + s_4$	19.0186963	4802	40.701

var.	i		ν_i (″/yr)	$A_i \times 10^6$	ϕ_i (°)
z_5	1	g_5	4.2488163	44119	30.676
	2	g_6	28.2206942	15750	308.112
	3	g_7	3.0895148	1800	121.362
	4	$-g_5 + 2g_6$	52.1925732	516	45.551
	5	$-g_5 + g_6 + g_7$	27.0613982	183	218.696
	6	$g_5 + g_6 - g_7$	29.3799573	178	217.460
	7	$-g_5 + 2g_6 + s_6 - s_7$	28.8679427	107	32.614
	8	$g_5 - s_6 + s_7$	27.5734578	95	43.733
	9	$2g_5 - g_7$	5.4070444	62	116.984
	10	g_8	0.6671228	58	74.116
z_6	1	g_6	28.2206942	48187	128.112
	2	g_5	4.2488163	33142	30.676
	3	$-g_5 + 2g_6$	52.1925732	1735	225.551
	4	g_7	3.0895148	1513	121.363
	5	$-g_5 + g_6 + g_7$	27.0614004	555	38.704
	6	$g_5 + g_6 - g_7$	29.3799880	549	37.558
	7	$-g_5 + 2g_6 + s_6 - s_7$	28.8679526	329	212.643
	8	$g_5 - s_6 + s_7$	27.5734600	288	223.742
	9	$2g_6 - g_7$	53.3518802	147	134.906
	10	$2g_5 - g_6$	−19.7230616	141	113.239
z_7	1	g_5	4.2488152	37351	210.671
	2	g_7	3.0895181	29107	121.370
	3	g_8	0.6671217	1641	74.103
	4	g_6	28.2206942	1540	308.111
	5	$-g_5 + 2g_7$	1.9298958	423	208.234
	6	$2g_5 - g_7$	5.4080991	411	301.393
	7	$g_8 - s_7 + s_8$	2.9771071	77	306.915
	8	$-g_5 + g_6 + g_7$	27.0613982	55	218.783
	9	$g_6 + s_6 - s_7$	4.8962208	48	115.763
	10	$2g_5 - g_6$	−19.7230660	40	293.231
z_8	1	g_8	0.6670822	9108	73.974
	2	g_7	3.0895148	3610	301.357
	3	g_5	4.2488152	1771	30.669
	4	$g_7 + s_7 - s_8$	0.7783945	156	65.196
	5	g_6	28.2206942	100	308.112
	6	$2g_5 - g_7$	5.4085913	48	127.645
	7	$-g_5 + g_7 + g_8$	−0.4921633	47	164.734
	8	$g_5 - g_7 + g_8$	1.8265408	42	163.848
	9	$-g_5 + 2g_7$	1.9313492	25	72.643

Note. In general, 20 terms are computed in the solutions, but only the first 10 largest terms are given here. Sometimes, solutions have fewer terms when the determination of the 10 first terms is not possible with full satisfaction. When possible, the identification of a term as a combination of the fundamental frequencies g_i, s_i and of the two libration frequencies θ and σ is given in the first column. The frequency of the term, its amplitude, and phase, as determined by the MFT (modified Fourier transform) are given in the next columns. On several occasions, due to resonances, several combinations of frequencies correspond to the same term. In this case, they are all given in column 1. The origin of time is J2000 (JD 2451545.0) and the reference frame is the ecliptic and equinox J2000.

TABLE III

var.	i		ν_i (″/yr)	$A_i \times 10^6$	ϕ_i (°)
ζ_1	1	s_1	−5.6173887	39957	349.155
	2	s_2	−7.0795195	30169	274.203
	3	$s_2 - \sigma$	−7.1949023	15844	104.982
	4	$s_1 + \sigma$	−5.5006436	15737	344.144
	5	s_5	0.0000000	13724	107.583
	6	$s_2 + \sigma$	−6.9609814	13393	97.943
	7	$s_2 + 2\sigma$	−6.8408877	11854	106.719
	8	$s_2 - 2\sigma$	−7.3326621	9580	196.543
	9		−5.0980781	4935	202.024
	10	$g_2 - g_1 + s_2$	−5.2139971	4810	26.013
ζ_2	1	s_5	0.0000000	13772	107.586
	2	s_3	−18.8511768	9544	60.007
	3	s_1	−5.6176699	6716	348.347
	4	s_4	−17.7482461	5759	123.906
	5	s_2	−7.0797656	4045	93.231
	6	$s_3 + 2\theta$; $-g_3 + g_4 + s_3$; $g_3 - g_4 + s_4$	−18.2999180	2980	90.470
	7	$s_1 + \sigma$	−5.5012852	2192	341.685
	8	$s_2 + 2\sigma$	−6.8439814	2116	276.764
	9	$s_2 + \sigma$	−6.9634072	2068	269.969
	10	$s_3 - 2\theta$; $g_3 - g_4 + s_3$	−19.4023477	1910	209.000
ζ_3	1	s_5	0.0000000	13773	107.586
	2	s_3	−18.8511680	8760	240.034
	3	s_1	−5.6176875	4960	348.307
	4	s_4	−17.7482549	4024	303.928
	5	s_2	−7.0797568	3431	93.267
	6	$s_3 + 2\theta$; $-g_3 + g_4 + s_3$; $g_3 - g_4 + s_4$	−18.2999619	2846	270.439
	7	$s_2 + 2\sigma$	−6.8410195	1736	286.932
	8	$s_2 + \sigma$	−6.9609463	1736	277.682
	9	$s_2 - \sigma$	−7.1949375	1610	285.060
	10	$s_3 - 2\theta$; $g_3 - g_4 + s_3$	−19.4022773	1607	29.348
ζ_4	1	s_4	−17.7481582	34689	303.401
	2	s_3	−18.8510273	15421	60.558
	3	s_5	0.0000000	13753	107.585
	4	$s_3 + 2\theta$; $-g_3 + g_4 + s_3$; $g_3 - g_4 + s_4$	−18.3004717	7481	90.022
	5	$s_4 + 2\theta$; $-g_3 + g_4 + s_4$	−17.1969609	5414	154.897
	6	s_6	−26.3302383	4579	127.270
	7	$s_4 - \theta$	−18.0110039	2619	72.168
	8		−17.6605312	2560	321.350
	9		−17.5475127	1889	65.444
	10		−17.8386855	1794	122.443
ζ_5	1	s_5	0.0000000	13775	107.586
	2	s_6	−26.3302328	3151	307.291
	3	s_8	−0.6918937	581	23.944
	4	s_7	−3.0055737	481	140.321
	5	$g_5 - g_6 + s_7$	−26.9774561	23	222.971
	6	$-g_5 + g_6 + s_6$	−2.3583505	16	44.728
	7	$2g_6 - s_6$	82.7716311	14	308.964
	8	$g_5 - g_7 + s_8$; $-g_5 + g_7 + s_8$	−1.8463348	11	37.109
	9	s_1	−5.6176381	11	168.467
	10	$-g_5 + g_7 + s_7$	−4.1647610	10	51.202
ζ_6	1	s_5	0.0000000	13774	107.586
	2	s_6	−26.3302328	7850	127.291
	3	s_8	−0.6918926	560	23.949
	4	s_7	−3.0055671	391	140.341
	5	$g_5 - g_6 + s_7$	−26.9774286	59	43.045
	6	$2g_6 - s_6$	82.7716256	34	128.948
	7	$g_5 + g_6 - s_6$	58.8001674	20	212.895
	8	$2g_5 - s_6$	34.8278774	16	294.100
	9	s_1	−5.6176469	14	168.653
	10	s_4	−17.7481769	13	124.072
ζ_7	1	s_5	0.0000000	13774	107.584
	2	s_7	−3.0055704	8871	320.330
	3	s_8	−0.6919222	563	203.676
	4	s_6	−26.3302328	348	307.291
	5	$-g_5 + g_6 + s_6$	−2.3583494	300	224.748
	6	$-g_5 + g_7 + s_7$	−4.1646797	188	231.632
	7	$g_5 - g_7 + s_8$; $-g_5 + g_7 + s_8$	−1.8462480	183	224.622
	8	$-g_7 + g_8 + s_8$	−3.1172552	59	146.941
	9	$g_6 - g_7 + s_6$	−1.1990610	26	314.001
	10	$2g_5 - s_7$	11.5031942	19	101.013
ζ_8	1	s_5	−0.0000033	13774	107.575
	2	s_8	−0.6918915	5819	203.963
	3	s_7	−3.0055616	1057	140.357
	4	$g_7 - g_8 + s_7$	−0.5803352	57	197.281
	5	$-g_5 + g_6 + s_6$	−2.3583109	56	44.901
	6	s_6	−26.3302328	38	307.291
	7	$-g_5 + g_7 + s_7$	−4.1645336	27	52.137
	8	$g_5 - g_7 + s_8$; $-g_5 + g_7 + s_8$	−1.8499471	18	92.446
	9	$g_5 - g_7 + s_8$	0.4654841	17	106.940
	10	$g_6 - g_7 + s_6$	−1.1989885	16	134.272

of the system is given when possible in column 1. As in Laskar (1988), very soon several terms are impossible to identify, but if we introduce the two additional frequencies

$$\theta = 0.28 \text{ arcsec/year}$$

$$\sigma = 0.12 \text{ arcsec/year}$$

then most of the leading features of the inner planets are recognized. These two arguments θ and σ will be discussed in Section 8. In the second columns of Tables II and III are the values of the frequency as determined by the MFT, and in the next two columns the values of the amplitude and phase (in degrees) of the term.

We have seen already that the modification to the secular system of the outer planets, adopted in order to take into account partially the contribution of the third order,

TABLE IV

COMPARISON OF THE AMPLITUDES OF THE PRINCIPAL TERMS OF THE SOLUTION OF JUPITER La88 (LASKAR 1988) AND La90 (PRESENT WORK) WITH THE RESULTS OBTAINED WITH DIRECT NUMERICAL INTEGRATION ADGSW (APPLEGATE *et al.* 1986), CMN (CARPINO *et al.* 1987), AND NMC (NOBILI *et al.* 1988)

	Jupiter	$z = e \exp \sqrt{-1}\varpi$				
		$A_j \times 10^6$				
j	ν_j	La88	La90	CMN	NMC	ADGSW
1	g_5	44130	44119	44166	44187	447⋆⋆
2	g_6	15746	15750	15693	15700	158⋆⋆
3	g_7	1798	1800	1835	1814	186⋆
4	$-g_5 + 2g_6$	522	516	571	574	575
5	$-g_5 + g_6 + g_7$	180	183	200	198	200
6	$g_5 + g_6 - g_7$	176	178	195	193	195
7	$-g_5 + 2g_6 + s_6 - s_7$	334	107	119	122	117
8	$g_5 - s_6 + s_7$	340	95	107	110	105
9	$2g_5 - g_7$	62	62	68	66	69
10	g_8	56	58	59	58	59
	Jupiter	$\zeta = \sin i/2 \exp \sqrt{-1}\Omega$				
		$A_j \times 10^6$				
j	ν_j	La88	La90	CMN	NMC	ADGSW
1	s_6	3148.1	3151.2	3153.3	3152.2	316⋆
2	s_8	580.5	580.9	584.5	576.4	575
3	s_7	483.0	481.3	484.8	482.4	479
4	$g_5 - g_6 + s_7$	42.8	23.0	23.8	23.5	23
5	$-g_5 + g_6 + l_6$	29.2	16.1	17.9	17.5	17
6	$2g_6 - s_6$	13.8	13.7	14.3	14.3	
7	$g_5 - g_7 + s_7$	11.3	11.3	11.7	10.5	11
8	s_1	11.2	10.7			
9	$-g_5 + g_7 + s_7$	9.6	9.8	9.8	9.9	10
10	$g_5 + g_6 - s_6$	8.5	8.5	8.5	8.5	9

improved the values of the derivatives at the origin (Table I). Here, we can directly compare the solutions with the output of the numerical integrations from Applegate *et al.* (1986), Carpino *et al.* (1987), or Nobili *et al.* (1989). The solutions from the present integrations of the modified system (denoted by La90) are now even closer to the results of the numerical integrations of the outer planets, compared to the same quantities from Laskar (1988). (These previous solutions are denoted La88.) As an example, in Table IV I give the comparison of the semianalytical solutions La88 and La90 of Jupiter with the corresponding solutions obtained by numerical integration. We can observe that due to the change of the frequency g_6, the amplitude of the terms of argument $2g_6 - g_5 + s_6 - s_7$, $g_5 - s_6 + s_7$, $g_5 - g_6 + s_7$, and $-g_5 + g_6 + s_6$ are now very close to the corresponding terms of CMN, ADGSW, or NMC, according to what was forecasted in Laskar (1988). This gives one more check of the accuracy of the secular system $\mathcal{S}$ and of the correction which was made. The same kind of comparison can be made on the solution of the other outer planets.

7. ANALYSIS OF THE PROPER MODES

In the following, we will be interested only in the determination of the fundamental frequencies of the system. I present here a simple method of further improving the determination of these frequencies.

The proper modes (Eq. (12)) are coordinates which are more suitable for the analysis of the solutions. In the case of a linear system, after transformation in polar symplectic coordinates, they are action angle variables; that is, the actions are constant and the angles linear functions of the time. In the case of a perturbed system, they are close to that. We will observe a leading periodic term of frequency g_i or s_i given mostly by the linear part C of Eq. (13) and smaller terms due to the nonlinear parts Ψ_3 and Ψ_5 of Eq. (13). These terms are usually much smaller than the principal terms. If we make the linear transformation to proper modes before the Fourier analysis, we have a main frequency which is isolated, and thus not possibly corrupted by the other main frequencies. This should improve the determination of the main frequencies although the spectral lines coming from the nonlinear part of Eq. (13) are still present.

In fact, the determination of the proper modes is rather loose. The only thing to find is a linear change of variables (Eq. (12)) which will produce variables closer to action angles variables (in case of an integrable system). This matrix can be chosen as the matrix of the eigenvectors of A (Eq. (10)), but due to the resonant terms of Eq. (10), the linear part of the solution will be somewhat different from A (Eq. (20)). We will thus prefer to use the proper modes of the solution rather than the proper modes of the secular system as a linear approxima-

TABLE V

MATRIX $\tilde{S}$ USED IN THE DEFINITION OF THE NORMALIZED PROPER MODES (EQ. (25))

	$\tilde{S} \times 10^8$							
	18544445	2770002	−158226	165869	3635272	10322	60545	717
	666756	−2073299	1167087	1346367	1963619	−40659	54130	1088
	424753	−1604706	−940557	−1315878	1891283	150580	56648	1259
	6826	−293957	−4013313	4903211	2030019	702976	73617	2008
z	−952	1346	101	58	4411911	−1574980	179992	5755
	−814	1379	675	604	3314181	4818663	151278	5760
	383	−350	−41	−34	−3735146	−154025	2910658	164066
	−4	−6	−3	−2	177129	−9956	−360992	910809
	3995691	3016884	−156939	−120029	1372387	13964	159813	70222
	671554	−404451	954443	575863	1377171	5577	99208	65885
	496004	−343135	−876009	−402447	1377264	133519	89258	64554
	105158	−77123	1542060	−3468927	1375325	457927	63897	60870
ζ	−1075	497	7	149	1377468	−315119	48134	58088
	−1373	711	198	1270	1377396	785009	39101	55969
	1027	280	−16	−111	1377366	−34771	−887144	−56304
	14	17	−2	−13	1377415	−3755	105694	−581911

Note. This matrix is the linear part of the solution, completed with some terms coming from Laskar (1988, 1984).

tion of the considered solution. Besides, the MFT analysis of the solution of Eq. (10) obtained by numerical integration gives us directly the main terms of this matrix. We have

$$\alpha = \tilde{S}\tilde{u} + \ldots \tag{24}$$

$\tilde{u}_i = \exp i(\nu_i t + \phi_i)$ with ν_i and ϕ_i being the fundamental frequencies and phases of the solution. $\tilde{S}$ is the real matrix of the amplitudes.

The new coordinates, will thus be defined as

$$u^{\bullet} = \tilde{S}^{-1}\alpha \tag{25}$$

In the remainder of the paper, we will call these variables the normalized proper modes. It should be noted that their amplitude is close to 1 (it is not exactly equal to 1 due to the presence of the nonlinear terms). The matrix $\tilde{S}$ is fixed in all the further analyses and is given in Table V. For the inner planets, the MFT does not give the amplitude of the terms which are two small, especially in the present case where we look for only the first 20 terms. Therefore, the matrix $\tilde{S}$ is completed with the corresponding terms obtained in Laskar (1988), and for the smaller terms in Laskar (1984, Table 24). This is similar to what was already actually used in Laskar (1987) and to the choice adopted by Nobili *et al.* (1989). The main difference in these previous studies is that here we have normalized these proper modes for practical reasons.

In a similar way as with the proper modes β (Eq. (14)), we will denote the normalized proper modes with the corresponding names of the variables:

$$u^{\bullet} = (z_1^{\bullet}, \ldots, z_8^{\bullet}, \zeta_1^{\bullet}, \ldots, \zeta_8^{\bullet})$$
$$\text{with} \begin{cases} z_i^{\bullet} = z_{0i} \exp\sqrt{\varpi_i^{\bullet}} \\ \zeta^{\bullet} = \zeta_{0i} \exp\sqrt{\Omega_i^{\bullet}} \end{cases} \tag{26}$$

The normalized proper modes $u_i^{\bullet}$ are then analyzed with the MFT and the results over 20 myr are given in Tables VI and VII. Like in Tables II and III, only the first 10 leading terms are given. These tables reflect the nonlinear character of the solutions. For a linear system, there would be only one periodic term for each proper mode. For the outer planets, the leading term is nearly equal to 1 and all the other terms are very small compared with the first one and decrease rapidly. We are nearly in action angle variables. In the inner planet solutions, the amplitudes of the main term are also very close to 1 (this is the effect of the normalization of the proper modes), but the

TABLES VI and VII

SOLUTIONS La90 OBTAINED OVER THE FIRST 20 MYR OF THE PRESENT WORK FOR THE NORMALIZED PROPER MODES z_i^* AND ζ_i^* EQS. (25–26)

var.	i		ν_i (″/yr)	$A_i \times 10^6$	ϕ_i (°)
z_1^*	1	g_1	5.5964355	988835	110.346
	2	$g_1 - \sigma$	5.4744873	372210	275.011
	3	$g_1 + \sigma$	5.7166963	282579	120.518
	4	$g_1 - 2\sigma$	5.3582256	151846	94.885
	5	$g_5 + \sigma$	4.3697461	68934	220.835
	6		6.9342275	62960	167.015
	7	$g_2 - s_1 + s_2$	5.9922686	55846	113.555
	8		5.6548477	53865	39.215
	9	$g_1 + s_1 - s_2$	7.0559473	44959	357.991
	10	$g_1 - 3\sigma$	5.2384131	44707	272.973
z_2^*	1	g_2	7.4559199	1005930	20.243
	2	$g_2 - \sigma$	7.3410293	84943	27.849
	3		6.9350977	84299	350.432
	4	$g_1 + s_1 - s_2$	7.0556309	71777	179.181
	5	$g_2 + \sigma$	7.5729902	59273	191.474
	6		6.8246807	27833	14.527
	7	$g_2 - 2\sigma$	7.2056338	17370	323.910
	8	$g_5 - \sigma$	4.3690605	12417	220.870
	9	g_6	28.2206162	11436	307.920
	10	$g_2 + 2\sigma$	7.7166299	10824	273.523
z_3^*	1	g_3	17.3646914	997897	123.952
	2	$2g_3 - g_4$	16.8128525	329788	272.708
	3	$g_3 - \theta$	17.0826592	277664	179.380
	4	$g_4 + 2\theta$ $2g_4 - g_3$ $g_3 - s_3 + s_4$	18.4679385	115466	191.159
	5	$g_4 + 4\theta$ $g_4 - s_3 + s_4$	19.0186963	99466	219.747
	6		17.1575156	95554	325.016
	7		16.5273135	57607	131.910
	8	$g_3 + \theta$	17.6356230	55475	33.836
	9		17.4768311	50243	260.258
	10		17.5523379	40434	197.646
z_4^*	1	g_4	17.9155020	999939	335.249
	2	$g_3 + \theta$	17.6308066	188226	198.726
	3	$g_4 + 2\theta$ $2g_4 - g_3$ $g_3 - s_3 + s_4$	18.4679385	182379	9.257
	4	$g_3 - 2\theta$ $2g_3 - g_4$ $g_4 + s_3 - s_4$	16.8123340	108249	89.873
	5		17.8108418	96343	58.564
	6	$g_4 + \theta$	18.1855283	88044	57.275
	7		17.7229336	72788	48.460
	8		18.0161104	61948	44.830
	9	$g_3 - 4\theta$ $g_3 + s_3 - s_4$	16.2612246	42913	58.890
	10		18.0862734	35767	356.170

var.	i		ν_i (″/yr)	$A_i \times 10^6$	ϕ_i (°)
z_5^*	1	g_5	4.2488152	1000001	30.672
	2	$2g_5 - g_7$	5.4081694	1870	120.743
	3	$-g_5 + 2g_6$	52.1925886	860	225.597
	4	$2g_5 - g_6$	−19.7230507	643	113.237
	5	$-g_5 + 2g_7$	1.9316843	399	33.324
	6	$g_6 + s_6 - s_7$	4.8953760	275	292.581
	7		3.6002889	123	121.395
	8	$-g_5 + 2s_6$	−56.9092214	110	44.107
	9	g_4	17.9154800	80	155.331
	10	$2g_6 - g_7$	53.3519769	70	134.986
z_6^*	1	g_6	28.2206942	999999	128.112
	2	$-g_5 + 2g_6$	52.1925732	35353	225.551
	3	$-g_5 + g_6 + g_7$	27.0614004	11546	38.704
	4	$g_5 + g_6 - g_7$	29.3799814	11375	37.537
	5	$-g_5 + 2g_6 + s_6 - s_7$	28.8679504	6821	212.636
	6	$g_5 - s_6 + s_7$	27.5734589	5991	223.738
	7	$2g_6 - g_7$	53.3518792	2999	134.906
	8	$2g_5 - g_6$	−19.7230616	2489	113.239
	9	$-2g_5 + 3g_6$	76.1644655	1550	323.030
	10	$-2g_5 + 2g_6 + g_7$	51.0333398	860	136.295
z_7^*	1	g_7	3.0895181	1000142	121.363
	2	$-g_5 + 2g_7$	1.9298958	13818	208.762
	3	$2g_5 - g_7$	5.4081826	11755	301.998
	4	$g_8 - s_7 + s_8$	2.9770598	2610	306.810
	5	$-g_5 + 2g_6$	52.1925062	1613	225.351
	6	$g_6 + s_6 - s_7$	4.8964680	1291	116.699
	7	$g_5 - g_6 + g_7$	−20.8823621	1232	203.933
	8	$-g_5 + g_6 + g_7$	27.0613894	1222	218.790
	9	$g_5 + g_6 - g_7$	29.3801594	852	37.763
	10	$g_5 - g_7 + g_8$	1.8212135	778	143.102
z_8^*	1	g_8	0.6670833	999969	73.977
	2	$g_7 + s_7 - s_8$	0.7783956	17255	65.099
	3	$-g_5 + g_7 + g_8$	−0.4921633	5013	164.735
	4	$g_5 - g_7 + g_8$	1.8339686	4993	189.390
	5	$-g_5 + 2g_7$	1.9359415	3726	203.683

Note. In general, 20 terms are computed in the solutions, but only the first 10 largest terms are given here. Sometimes, solutions have fewer terms when the determination of the 10 first terms is not possible with full satisfaction. When possible, the identification of a term as a combination of the fundamental frequencies g_i, s_i and of the two libration frequencies θ and σ is given in the first column. The frequency of the term, its amplitude, and phase, as determined by the MFT (modified Fourier transform) are given in the next columns. On several occasions, due to resonances, several combinations of frequencies correspond to the same term. In this case, they are all given in column 1.

TABLE VII

var.	i		ν_i (″/yr)	$A_i \times 10^6$	ϕ_i (°)
	1	s_1	−5.6175469	1000032	348.703
	2	$s_1+\sigma$	−5.5009775	356587	342.887
	3	$s_1-2\sigma$	−5.8501670	120591	165.472
	4	$s_1+3\sigma$	−5.2160977	90262	18.911
$\zeta_1^\bullet$	5	$s_1+2\sigma$	−5.3717783	89798	35.479
	6		−5.1002490	87880	195.380
	7	$s_1-3\sigma$	−5.9689863	51511	350.642
	8		−6.8430674	41773	281.639
	9		−6.1549014	37298	89.771
	10		18.1498359	13495	111.186
	1	s_2	−7.0796338	999688	273.772
	2	$s_2-\sigma$	−7.1948672	498986	105.162
	3	$s_2+\sigma$	−6.9609199	473858	97.954
	4	$s_2+2\sigma$	−6.8409141	447180	107.121
$\zeta_2^\bullet$	5	$s_2-2\sigma$	−7.3326357	289241	196.750
	6		−6.7384160	94787	44.496
	7		−7.4053564	87624	233.348
	8		−7.4877979	81403	47.947
	9		−6.5601650	76503	303.466
	10		−8.4234199	63724	211.208
	1	s_3	−18.8511504	999395	60.426
	2	$\{s_3+2\theta$; $-g_3+g_4+s_3$; $g_3-g_4+s_4$	−18.3000674	348764	89.583
	3	$\{s_3-2\theta$; $g_3-g_4+s_3$	−19.4022422	144384	209.568
	4	$s_3-\theta$	−19.1307480	92428	305.899
$\zeta_3^\bullet$	5	$\{s_4+2\theta$; $-g_3+g_4+s_4$	−17.1965566	83934	156.844
	6		−18.9700137	38404	73.358
	7		−18.6974297	20178	221.700
	8		−18.7793350	19780	222.831
	9		−18.2268105	19753	46.302
	10		−19.0654365	17694	50.212
	1	s_4	−17.7481758	998857	123.278
	2	$\{s_4+2\theta$; $-g_3+g_4+s_4$	−17.1970928	119266	334.279
	3	$\{s_3-2\theta$; $g_3-g_4+s_3$	−19.4025586	88857	207.832
	4	$s_4-\theta$	−18.0111445	75747	242.092
$\zeta_4^\bullet$	5		−17.6609355	71819	138.927
	6	$\{s_3+2\theta$; $-g_3+g_4+s_3$; $g_3-g_4+s_4$	−18.3019922	59965	267.740
	7		−17.8385713	58410	289.132
	8		−17.5463613	52055	246.709
	9		−17.9440400	36639	212.259
	10		−18.5956260	33620	98.105

var.	i		ν_i (″/yr)	$A_i \times 10^6$	ϕ_i (°)
	1	s_5	0.0000000	999999	107.586
	2	$-g_5+g_6+s_6$	−2.3583505	277	224.750
	3	$g_5-g_6+s_6$	−50.3021184	162	29.832
	4	s_2	−7.0796865	31	93.568
$\zeta_5^\bullet$	5	s_3	−18.8511372	22	240.143
	6	$\{g_5-g_7+s_8$; $-g_5+g_7+s_8$	−1.8462382	21	54.564
	7	$-g_5+g_7+s_6$	−4.1648247	20	231.865
	8	$g_6-g_7+s_6$	−1.1990665	19	134.113
	9	$s_2-\sigma$	−7.1949254	16	285.366
	10	$s_2+\sigma$	−6.9609353	15	277.916
	1	s_6	−26.3302328	1000000	127.291
	2	$g_5-g_6+s_7$	−26.9774352	7458	43.030
	3	$2g_6-s_6$	82.7716267	4353	128.951
	4	$g_5+g_6-s_6$	58.8001674	2552	212.896
$\zeta_6^\bullet$	5	$2g_5-s_6$	34.8278818	2017	294.115
	6	$-g_5+g_6+s_6$	−2.3583153	1945	224.855
	7	$g_5-g_6+s_6$	−50.3021151	1486	209.839
	8	$-g_5+g_7+s_6$	−27.4893475	1292	218.527
	9	$g_5-g_7+s_6$	−25.1711565	1221	215.937
	10	$-g_6+g_7+s_7$	−28.1365554	811	314.075
	1	s_7	−3.0055704	999997	140.330
	2	$-g_5+g_6+s_6$	−2.3583483	33727	44.751
	3	$-g_5+g_7+s_6$	−4.1646742	21196	51.668
	4	$\{g_5-g_7+s_8$; $-g_5+g_7+s_8$	−1.8462480	20499	45.092
$\zeta_7^\bullet$	5	$-g_7+g_8+s_8$	−3.1172498	6679	326.969
	6	$g_6-g_7+s_6$	−1.1990599	3079	134.003
	7	$2g_5-s_7$	11.5031942	2135	281.015
	8	$g_5+g_7-s_7$	10.3438938	1383	191.704
	9	$g_5-g_6+s_7$	−26.9764761	876	45.785
	10	$-g_5+g_6+s_7$	20.9663086	828	57.783
	1	s_8	−0.6918915	1000003	23.961
	2	$g_7-g_8+s_7$	−0.5803264	9723	17.325
	3	$-g_5+g_6+s_6$	−2.3582615	4204	225.099
	4	$\{g_5-g_7+s_8$; $-g_5+g_7+s_8$	−1.8491221	3434	346.114
$\zeta_8^\bullet$	5	$g_5-g_7+s_8$	0.4654655	2797	286.876
	6	$g_6-g_7+s_6$	−1.1989666	2082	314.354
	7	$g_5-g_8+s_7$	0.5782852	968	103.718
	8	$-g_5+g_7+s_6$	−4.1639216	810	234.206
	9	$g_5+g_7-s_7$	10.3438696	602	12.479
	10	$\{2g_5-s_8$; $2g_7-s_7$	9.1884727	550	1.150

other terms are still large, especially for the proper modes $z_1^\bullet$, $z_3^\bullet$, $z_4^\bullet$, $\zeta_1^\bullet$, $\zeta_2^\bullet$, $\zeta_3^\bullet$, $\zeta_4^\bullet$. This reflects the importance of nonlinearities in the inner Solar System. As with the solutions for the variables z_i, ζ_i, the identification of the principal arguments as combinations of the fundamental frequencies g_i, s_i and of the two arguments θ, σ (see next section) can be made and is given, when it is possible in the first column. As previously, the solutions containing the first 20 terms are reconstructed and analyzed again with the MFT, after an offset of a few million years. The comparison with the solution gives the precision of the MFT. This precision is always very good, and is summarized for the main frequencies of the system in Table VIII with the values of these frequencies obtained over 20 myr. We can see that these frequencies are very well de-

TABLE VIII

FUNDAMENTAL FREQUENCIES OF THE PRESENT WORK La90 (ν) OBTAINED OVER THE FIRST INTERVAL OF ABOUT 20 MYR OF THE SOLUTION WITH THE UNCERTAINTY IN THEIR DETERMINATION ($\delta\nu$)

	ν	$\delta\nu$
g_1	5.5964355	0.0004
g_2	7.4559199	0.00004
g_3	17.3646914	0.00006
g_4	17.9155020	0.001
g_5	4.2488152	0.000001
g_6	28.2206942	0.000001
g_7	3.0895181	0.00001
g_8	0.6670833	0.00007
s_1	−5.6175469	0.00006
s_2	−7.0796338	0.0005
s_3	−18.8511504	0.0005
s_4	−17.7481758	0.002
s_5	0.0000000	0.000001
s_6	−26.3302328	0.000001
s_7	−3.0055704	0.00003
s_8	−0.6918915	0.00004

Note. This uncertainty reflects the precision of the MFT (modified Fourier transform) but not the actual accuracy of the frequencies.

termined. In the cases of g_5, g_6, s_5, s_6, the precision reaches the limit of 0.000001 arcsec/year which was asked in the MFT. The frequencies are determined by the MFT much more accurately than with a mere FFT. (The resolution in frequency of the FFT is given by the length of the data. In the case of our analysis over 20 myr of data, it would be only 0.05 arcsec/year.) In fact, I will show in Section 9 that the fundamental frequencies are slowly changing with time, so it is not possible to determine them in the usual sense. What I determine here is an *average* frequency over the given time span and the precisions $\delta\nu$ given in Table VIII are the evaluations of the precisions of these determinations using the MFT. It should be noted that for the outer planets these frequencies are very close to those given by Applegate *et al.* (1986) and Carpino *et al.* (1987): maximum discrepancies are of about 0.01 arcsec/year. Differences with Nobili *et al.* (1989) reach 0.02 arcsec/year, but in their modeling of the inner planets, they introduce a moving plane related to the angular momentum of the system with frequency −0.01 arcsec/year, and the results are thus more difficult to compare.

8. THE SECULAR RESONANCES IN THE SOLAR SYSTEM

In Laskar (1987, 1988), the existence of secular resonances in the inner Solar System was detected but there were still several features in the solutions which were not completely identified. One very important resonance is

$$2(g_4 - g_3) - (s_4 - s_3). \qquad (27)$$

In the previous papers (Laskar 1987, 1988), this resonance was analyzed over 30 myr and was found to be a libration. This resonance was a good candidate as an explanation for the appearance of chaotic motion in the Solar System. I investigated the evolution of the corresponding normalized proper modes

$$z_4^{\bullet 2}\bar{z}_3^{\bullet 2}\bar{\zeta}_4^{\bullet}\zeta_3^{\bullet} \qquad (28)$$

whose argument is

$$2(\varpi_4^{\bullet} - \varpi_3^{\bullet}) - (\Omega_4^{\bullet} - \Omega_3^{\bullet}). \qquad (29)$$

The variation of (29) over nearly 200 myr is given in Fig. 2, and the variations of Eq. (28) in the complex plane is given in Fig. 3 for a selection of sections of 10 myr (the full interval was split in order to make the figures more intelligible). In these figures, we can see in a very apparent manner that after 30 myr, there is a first transition from libration around 0° to circulation. We then have other transitions to libration around 0°, circulation, and also libration around 180°, and so on.

It has been shown (see Chirikov (1979) for a thorough review) that the motion near a resonance is in general like the motion of a pendulum with libration, separatrix, and rotation motion. Under a perturbation, the separatrix splits and gives rise to a stochastic region. The motion in this region which is close to the separatrix is extremely unsta-

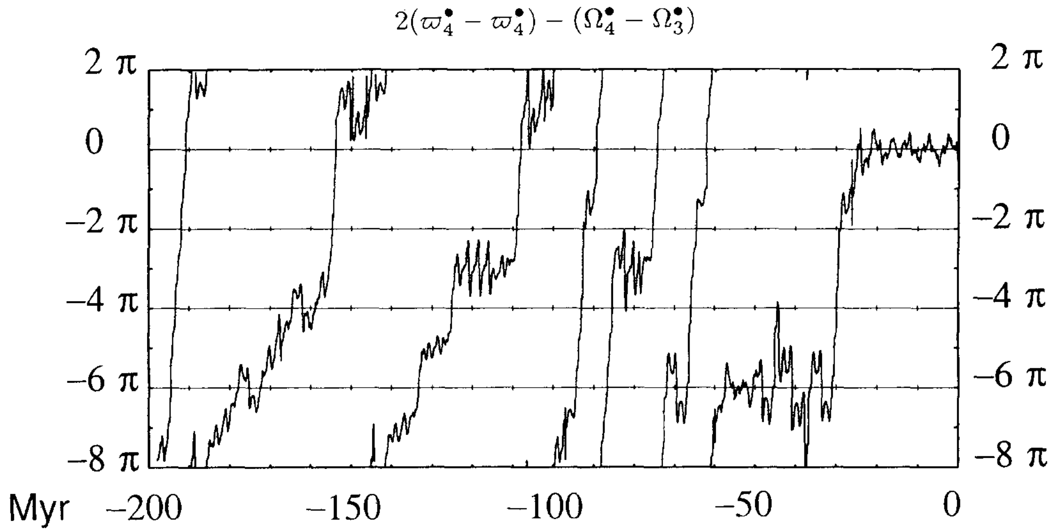


FIG. 2. Evolution of the resonance argument $2(\varpi_4^\bullet - \varpi_3^\bullet) - (\Omega_4^\bullet - \Omega_3^\bullet)$ over 200 myr. For clarity of the figure, the evolution of this argument is followed from 2π to -8π. We can observe very distinctly the transitions of this argument from libration to circulation. At the beginning the argument is in libration around 0°. The first transition occurs after about 20 myr and then we have successive libration around 0°, circulation, and libration around 180°.

ble since it can wander from libration to circulation and thus give rise to exponential divergence of nearby orbits and positive Lyapunov exponents. The resonant argument (Eq. (29)) is therefore most probably at the origin of the exponential divergence of nearby orbits in the solutions of the secular system $\mathcal{S}$.

Besides, if we just consider the first 20 myr of the study of the previous sections, the argument stays in libration. The frequency of libration of Eq. (28) becomes a new fundamental frequency of the system. The period of the libration can be measured on the plot and appears to be of about 4.6 myr which corresponds very closely to the value of the frequency $\theta = 0.28$ arcsec/yr which I was obliged to introduce in Tables II and III and VI and VII in order to properly identify the arguments of the frequencies given by the MFT. This is what happens when we have a resonance (or more precisely a libration): one of the fundamental frequencies becomes a combination of the other frequencies, and the libration frequency is the new corresponding independent frequency. In the present case, we have even something more: the frequency of libration θ is itself in resonance with $g_4 - g_3$ ($2\theta \approx g_4 - g_3$). With these relations, we can identify properly nearly all the leading terms of the solutions of Mars and the Earth (Tables II and III).

The previous resonance was identified quite early, but there was still a problem with the very leading terms of the solutions of Mercury and Venus. After investigation, I found another candidate for resonance which is the argument

$$(\varpi_1^\bullet - \varpi_5^\bullet) - (\Omega_1^\bullet - \Omega_2^\bullet), \qquad (30)$$

corresponding to the frequency

$$(g_1 - g_5) - (s_1 - s_2) \qquad (31)$$

and to the combination of normalized proper modes

$$z_1^\bullet \bar{z}_5^\bullet \bar{\zeta}_1^\bullet \zeta_2^\bullet. \qquad (32)$$

This argument is very interesting. It involves the proper modes related to Mer-

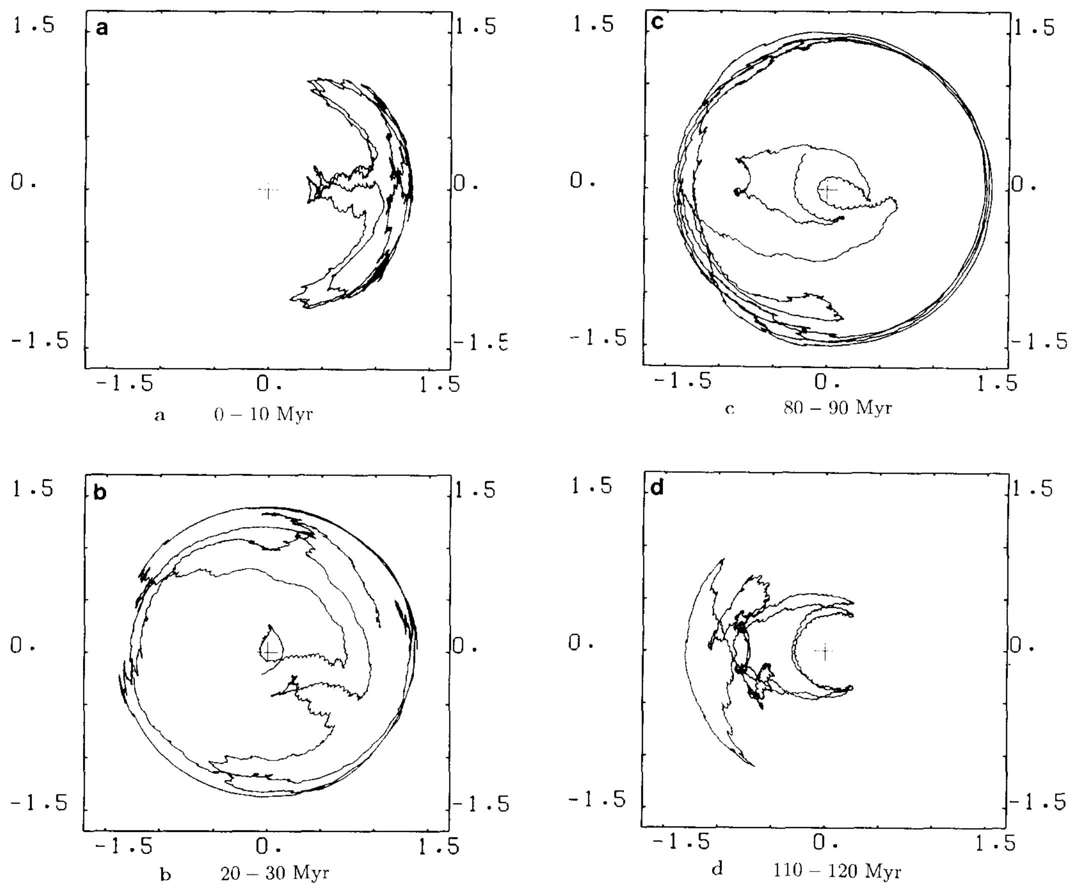


FIG. 3. (a–d) Examples of transition from libration around 0° (a) to circulation (b), and from circulation (c) to libration around 180° for the argument $2(\varpi_4^* - \varpi_3^*) - (\Omega_4^* - \Omega_3^*)$. The quantity which is actually plotted in the complex plane is $z_{04}\exp i(2(\varpi_4^* - \varpi_3^*) - (\Omega_4^* - \Omega_3^*))$ (Eq. (26)).

cury, Venus, and Jupiter. In Laskar (1984), when I integrated analytically the secular system (Eq. (10)), I already found that this combination leads to a small divisor and increases the amplitude of one of the leading terms in the solutions of Mercury and Venus. As previously, we can plot the evolution of this argument (Eq. (30)) over 200 myr (Fig. 4) or the combination (Eq. (31)) on selected sections of 20 myr (Fig. 5). This argument is truly in resonance. We have a libration of the argument (Eq. (30)) around 180° with an amplitude of about 135° in the first 60 myr and down to about 80° after. We do not observe here any transition from libration to circulation. The period of libration can be obtained easily directly on the plot (one could also make a MFT of the plot) and is found to be of about 10 myr which corresponds very closely to the unidentified frequency $\sigma \approx 0.12$ arcsec/year of Tables II and III and VI and VII. This second secular resonance in the Solar System allows us now to identify all the leading terms of the solutions of Mercurcy and Venus. Due to the librations, these terms are not combinations of the fundamental frequencies as given in Table VIII, but they are combinations of the two librations frequencies θ and σ.

In the outer planet solutions, I also found the resonance τ

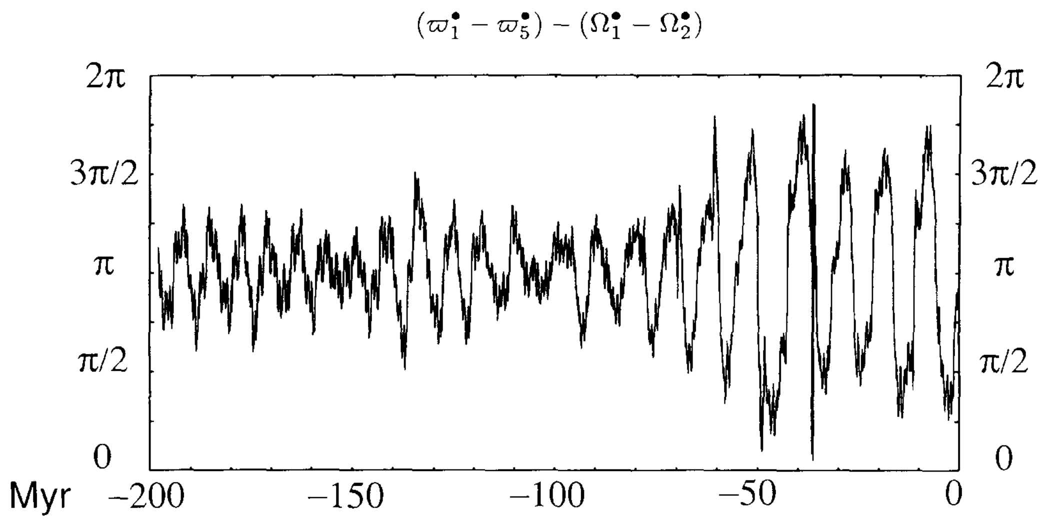


FIG. 4. Evolution of the resonant argument $(\varpi_1^\bullet - \varpi_5^\bullet) - (\Omega_1^\bullet - \Omega_2^\bullet)$ over 200 myr. This argument of the proper modes stays in libration around 180° during the entire time span with a very large amplitude at the beginning reaching 135° and then a smaller amplitude about 85°. Nearly −40 myr we can observe a very sharp feature which is not significant but corresponds to a passage around the origin due to some other periodic term (Fig. 5b) (we must not forget that we are in a dynamical system with 15 degrees of freedom).

$$2(g_5 - g_7) - (s_8 - s_7) \approx 0.005 \text{ arcsec/year} \qquad (33)$$

which was already reported by Nobili *et al.* (1989). Although the present value of this argument is much closer to zero than in Nobili *et al.* (1989) (they found about 0.04 arcsec/year due to slight changes in the values of the frequencies), the corresponding argument of the proper modes

$$2(\varpi_5^\bullet - \varpi_7^\bullet) - (\Omega_8^\bullet - \Omega_7^\bullet) \qquad (34)$$

is circulating over 200 myr with the same frequency (Eq. (33)) (Fig. 6) according to the results of Nobili *et al.* (1989). This resonance does not introduce any new frequency in the solutions, but it prevents one from distinguishing between the two arguments $g_5 - g_7 + s_7$ and $g_7 - g_5 + s_8$ or $2g_5 - s_8$ and $2g_7 - s_7$ in the MFT of the solution over 20 myr of ζ_7 and ζ_8 (Table VI and VII).

All these resonances are summarized on Table IX. From this analysis, we can conclude that in the Solar System, we have two very important secular resonances θ (Eq. (27)) which essentially involve Mars and the Earth, and σ (Eq. (31)) among Mercury, Venus, and Jupiter (or more precisely the coupled Jupiter–Saturn). These two resonances completely change the aspect of the long term evolution of the Solar System as it was given in Brouwer and Van Woerkom (1950) or Bretagnon (1974). Even more, the secular resonance θ between Mars and Earth is undoubtedly responsible for the appearance of chaotic motion in the inner Solar System and explains the Lyapounov exponent of 1/(5 myr) obtained in Laskar (1989a).

9. THE EVOLUTION OF THE MAIN FREQUENCIES: A MEASURE OF THE CHAOS

In the previous sections, I described the secular resonances which were responsible for the appearance of the chaos in the solutions of the secular system $\mathcal{S}$. In this section, I present a new method for the numerical determination of the size of the chaotic zones, on the basis of the analysis of the

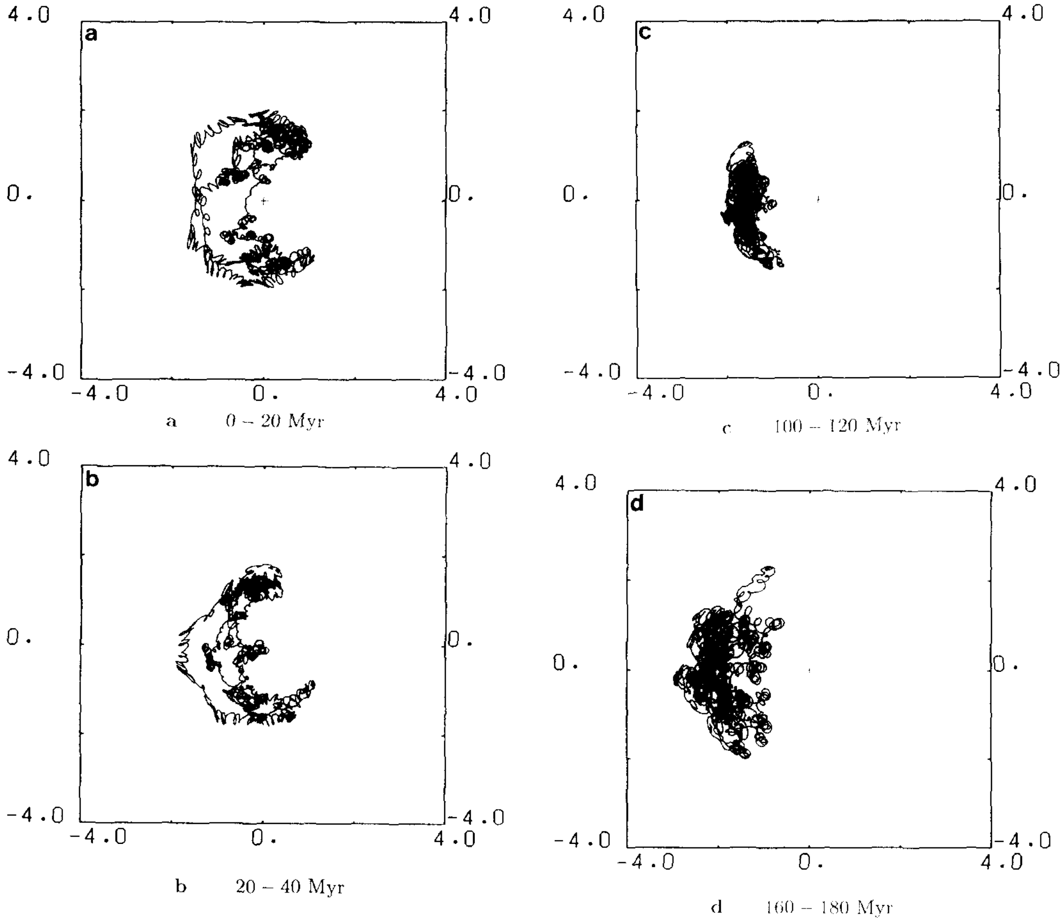


FIG. 5. (a–d) Selection of pictures of the libration around 180° for the argument $(\varpi_1^* - \varpi_5^*) - (\Omega_1^* - \Omega_2^*)$. The quantity which is actually plotted in the complex plane is $z_1^* \bar{z}_5^* \bar{\zeta}_1^* \zeta_2^*$.

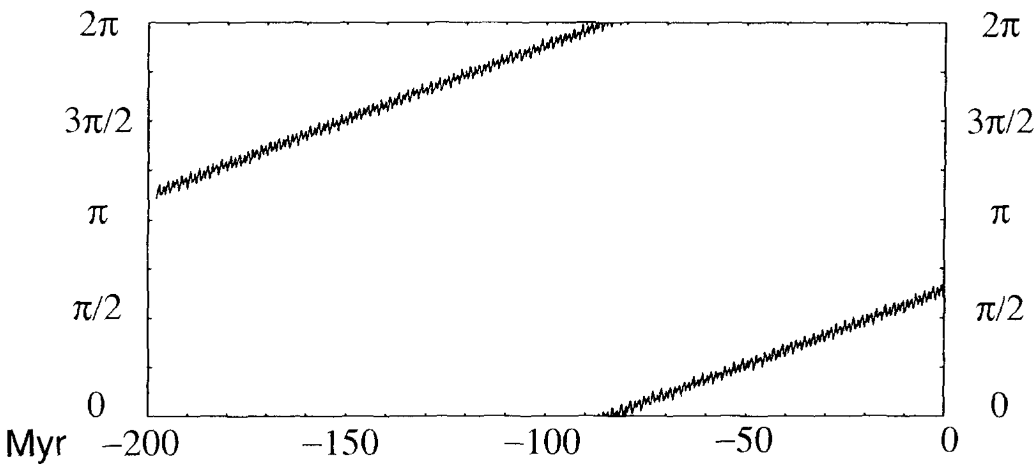


FIG. 6. Evolution of the argument $2(\varpi_5^* - \varpi_7^*) - (\Omega_8^* - \Omega_7^*)$ over 200 myr. This argument of the proper modes is circulating during the entire time span with a frequency equal to $2(g_5 - g_7) - (s_8 - s_7)$.

TABLE IX

THE SECULAR RESONANCES IN THE SOLAR SYSTEM (EXCLUDING PLUTO)

res.	θ	σ	τ
comb. of frequencies	$2(g_4 - g_3) - (s_4 - s_3)$	$(g_1 - g_5) - (s_1 - s_2)$	$2(g_5 - g_7) - (s_8 - s_7)$
value over 20 *Myr*	$-0.001''/yr$	$-0.1''/yr$	$0.005''/yr$
argument	$2(\varpi_4^\bullet - \varpi_3^\bullet) - (\Omega_4^\bullet - \Omega_3^\bullet)$	$(\varpi_1^\bullet - \varpi_5^\bullet) - (\Omega_1^\bullet - \Omega_2^\bullet)$	$2(\varpi_5^\bullet - \varpi_7^\bullet) - (\Omega_8^\bullet - \Omega_7^\bullet)$
0 – 20 *Myr*	*libration of* 80° *around* 0° *period* ≈ 4.6 *Myr*	*libration of* 135° *around* 180° *period* ≈ 10 *Myr*	*circulation*
0 – 200 *Myr*	{ *libration around* 180° / *circulation* / *libration around* 0°	*libration of* 85° *to* 135° *around* 180° *period* ≈ 10 *Myr*	*circulation*

Note. The first resonance θ is responsible for the chaotic behavior of the solutions of the secular system, due to the transitions from libration to circulation. The second resonance σ stays in libration over 200 myr and affects very much the form of the solutions of the inner planets, especially of Mercury and Venus. The third resonance τ is not a libration, but prevents the differentiation of several arguments in the MFT of the outer planets.

evolution of the main frequencies of a system with time.

In the case of an integrable system with n degrees of freedom, after reduction to action angle variables (J_j, θ_j), the Hamiltonian depends only on the actions $J_1, J_2, \ldots, J_n$.

$$H(J, \theta) = H_0(J), \tag{35}$$

where (J) stands for $(J_1, J_2, \ldots, J_n)$ and (θ) for $(\theta_1, \theta_2, \ldots, \theta_n)$. The equations of motion are thus for all $j = 1, \ldots, n$

$$\dot{J}_j = 0, \quad \dot{\theta}_j = \frac{\partial H_0(J)}{\partial J_j} = \nu_j(J). \tag{36}$$

The motions in the phase space follow nice tori with constant radii J_j which are described at constant velocity $\nu_j(J)$ (Fig. 7a). If we plot the action (the radius of the tori) against time, we obtain horizontal lines. Now suppose that we are still in an integrable system, but not in the action angle variables, although close to them. For example, we can make the change of variables $z_j = J_j \exp i\theta_j \mapsto z_j' = J_j' \exp i\theta_j' = f_j(z_1, z_2, \ldots, z_n)$, where f_j is an analytic function close to identity. Every motion will still lie on a torus, but the projections of the orbits on the planes (J_j', θ_j') will not look very circular (Fig. 7b). If we plot the action-like variable J_j' against time we will obtain a curve in which variations do not represent the variations of the action J_j which is actually constant. We are not in the proper variables and the constancy of the action J_j is not directly visible on these plots. On the contrary, the analysis of the frequencies of the system is still possible.

In the case of a nondegenerate integrable system (i.e., with nonvanishing torsion $|\partial^2 H_0/\partial J^2| \neq 0$ on the domain of interest), we can invert the relations (Eq. (36)) and use the frequencies of the system as parameters

$$J_j = F_j(\nu_1, \nu_2, \ldots, \nu_n). \tag{37}$$

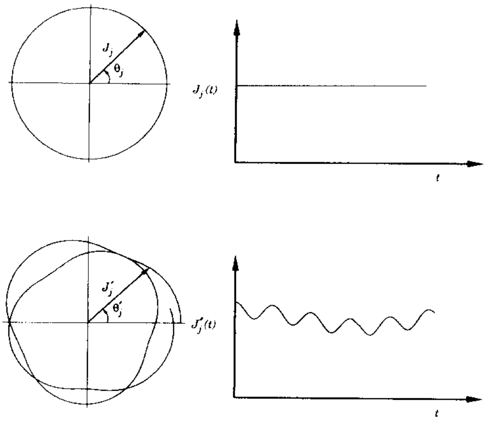


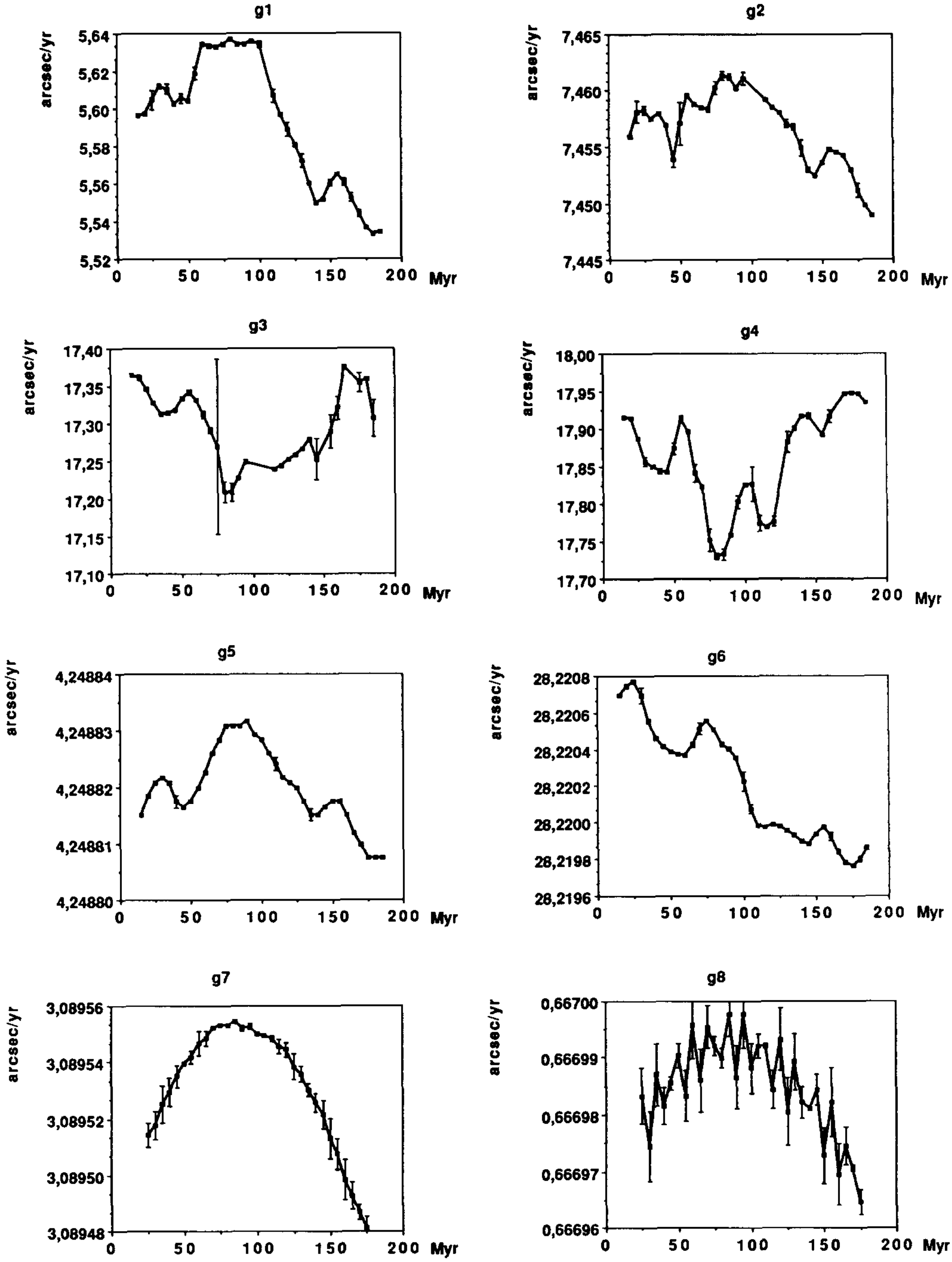
FIG. 7. In the case of an integrable system with n degrees of freedom with Hamiltonian $H_0(J)$, the motion in the phase space follows the tori nicely with constant radii J_j which are described at constant velocity $\nu_j(J)$ (a). If we plot the actions (the radii of the tori) against time, we obtain horizontal lines. Suppose we make a change of variables close to identify into the new variables (J_j', θ_j'). The motion will still be on tori, but the projections of the orbits on the planes (J_j', θ_j') will not look very circular (b). If we plot the action-like variable J_j' against time we will obtain a curve in which variations do not represent the variations of the action variables which actually are constant. We are not in the proper variables and it is not possible to observe any evolution of the actions on these plots. On the contrary, the analysis of the frequencies of the system is still possible.

The frequencies are constant on the orbits, and the Fourier analysis of $z_j(t)$ will give us directly the frequency ν_j, and thus through Eq. (37) an equivalent measure of the actions. The major advantage of this method is that it will still be valid if we are in the modified variables J_j', θ_j'. The Fourier analysis of $z_j'(t)$ will still give us a measure of the fundamental frequencies ν_j and thus of the actions J_j, although we are not in action angle variables (ν_j is still the dominant frequency of $z_j'(t)$).

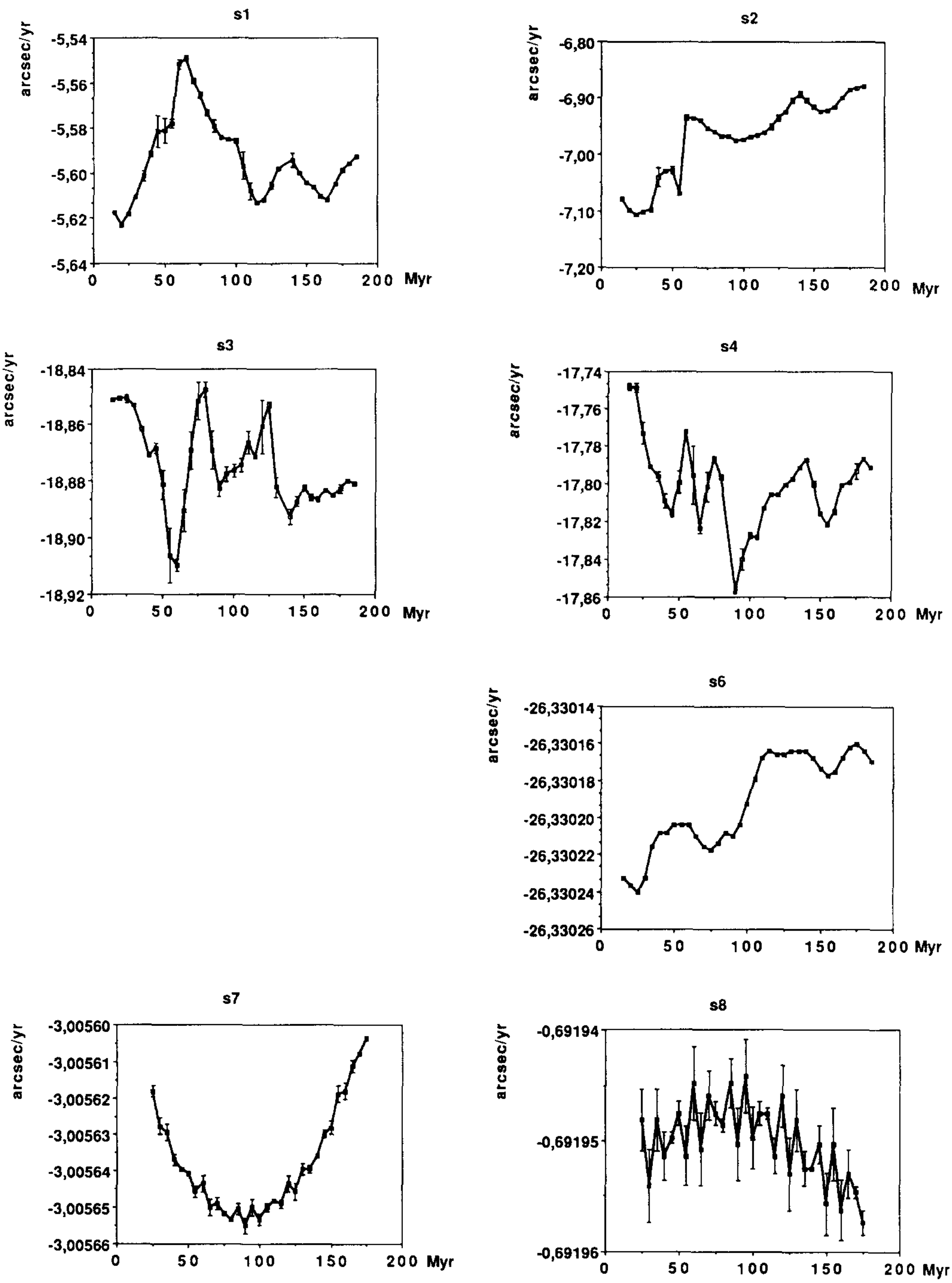
Suppose now that we can observe our system over a very long time span. We can then split this time span into several sections and make a separate Fourier analysis of $z_j'(t)$ in each section. In the case of an integrable system, the frequencies ν_j will not vary, but if we are in a chaotic zone, the action variables will evolve with time and as the motion progress will move in the complementary of the invariant tori. The frequencies will thus change slowly with time. This gives us a way to obtain a measure of the size of the chaotic zone, by the measure of the variation of the main frequencies of the system: We use the frequencies as variables instead of the actions for the parametrization of the size of the chaotic regions.

Of course, this method cannot be used in every case. First, we need a method to compute the frequencies very accurately, but this is given by the MFT described above. Then, the motion needs to be only weakly chaotic and not strongly chaotic: the frequencies must not change too quickly so that we can have a good measure of them by the MFT analysis over a long time span.

I have applied this method to the solutions of the secular system $\mathcal{S}$ over nearly 200 myr. The frequencies are computed with the MFT on a time step of 20 myr after the linear transformation to the normalized proper modes Eq. (25), always using the same matrix of Table V. As already stated, the determination of the proper modes is rather loose, and we just try to be much closer to action angle variables. I thus found it was preferable to always take the same linear transformation rather than to compute a new transformation at each step. As in the previous sections, the precision in the determination of the fundamental frequencies is computed by using a new MFT analysis of the approximate solution corresponding to the first 20 periodic terms of the solutions, after an offset of a few million years. This computation is applied on the whole 200-myr interval, by sections of 20 myr except for g_7, g_8, s_7, s_8 where the span of time was extended to 50 myr in order to improve the accuracy of the determination. These sections overlap since the offset from one section to another is only of 5 myr, which introduces a smoothing effect



FIGS. 8 AND 9. Evolution of the secular frequencies g_i, s_i over 200 myr backward. The fundamental frequencies of the secular system are determined by the MFT analysis of the normalized proper modes over segments of about 20 myr except for g_7, g_8, s_7, s_8 where the MFT analysis is made over 50 myr. The different spans of time are overlapping since we make an offset of only 5 myr between two successive segments. During the MFT analysis, when possible, the 20 leading terms of the normalized proper modes are identified. A test solution is then reconstructed with these 20 periodic terms and the



test solution is analyzed again with the MFT after an offset of a few million years. The uncertainty on each determination of the frequencies is obtained as the difference of the determination with the initial solution and with the test solution. This uncertainty is represented as the error bar on the plots. When the error bar is not visible, it just means that it is too small to be seen. The evolution of the fundamental frequencies gives a lower bound to the size of the chaotic zones in the directions of the action variables related to the corresponding fundamental frequencies.

on the results. The results are displayed in Fig. 8 for all the fundamental frequencies and summarized in Table X. Each determination of the frequencies is given with an error bar corresponding to the accuracy of the determination of this frequency. When this bar does not appear, it just mean that it is too small to be visible on the figures.

The results are quite spectacular and reflect, I believe, a proper representation of the behavior of the Solar System. We can see that the variations of the frequencies for the inner planets are very large, as much as 0.2 arcsec/year for the frequencies g_3, g_4, and s_2 related to the perihelion of the Earth and Mars and to the node and inclination of Venus. (Let me remind you that 0.1 arcsec/year corresponds to about one full rotation every 10 myr.) This is compatible with a Lyapounov exponent of 1/5 myr^{-1} and explains in another way why it is not possible to obtain a quasi-periodic representation of the solution over 100 myr. The variations of the frequencies g_1, s_1, s_3, s_4 are also large,

TABLE X

MEAN VALUES OF THE FUNDAMENTAL FREQUENCIES ($\bar{\nu}$) OVER 200 MYR, AND NUMERICAL LOWER ESTIMATES Z_ν OF THE SIZE OF THE CHAOTIC ZONES, OBTAINED AS THE AMPLITUDES OF THE VARIATIONS OF THE FUNDAMENTAL FREQUENCIES (FIGS. 8 AND 9)

	ν_0 (arcsec/yr)	$\bar{\nu}$ (arcsec/yr)	Z_ν (arcsec/yr)
g_1	5.86046	5.59	0.10
g_2	7.46041	7.455	0.013
g_3	17.46509	17.30	0.17
g_4	18.11381	17.85	0.20
g_5	4.12866	4.24882	0.00002
g_6	23.47280	28.2203	0.0010
g_7	2.98980	3.08952	0.00007
g_8	0.65270	0.66698	0.00003
s_1	−5.20087	−5.59	0.08
s_2	−6.57027	−7.00	0.23
s_3	−18.74453	−18.88	0.06
s_4	−17.63461	−17.80	0.12
s_5	0.0000	0	
s_6	−25.67345	−26.33020	0.00008
s_7	−2.92885	−3.00563	0.00006
s_8	−0.68277	−0.69195	0.00001

Note. In the first column, ν_0 are the values of the fundamental frequencies of the linear part of the secular system, taking the initial corrections of Section 5 into account.

although not as important. These results show that the chaotic zones in the inner Solar System are sufficiently large to induce macroscopic effects over even 100 myr, especially if we consider that here we can only give lower estimates. Moreover, a small change in the initial conditions, or a small additional perturbation will not be sufficient to remove the system from its chaotic zone and thus we can believe that the chaotic nature of the inner Solar System is robust against some small changes in the model.

On the contary, the frequencies of the outer planets are nearly constant. Apart from g_6 which shows an evolution which is significiant over 1 byr, the other frequencies g_5, g_7, g_8, s_6, s_7, s_8 could be nearly considered as integrals of the motion over 5 byr. There should still be some chaotic zones around these frequencies (at least because the outer planet system is coupled with the inner planet system which is chaotic), but the width of the zones is narrow. If we look at the results of numerical integration of the outer planets over 200 myr of the LONGSTOP team (Nobili *et al.* 1989), we can see that their solutions present a cluster of lines around g_6. This tends to indicate that the chaotic zone around g_6, although much smaller than that in the inner planets case, should also come from intrinsic stochasticity of the outer planets themselves.

In the first column of Table X, the values ν_0 of the linear part of the differential system are also given. These values correspond to the values of the frequencies when the values of the action like variables are zero. The size of the chaotic zone in each direction Z_ν should thus be compared to the distance $\bar{\nu} - \nu_0$. For the inner planets, these two quantities are comparable, while for the outer planets, $Z_\nu/(\bar{\nu} - \nu_0)$ varies from about 1/500 to 1/8000. If one thinks of a usual plot of surface of a section of about 10 cm in width, the zone around g_6 will look like a very thin line 0.01 mm wide, which is much smaller than what any plotter device

can draw. On the contrary, in the direction of the inner planet frequencies, the chaotic zone will fill a large area of the plot: most of the KAM invariant tori are destroyed.

10. DISCUSSION

After the detection of chaotic motion in the inner Solar System (Laskar 1989a), it was very important to locate the mechanism which was responsible for this chaotic motion. The answer to this question is given here. The appearance of chaos in the inner Solar System is due to the transition from libration to circulation of the resonant argument θ (Table IX) involving the proper modes related to the Earth and Mars. This argument is present in the very leading terms of the solutions of Mars and the Earth (Tables II and III) which explains why its effect is so important. We also have a very important resonance σ among the proper modes of Mercury, Venus, and Jupiter. This resonance is also present in the leading terms of the solutions of Mercury and Venus, but stays in libration and thus does not seem to induce exponential divergence of nearly orbits, although the change in the amplitude of the libration suggests that we are also in a chaotic zone and that transition to circulation could occur in a longer span of time.

Another important problem was to evaluate the size of the region of the phase space where the chaotic motion takes place. I introduce here a method which I believe is new for the determination of a numerical lower estimate of the chaotic zone of a weakly chaotic system by the MFT analysis of the evolution the fundamental frequencies of the system over the time. In the present case, this shows that for the frequencies g_3, g_4, s_3, s_4 related to the Earth and Mars, we have a very large variation of the frequencies, which means a very large chaotic zone. This implies that a small change of initial conditions will not move the system solution outside from this zone. This is thus a very strong indication that the appearance of chaos in the inner Solar System is robuts against a small change of initial conditions or perturbations. On the contrary, the frequencies related to the outer planets show very small variations over 200 myr. The outer planet solution is thus close to a quasi-periodic motion over 200 myr. Nevertheless, the variation of g_6 is significant over 1 billion byr, and there is still the possibility of intrinsic chaos in the outer planet system over 5 byr, but the chaotic zone will be thin.

The large variations in the fundamental frequencies of the inner Solar System in the course of its history implies a new vision in the study of several long term phenomena in the history of the Solar System. Among them is the long term evolution of the precession and obliquity of the Earth and Mars and the possibility of secular spin–orbit resonances, which is critical for the Milankovitch theory of the paleoclimates of the Earth and Mars (Berger *et al.* 1984, Ward *et al.* 1979). The secular frequencies of the inner Solar System are not fixed during the Solar System history but are sweeping large portions of the phase space, which should also be of importance for the secular evolution of the minor planets related to the inner planet secular frequencies (Williams and Hierath 1987).

The picture of the dynamics of the Solar System which is given here is very different from the regular quasiperiodic motion described by Laplace and Lagrange 200 years ago. It is much more complicated and several points which are still under investigation will be presented in a forthcoming paper: The large chaotic region around the secular frequencies related to Mars and the Earth suggests that this region is not restricted to the chaotic layer around a separatrix but that we are in the presence of resonance overlap and that there is a possibility of transition from one resonance to another. Besides, I mentioned that the libration frequency θ is also in resonance with the circulation of the argument related to $g_3 - g_4$. This resonance is similar to the second-order resonance reported by Tite-

more and Wisdom (1989) in the possible evolution of the Uranian satellites or by Milani *et al.* (1989) in the motion of Pluto. It needs further investigation and another such resonance is also possibly related to the resonance σ.

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