Problem Set 4

1. Prove that $< \mathbf{e}_k^a (\mathbf{w}_k)^{\mathrm{T}} >= 0$, where
$$\mathbf{e}_{k+1}^f = \boldsymbol{\phi}_k \mathbf{e}_k^a - \mathbf{w}_k$$
and
$$\mathbf{e}_k^a = (\mathbf{I} - \tilde{\mathbf{K}}_k \mathbf{H}_k)\mathbf{e}_k^f + \tilde{\mathbf{K}}_k \mathbf{v}_k.$$
You will need to use the basic assumptions about the model and observation error made when deriving the Kalman Filter: $< \mathbf{w}_k (\mathbf{w}_l)^{\mathrm{T}} >= \mathbf{Q}_k \delta_l^k$, $< \mathbf{w}_k (\mathbf{v}_l)^{\mathrm{T}} >= 0$ for all k and l and $< \mathbf{e}_0^f (\mathbf{w}_k)^{\mathrm{T}} >= 0$ for all k.

2. Scalar KF. Consider the following scalar system and measurement equations:
$$\mathrm{x}_{k+1} = m\mathrm{x}_k + \mathrm{w}_k \tag{1}$$
$$\mathrm{z}_k = \mathrm{H}\mathrm{x}_k + \mathrm{v}_k \tag{2}$$
where the system noise w_k is $\mathcal{N}(0, \mathrm{Q})$ and v_k is $\mathcal{N}(0, \mathrm{R})$. Both noises are assumed to be white in time, and uncorrelated with each other at all times, and uncorrelated with the initial state error.

   (a) Write the KF equations for this system.

   (b) Remove the analysis step and obtain only 3 equations for K_k, P_{k+1}^f and x_{k+1}^f.

   (c) Write a MATLAB script which integrates the KF in time. Here are some steps to help you do this. Note that when dealing with random numbers, each random sequence is different (unless you initialize the random number generator). Thus each time you run your script you will get a different realization of the errors. Thus, when doing the experiments, run the same one many times.

      i. Define parameters, Q, R, m=1, H=1, the total number of time steps (50).

      ii. Define the initial conditions: $\mathrm{x}_0^f = 0$, $\mathrm{P}_0^f = 1$ and $\mathrm{x}_0^t = 10$.

      iii. Loop in time.

         A. First generate the obs by perturbing the truth using the measurement equation. Assume an obs is available every time step.

         B. Compute K_k, P_{k+1}^f, x_{k+1}^f, x_{k+1}^t.

      iv. Plot x_k^f, x_k^t and z_k as a function of k.

      v. Plot the KF predicted error $(\mathrm{P}_k^f)^{1/2}$ and the actual error $|\mathrm{x}_k^f - \mathrm{x}_k^t|$ as a function of time step k. Also plot K_k on the same frame.

   Include a hardcopy of your script, or email it to Lisa when handing the problem in.

(d) Estimation of a constant. For this case, set Q=0, R=1. Does the KF work? How do you know? What happens to the gain K_k with time? Why? Increase the obs error variance to R=3. Does the KF still work?

(e) Neutral dynamics. Now let Q=1, and try R=0.1, 1, 3. What happens to K_k and P_k^f as R increases? Does the KF work? How do you know?

(f) Now consider the usual case, where our model is incorrect. Thus, the true model parameter m^t=1.005 but we think (so the KF uses) m=1. Now what happens to K_k and P_k^f as R increases? Does the KF work? How do you know? If it does not work, can you make it work by choosing appropriate parameters values?

3. (Todling ch. 5, #9)

MATLAB exercise. Consider the following linear dynamical process

$$\mathbf{x}_k = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = \begin{pmatrix} 1 & T \\ -\omega^2 T & 1-2\alpha T \end{pmatrix} \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \end{pmatrix} + \begin{pmatrix} w_1(k-1) \\ w_2(k-1) \end{pmatrix} \tag{3}$$

and the following observation process

$$z(k) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + v(k) \tag{4}$$

for $\mathbf{w}(k) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$, $v(k) \sim \mathcal{N}(0, r/T)$ and both uncorrelated from each other at all times. Here the (co)variance $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}.$$

Use the following parameters to address the questions below.

ω	α	r	T
0	-0.1	0.02	0.02

(a) Is the dynamical system stable or unstable?

(b) Using MATLAB, simulate the stochastic dynamical system from k=0 to k=500 starting from $\mathbf{x}_0 = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}$. Plot the state $\mathbf{x}_k$ against k.

(c) Using the linear Kalman filter, simulate the evolution of the error covariance matrix, starting from the initial condition $\mathbf{P}_0^a = \mathbf{I}$, where $\mathbf{I}$ is the 2x2 identity matrix. Plot the analysis error variance, in both variables, for the same time interval as in the previous item.

(d) Is the filter stable or unstable. Explain.

(e) Are your answers to questions (a) and (d) incompatible? Explain.

(f) Plot the true state evolution together with the analysis estimate for both variables and for the time interval in item (b). Note that your initial estimate should be a realization of the initial state p.d.f. which is $\mathcal{N}(\mathbf{0}, \mathbf{P}_0^a)$:

$$\mathbf{x}_0^a = \mathbf{x}_0 + \boxed{\mathrm{chol}(\mathbf{P}_0^a)} * \boxed{\mathrm{randn}(:)}.$$

(g) *Suboptimal filters:* Let us now study the behaviour of two suboptimal filters. Before starting, however, you must replace the analysis error covariance equation by Joseph's formula, if you weren't already using it. Recall that Joseph's formula is valid for all gain matrices, not just the optimal one. Thus we can use it to evaluate the performance of suboptimal filters.

(i) Assuming the calculation of the forecast error covariance is computationally too costly for the present problem, we want to construct a suboptimal filter that somehow replaces the calculation of $\mathbf{P}_k^f$ by a simpler equation. Let us first try the simple alternative, $\mathbf{P}_k^f = \mathbf{I}$. With this choice of forecast error covariance, it is simple to see that the gain matrix becomes

$$\tilde{\mathbf{K}}_k = \mathbf{H}^{\mathrm{T}}(\mathbf{H}\mathbf{H}^{\mathrm{T}} + r/T)^{-1} = \frac{1}{1 + r/T}\mathbf{H}^{\mathrm{T}}$$

where we used explicitly that $\mathbf{H}$=(1 0) for the system under consideration. Keeping the equation for $\mathbf{P}_k^f$, in your MATLAB code as dictated by the Kalman filter, replace the expression for the optimal gain by the one given above. This turns the state estimate into a suboptimal estimate. Also, since you have kept the original expression for the forecast error covariance evolution, and you are using Joseph's formula for the analysis error covariance, these two quantities provide now filter performance information due to suboptimal choices of gains. With the "approximate" gain matrix above, is the resulting filter stable or unstable? Explain. If this is not a successful choice of gain matrix, can you explain why that is?

(ii) Let us now build another suboptimal filter that replaces the gain by the asymptotic gain obtained from the optimal run in item part (b). To obtain the optimal asymptotic gain, you need to run the experiment in part (b) again, output the gain matrix at the last time step from that run, and use it as a suboptimal choice for the gain matrix in this iterm. You should actually make sure that the gain has asymptoted by looking at its value for a few time steps before the last time step, and verifying that these values are indeed the same. Now run an experiment similar to that in (i), but using the asymptotic gain for the suboptimal gains at all time steps. Is the resulting filter stable or unstable? (Note: This choice of gain corresponds to using the so-called Wiener filter.)

4. KF divergence. Suppose the true (scalar) system is

$$\mathrm{x}_{k+1}^t = \mathrm{x}_k^t + \mathrm{w}_k, \tag{5}$$

with $< \mathrm{w}_k >= 0$ and $< \mathrm{w}_k(\mathrm{w}_k)^{\mathrm{T}} >= \mathrm{Q}_k\delta_l^k$, but we believe the true system is

$$\mathrm{x}_{k+1}^d = \mathrm{x}_k^d. \tag{6}$$

Here the superscript d refers to the "design" system. So, although the true system is corrupted by random noise, we believe the true state is simply a constant. The initial condition is unbiased, with variance 1. Thus,

$$< (\mathrm{e}_0^d) >= 0, \qquad < (\mathrm{e}_0^d)(\mathrm{e}_0^d)^{\mathrm{T}} >= 1.$$

The measurement equation is

$$\mathrm{z}_k = \mathrm{x}_k + \mathrm{v}_k. \tag{7}$$

where $< \mathrm{v}_k(\mathrm{v}_k)^{\mathrm{T}} >= \delta_l^k$, $< \mathrm{v}_k >= 0$. Show that the design forecast error variance is $(\mathrm{P}_k^f)^d = 1/(k+1)$. Show that the actual error variance diverges as fast as k. Similarly, even if the true system were biased, i.e.

$$\mathrm{x}_{k+1}^t = \mathrm{x}_k^t + \mathrm{c},$$

using the above design system, show that the actual error variance diverges.