Problem Set 2

1. Let us examine the case of two observations on a 1-D grid from section 3.2. As in the text, obs 1 is located at x/L=-2.0, the analysis is done at x/L=0 and obs 2 varies over x/L=[-4,4]. Also, $\alpha = (\sigma^r/\sigma^b)^2$=0.25 and

$$\rho^b(\Delta x) = \left(1 + \frac{|\Delta x|}{L}\right) \exp\left(-\frac{|\Delta x|}{L}\right).$$

Obtain MATLAB script **prob3p2.m**. This script plots the weights w_1 and w_2 from (3.20) and (3.21) as well as the normalized analysis error for (3.22). Running **prob3p2.m** will reproduce Fig. 4.7 of Daley (1991). In this problem, we consider how changes to the free parameters of this system, the position of obs 1 and α affect the results.

i) Consider the impact of changing α on the results. Try for example, α=0.1,0.25,0.5,1.0 .

(a) What happens to the analysis error as α increases? Why?

(b) What happens to the phenomenon of observation screening?

ii) Consider how the results vary when obs 1 is moved. Plot the results when α=0.25 and obs 1 is located at x/L=-3.0, -2.0, -1.0, 0.0 .

(a) When obs 1 is located at x1, for what positions of obs 2 are the weights equal?

(b) What locations of obs 2 have no effect on the analysis?

(c) What is the best location for obs 1? For this location, over what length scale does obs 2 influence the analysis?

2. In this problem we generate a correlation matrix for a specific grid and examine its properties. Consider the interval $(-L_x, L_x]$ and divide it into J gridpoints. Consider the homogeneous and isotropic, Gaussian correlation function in 1D:

$$\rho(x, y) = \rho(r = |x - y|) = \exp(-\frac{1}{2}(x - y)^2/L_d^2). \tag{1}$$

r is the distance between two points in the domain and L_d is the decorrelation length. Points in the discrete domain are defined by

$$x_j = j\Delta x$$

where $\Delta x = 2L_x/J$ for $j \in \{-J/2 + 1, J/2\}$, and the elements of the homogeneous, isotropic correlation matrix, $\mathbf{B}$ are given by

$$B_{ij} = \rho(x_i, y_j).$$

a) Construct a MATLAB function that returns the correlation matrix $\mathbf{B}$, given the half-length of the domain, L_x, the number of gridpoints J, and the decorrelation length, L_d. For $(L_x, L_d, J) = (1, 0.2, 32)$, compute $\mathbf{B}$ using this function. Make a contour plot of the correlation array. (Note: use `meshgrid` to generate this matrix. Matlab hints: A*B is the usual matrix multiplication of nxr matrix A by rxm matrix B. C.*D is the product $C_{ij}D_{ij}$. i.e. an elementwise multiplication of the matrices.)

b) For the parameters of part (a), plot the correlation functions at the following two specific locations: $x_j \in \{0, L_x\}$. There is a MATLAB function `plot`.

c) Is the $\mathbf{B}$ obtained above an *acceptable* correlation matrix? Why or why not? Hint: check its eigenvalues. MATLAB has a function called `eig`.

d) From the figures constructed in parts (a) and (b), we see that the correlations decrease very quickly toward zero. What if we actually set some of these small value to zero in order to save some storage space? Without actually changing the storage of the matrix, define a new matrix $\mathbf{B}_c$ by setting elements corresponding to $|r| > L_c$ to zero. Use the same parameters as in (a) and a cutoff value of $L_c = 3L_d$. Make a contour plot, and plot the correlation function at two locations as in part (b). Is $\mathbf{B}_c$ an acceptable correlation matrix?

e) Repeat parts (a) to (c) for the Triangular correlation function,

$$T(x,y) = \begin{cases} 1 - |x-y|/L_c, & \text{for} |x-y| \le L_c \\ 0 & \text{otherwise} \end{cases}$$

f) Construct a matrix $\mathbf{Q}$ as the Hadamard product of the matrices $\mathbf{B}$ and $\mathbf{T}$ by multiplying the matrices element by element:

$$\mathbf{Q} = \mathbf{B} \circ \mathbf{T} = [B_{ij}T_{ij}].$$

MATLAB can do this product trivially ($Q = B. * T$). Make a contour plot of $\mathbf{Q}$. Is $\mathbf{Q}$ an acceptable correlation matrix? Plot the correlation functions from $\mathbf{Q}$, $\mathbf{B}$ and $\mathbf{T}$ on the same graph, for x=0.

3. Let us now consider how we can generate an ensemble of vectors that have a specified covariance matrix. That is, we want to find vectors, $\mathbf{w}$ such that

$$E(\mathbf{w}\mathbf{w}^{\mathrm{T}}) = \mathbf{Q}$$

where $\mathbf{Q}$ is specified and can be a full matrix.

First assume we can generate N samples, ϵ_i, $i = 1, 2, \ldots, N$ from a normal distribution $\mathcal{N}(0, \sigma^2)$. Put these into a vector called $\mathbf{v} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_N)^{\mathrm{T}}$. Note that the sample covariance matrix for the vectors so formed should be:

$$\begin{aligned} < \mathbf{v}\mathbf{v}^{\mathrm{T}} > &= < \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_N \end{pmatrix} \begin{pmatrix} \epsilon_1 & \epsilon_2 & \ldots & \epsilon_N \end{pmatrix} >= \begin{bmatrix} <\epsilon_1\epsilon_1> & <\epsilon_1\epsilon_2> & \ldots & <\epsilon_1\epsilon_N> \\ <\epsilon_2\epsilon_1> & <\epsilon_2\epsilon_2> & \ldots & <\epsilon_2\epsilon_N> \\ \vdots & \vdots & \ddots & \vdots \\ <\epsilon_N\epsilon_1> & <\epsilon_N\epsilon_2> & \ldots & <\epsilon_N\epsilon_N> \end{bmatrix} \\ &= \begin{pmatrix} \sigma^2 & 0 & \ldots & 0 \\ 0 & \sigma^2 & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ldots & \sigma^2 \end{pmatrix} = \sigma^2 \mathbf{I}. \end{aligned} \tag{2}$$

a) What if each element was drawn from a different Normal distribution, $\mathcal{N}(0, \sigma_i^2)$? What form does the sample covariance matrix, $\mathbf{R} = <\mathbf{v}\mathbf{v}^{\mathrm{T}}>$ take now?

b) Now consider an arbitrary covariance matrix, $\mathbf{Q}$, where $\mathbf{Q} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$. What is the covariance matrix for $\mathbf{Lv}$ if $<\mathbf{v}\mathbf{v}^{\mathrm{T}}>=\mathbf{I}$? i.e. elements of $\mathbf{v}$ are drawn from a $\mathcal{N}(0,1)$ distribution.

c) Finally consider

$$\mathbf{Q} = \mathbf{E}\mathbf{D}\mathbf{E}^{\mathrm{T}}.$$

How would you generate vectors, $\mathbf{w}$, having the covariance matrix $\mathbf{Q}$, given an ensemble of vectors, $\mathbf{v}$, where $\mathbf{v}$ are drawn from a $\mathcal{N}(0,1)$ distribution?

d) Using MATLAB, contour plot sample correlation matrices for $\mathbf{Q}$ where the correlation function is

$$\rho(r = |x-y|) = \exp(-\frac{1}{2}(x-y)^2/L_d^2)$$

for $L_d = 0.2$. Use the same parameters as in prob. 2 for the mesh and other constants. Assume variances are all equal to 1. Use 100, 1000 and 1000 samples in the ensemble. Now contour plot the Identity matrix for the same 32-dimensional space using 1000 samples. How does this plot differ from that for $\mathbf{Q}$? What assumption is really being made when an Identity covariance matrix is used?

4. Filtering properties. In this problem, we are going to examine the structure of the weight matrix and verify the assertion of section 3.8 that the filtering of observation increments is controlled by the background error covariance matrix. Start by defining a grid as in prob. 2. Consider the interval $(-L_x, L_x]$ and divide it into J gridpoints. We will use the homogeneous and isotropic, Gaussian correlation function in 1D:

$$\rho(|x-y|) = \exp(-\frac{1}{2}(x-y)^2/L_d^2). \tag{3}$$

r is the distance between two points in the domain and L_d is the decorrelation length. Points in the discrete domain are defined by

$$x_j = j\Delta x$$

where $\Delta x = 2L_x/J$ for $j \in \{-J/2+1, J/2\}$. Form the the homogeneous, isotropic correlation matrix, $\mathbf{Q}$ using

$$\mathbf{Q}_{ij} = \rho(x_i, y_j).$$

(a) Find the eigenvalues and eigenvectors of $\mathbf{Q}$. Plot the eigenvalues as a function of index. Plot the eigenvectors corresponding to the 6 largest eigenvalues, as a function of x. Draw a zero-line and count the number of zero crossings for each eigenvector plotted. (Type **help eig** for help on how to use MATLAB's eigenvalue function.)

(b) Now compute

$$\mathbf{A} = \mathbf{B}(\mathbf{B}+\mathbf{R})^{-1}$$

where $\mathbf{R} = (\sigma^r)^2\mathbf{I}$, $\mathbf{B} = \mathbf{DQD}$ and $\mathbf{D}$ is a diagonal matrix with each diagonal element equal to σ^b. Let σ^r=1 and σ^b=2. Compute the eigenvalues and eigenvectors of $\mathbf{A}$ making sure to sort by magnitude. Compare the eigenvalues of $\mathbf{A}$ to $(1+\alpha/\lambda_q)^{-1}$ when $\alpha = (\sigma^r/\sigma^b)^2$ as before and λ_q is an eigenvalue of $\mathbf{Q}$. Plot the eigenvectors of $\mathbf{A}$. Are they the same as those for $\mathbf{Q}$?

(c) Now let's create some observation increments. To avoid interpolation, place an obs at every grid point. To see the filtering aspects, define an obs increment by

$$y = \cos(c\pi x)$$

where the x-grid values are from (-1,1]. Parameter c determines the wavenumber and hence the spatial scale of the obs increment. Create an obs increment vector, and compute the analysis increment using

$$\mathbf{d} = \mathbf{A}\mathbf{y}.$$

Plot $\mathbf{y}$, $\mathbf{d}$ versus x for various values of c. Which waves are filtered the least? For which waves is the amplitude dropped by 80% or more?

(d) How are filtering properties affected by a change in L_d?