Chapter 7

Basic Concepts of Atmospheric Dynamics

In this lecture the basic equations that govern atmospheric dynamics are introduced. The main goal here is to outline the necessary concepts for a better understanding of the problem of atmospheric data assimilation to be treated in the following lectures. Much of the content of this lecture can be found in meteorology text books such as: Daley [39], Ghil & Childress [62], Haltiner & Williams [70], Holton [82], and Pedlosky [114].

In Section 7.1, we introduce the governing equations. In Section 7.2, we make a scale analysis of the governing equations for synoptic scale problems, which leads us to introduce the notions of hydrostatic and geostrophic approximations, which are discussed in Section 7.3. Notions on vertical stratification notion are introduced in Section 7.4. In Section 7.5 we solve the equations of motion for the simple in which the atmosphere is approximated by the linearized shallow water equations. Finally in Section 7.6, we discretize the shallow–water equations using a relatively simple finite difference scheme.

7.1 Governing Equations

(I) Momentum Equation:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + 2\boldsymbol{\Omega}\times\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{f} \tag{7.1}$$

where $\mathbf{v}$ is the velocity vector of the atmospheric "fluid" in three dimensions, in a rotating frame of reference; $\boldsymbol{\Omega}$ is the three–dimensional angular velocity vector (velocity with which the rotating frame of reference moves); ρ is the density of the atmospheric "fluid"; p is its pressure; $\mathbf{g}$ is the gravitational acceleration vector in three dimensions; $\mathbf{f}$ represents the three–dimensional friction force (e.g. between the atmosphere and the earth surface); and ∇ is the gradient vector in three spatial dimensions.

(ii) Continuity Equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{7.2}$$

This equation means that the rate of change of the local density is equal to the negative of the (mass) density divergence. It is common to re–write this equation as:

$$\frac{1}{\alpha}\frac{D\alpha}{Dt} + \nabla \cdot \mathbf{v} = 0\,, \tag{7.3}$$

where $\alpha = 1/\rho$ is the specific volume, and the operator D/Dt is formally defined as:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\,. \tag{7.4}$$

(iii) First Law of Thermodynamics:

This law states that the change in internal energy of the system is equal to the difference between the heat added to the system and the work done by the system. For the atmosphere, the first law of thermodynamics translates into:

$$\frac{De}{Dt} = -p\frac{D\alpha}{Dt} + Q\,, \tag{7.5}$$

where e is the specific internal energy, which is only a function of the temperature T of the system, and Q is the heat per unit of mass. It is worth noticing that the temperature is a function of the space variables, as well as of time.

Introducing the specific heat at constant volume for the dry air $c_v \equiv e/T$, we have that

$$c_v\frac{DT}{Dt} = -p\frac{D\alpha}{Dt} + Q\,. \tag{7.6}$$

where T is the temperature of the system.

In this lecture, we refer very little to the thermodynamics equation and therefore the equations above are sufficient.

7.2 Scales of the Equations of Motion

In spherical coordinates (λ, φ, z) the three components of the momentum equation (Newton's equations) can be written as (e.g., Washington & Parkinson [138]):

$$\frac{du}{dt} - \frac{uv\tan\varphi}{r} + \frac{uw}{r} = -\frac{1}{\rho r\cos\varphi}\frac{\partial p}{\partial \lambda} + fv - \hat{f}w + f_\lambda \tag{7.7}$$

$$\frac{dv}{dt} - \frac{u^2\tan\varphi}{r} + \frac{vw}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} - fu + f_\varphi \tag{7.8}$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \hat{f}u + f_z \tag{7.9}$$

Table 7.1: Definition of the scale parameters

$U \sim 10$ m/s	horizontal velocities
$W \sim 1$ cm/s	vertical velocities
$L \sim 10^6$ m	length (horizontal; $1/2\pi$ wave length)
$H \sim 10^4$ m	depth (vertical)
$\Delta P/\rho \sim 10^3$ m^2/s^2	horizontal pressure fluctuations
$L/U \sim 10^5$ s	time

where we use the definitions $\mathbf{v} \equiv (u, v, w)^T$ and $\mathbf{f} \equiv (f_\lambda, f_\varphi, f_z)^T$, and we notice that

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{7.10}$$

$$= \frac{\partial}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial z}. \tag{7.11}$$

The parameters f and $\hat{f}$ are defined as:

$$f = 2\Omega\sin\varphi \tag{7.12}$$

$$\hat{f} = 2\Omega\cos\varphi \tag{7.13}$$

where f is known as the Coriolis parameter, and Ω is the magnitude of the vector $\mathbf{\Omega}$. Moreover, $r = a + z$, with a representing the radius of the earth and z the height, starting from the surface.

In this lecture, we are interested in synoptic scale dynamics, and therefore we introduce scale variables that refer to synoptic atmospheric systems as in Table 7.2 (see Holton 1979, p. 36 for more details). In particular, notice that the time scale is on the order of days; this scale is called advective time scale, where pressure systems move approximately with the horizontal winds.

Disregarding the friction force $\mathbf{f}$ from this point on, we can proceed with the scale analysis of the equations (7.7)–(7.9), noticing that an estimate of the scale of the Coriolis parameter can be obtained for the mid–latitude $\varphi = \varphi_0 = 45^o$ as:

$$[f] = [\hat{f}] = f_0 = 2\Omega\sin\varphi_0 = 2\Omega\cos\varphi_0 \tag{7.14}$$

$$= 2\left(\frac{2\pi}{86164}\right)\cos(45^o) \sim 10^{-4}, \tag{7.15}$$

where the notation [.] is used to indicate the scale of the quantity between the curly brackets.

The requirement of synoptic dynamics imposes a restriction in the horizontal direction. To define scales in the vertical direction it is necessary to establish at what height we are interested in describing the atmosphere. For tropospheric dynamics, the pressure gradient can be represented by the scale defined by P_0/H, where P_0 ($\sim$ 1000 mb = 1 atm) is the pressure at the surface and H is the troposphere depth introduced above.

Table 7.2 shows the results of the scale analysis, where the magnitude of each term in the equations (7.7)–(7.9) is indicated. We see directly from the table that the horizontal and

Table 7.2: Scale analysis of the components of the momentum equation

Horizontal scale analysis						
zonal	$[\frac{du}{dt}]$	$[fv]$	$[\hat{f}w]$	$[\frac{uw}{r}]$	$[\frac{uv\tan\varphi}{r}]$	$[\frac{1}{\rho r\cos\varphi}\frac{\partial p}{\partial\lambda}]$
meridional	$[\frac{dv}{dt}]$	$[fu]$		$[\frac{vw}{r}]$	$[\frac{u^2\tan\varphi}{r}]$	$[\frac{1}{\rho r}\frac{\partial p}{\partial\varphi}]$
scales	$\frac{U^2}{L}$	f_0U	f_0U	$\frac{UW}{[r]}$	$\frac{U^2}{[r]}$	$\frac{\Delta P}{\rho L}$
magnitude (m/s^2)	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}
Vertical scale analysis						
vertical	$[\frac{dw}{dt}]$	$[\hat{f}u]$			$[\frac{u^2+v^2}{r}]$	$[\frac{1}{\rho}\frac{\partial p}{\partial z} - g]$
scales	$\frac{UW}{L}$	f_0U			$\frac{U^2}{[r]}$	$\frac{P_0}{\rho H} + [g]$
magnitude (m/s^2)	10^{-7}	10^{-3}			10^{-5}	10

vertical scales are independent. This fact is exactly what allows us to distinguish between horizontal and vertical motion as approximately separate entities.

7.3 Geostrophic and Hydrostatic Approximations

The scale analysis of the momentum equations in the horizontal direction shows that synoptic dynamics are dominated by the Coriolis term and by the pressure gradient term. In this way, to first order the horizontal equations can be approximated by

$$fv \approx \frac{1}{\rho}\frac{\partial p}{\partial\lambda} \tag{7.16}$$

$$-fu \approx \frac{1}{\rho}\frac{\partial p}{\partial\varphi} \tag{7.17}$$

This approximation motivates us to define the so–called geostrophic winds as those satisfying exactly the relation:

$$\mathbf{v}_g \equiv \mathbf{k} \times \frac{1}{f\rho}\nabla p \tag{7.18}$$

where $\mathbf{k}$ is the unit vector in vertical direction.

The table 7.2 also indicates that a reasonable simplification of the vertical component of the momentum equation is

$$\frac{1}{\rho}\frac{\partial p}{\partial z} \approx -g\,, \tag{7.19}$$

meaning that the pressure field is nearly in hydrostatic balance. In other words, the pressure at a point is approximately equal to the weight of the air column above the point. In the same way that we were motivated to introduce geostrophic winds, we can define a standard pressure $\bar{p}$ as the one that follows exactly the hydrostatic relation:

$$\frac{d\bar{p}}{dz} = \bar{\rho} g \tag{7.20}$$

where $\bar{\rho}$ is a standard density. Notice that, by simplifying the vertical component of the momentum equation the vertical winds disappear. This means that at synoptic scales these winds are negligible.

7.4 Vertical Stratification

Let us examine the hydrostatic approximation introduced in the previous section in more detail. Since $g\rho > 0$, the pressure p monotonically decreases with the height z. Moreover, within the tropospheric layer $g \approx const.$, which means that given a density function $\rho = \rho(z,p)$, the hydrostatic approximation

$$\frac{dp}{dz} = -g\rho\,, \tag{7.21}$$

when satisfied exactly, provides a model for the vertical atmosphere.

A simple atmospheric model, one called homogeneous, is that for which the density $\rho = \bar{\rho}$ is constant (independent of height and pressure). In this case,

$$p = \bar{p} - g\bar{\rho}(z - \bar{z})\,, \tag{7.22}$$

where the quantities with a bar are standard quantities, defined generally at sea level.

A more realistic model, not homogeneous, is found when we consider the atmosphere as an ideal gas. In this case, the pressure and density are related by the ideal gas law.

$$\frac{p}{\rho} = RT\,, \tag{7.23}$$

where T is the temperature and R is the gas constant for the dry air.

In this way, the hydrostatic balance can be written as:

$$\frac{dp}{p} = -\frac{g}{R}\frac{dz}{T_0 - \Gamma z} \tag{7.24}$$

where we use the fact that in the troposphere the rate of temperature decrease is approximately constant: $dT/dz = -\Gamma$, for Γ being the lapse rate, and T_0 the temperature of an isothermal atmosphere.

One of the conventional ways of taking measurements of the atmosphere is by means of balloons. They usually measure the temperature, pressure and wind. That is, the temperature and wind are functions of pressure, in particular $T = T(p)$. The hydrostatic relation, written as:

$$\frac{dz}{dp} = -\frac{R}{g}\frac{T(p)}{p} \tag{7.25}$$

can be used to obtain the temperature, pressure and density profiles as functions of the height. This information can then be used in the solution of the governing equations.

In fact, we can simplify this transformation procedure by introducing pressure as the vertical coordinate, instead of the height z. By defining the geopotential function $\phi \equiv gz$, the hydrostatic equation becomes:

$$\frac{d\phi}{dp} = -\frac{RT}{p}\,. \tag{7.26}$$

The governing equations can be written using pressure as the vertical coordinate (e.g., Haltiner & Williams [70], Section 1.9).

7.5 Linearized Shallow–Water Equations

The system of shallow–water equations, in cartesian coordinates, can be written as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial h}{\partial x} = 0 \tag{7.27a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial h}{\partial y} = 0 \tag{7.27b}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{7.27c}$$

where x and y indicate the zonal and meridional directions, respectively, and we consider the Coriolis parameter $f = f_0$ to be constant. The boundary conditions that interest us at this moment are periodic in both directions and the extent of the domain is taken as $2\pi a$, where a is the radius of the earth.

A simple linearization that we can use for the system above, with relevant meaning, is when the reference state (or basic state) consists of a null winds, i.e., state of rest, and of a free surface height, i.e., independent of space and time. That means, the basic state is defined as:

$$u = 0 + u' \tag{7.28a}$$

$$v = 0 + v' \tag{7.28b}$$

$$h = H + h' \tag{7.28c}$$

where $H = const.$ e $(.)'$ is used to indicate perturbations. In this case, the equations (7.27a)–(7.27c) are reduced to the equations

$$\frac{\partial u}{\partial t} - f_0 v + \frac{\partial \phi}{\partial x} = 0 \tag{7.29a}$$

$$\frac{\partial v}{\partial t} + f_0 u + \frac{\partial \phi}{\partial y} = 0 \tag{7.29b}$$

$$\frac{\partial \phi}{\partial t} + \Phi(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0 \tag{7.29c}$$

where we eliminate the notation $(\cdot)'$ so that u, v and ϕ in the system of equations above refer to perturbations; moreover, we introduce the basic geopotential height $\Phi = gH$ and its correspondent perturbation $\phi = gh$.

One of the ways to solve the equations above is to introduce the stream function ψ and the potential velocity χ by means of the Helmholtz theorem:

$$u = -\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x} \tag{7.30a}$$

$$v = \frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y} \tag{7.30b}$$

from where it follows that

$$\nabla^2 \psi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{7.31a}$$

$$\nabla^2 \chi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{7.31b}$$

Then, the equations (7.27a) and (7.27b) can be transformed into equations for the relative vorticity and divergence:

$$\frac{\partial \nabla^2 \psi}{\partial t} + f_0 \nabla^2 \chi = 0 \tag{7.32a}$$

$$\frac{\partial \nabla^2 \chi}{\partial t} - f_0 \nabla^2 \psi + \nabla^2 \phi = 0\,, \tag{7.32b}$$

where $\nabla^2 \psi$ is the vertical component of relative vorticity, and $\nabla^2 \chi$ is the divergence (see Holton [82] pp. 73, 83–86 for more details). Using (7.30a) and (7.30b) we can re–write the equation for the perturbation geopotential height as:

$$\frac{\partial \phi}{\partial t} + \Phi \nabla^2 \chi = 0 \tag{7.33}$$

The expressions (7.32)–(7.33) form a system of coupled, constant–coefficient, linear partial differential equations, which can be solved by normal mode expansion (i.e., Fourier series).

In this way we write:

$$\begin{pmatrix} \psi(x,y,t) \\ \chi(x,y,t) \\ \phi(x,y,t) \end{pmatrix} = \begin{pmatrix} \hat{\psi}(t) \\ i\hat{\chi}(t) \\ f_0\sqrt{k}\hat{\phi}(t) \end{pmatrix} \exp\left\{ i \left[\frac{(mx + ny)}{a} \right] \right\} \tag{7.34}$$

where m is the zonal wave number, n is the meridional wave number, $i = \sqrt{-1}$, and the constant k is given by

$$k = \frac{(m^2 + n^2)\Phi}{a^2 f_0^2}\,. \tag{7.35}$$

Then, we see that

$$\nabla^2 \psi = -\frac{f_0^2}{\Phi} k \psi \tag{7.36a}$$

$$\nabla^2 \chi = -\frac{f_0^2}{\Phi} k \chi \tag{7.36b}$$

$$\nabla^2 \phi = -\frac{f_0^2}{\Phi} k \phi \tag{7.36c}$$

Therefore, substituting (7.34) and (7.36) in equations (7.32)–(7.33) we obtain:

$$\frac{d\hat{\psi}}{dt} + if_0\hat{\chi} = 0 \tag{7.37a}$$

$$i\frac{d\hat{\chi}}{dt} - f_0\hat{\psi} + f_0\sqrt{k}\hat{\phi} = 0 \tag{7.37b}$$

$$f_0\sqrt{k}\frac{d\hat{\phi}}{dt} - if_0^2k\hat{\chi} = 0 \tag{7.37c}$$

These equations can be written in compact form,

$$\frac{d\hat{\mathbf{w}}(t)}{dt} = -if_0\hat{\mathbf{L}}\hat{\mathbf{w}}(t)\,, \tag{7.38}$$

where the vector $\hat{\mathbf{w}} \equiv (\hat{\psi}, \hat{\chi}, \hat{\phi})^T$, and the matrix $\hat{\mathbf{L}}$ is given by

$$\hat{\mathbf{L}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -\sqrt{k} \\ 0 & -\sqrt{k} & 0 \end{pmatrix}. \tag{7.39}$$

The solution of equation (7.38) is

$$\hat{\mathbf{w}}(t) = e^{-if_0\hat{\mathbf{L}}t}\hat{\mathbf{w}}(0)\,, \tag{7.40}$$

where $\hat{\mathbf{w}}(0)$ represents the initial condition vector. This expression can be written in a more convenient way if we expand the vector $\hat{\mathbf{w}}(0)$ in terms of the eigenvectors of $\hat{\mathbf{L}}$. These eigenvectors can be determined by solving the equation:

$$(\hat{\mathbf{L}} - \sigma_\ell\mathbf{I})\hat{\mathbf{v}}_\ell = \mathbf{0} \tag{7.41}$$

where σ_ℓ refers to the eigenvalue corresponding to the eigenvector $\hat{\mathbf{v}}_\ell$, and $\mathbf{I}$ represents the 3×3 identity matrix. Notice that writing the solution as in (7.34) produces a matrix $\hat{\mathbf{L}}$ which is real and symmetric.

It is simple to show that the eigenvalues of the matrix $\hat{\mathbf{L}}$ are determined by solving the characteristic equation,

$$\sigma_\ell^3 - (1 + k)\sigma_\ell = 0 \tag{7.42}$$

whose solutions can be written as: $\sigma_\ell \in \{\sigma_G^- = -\sqrt{1+k}, \sigma_R = 0, \sigma_G^+ = +\sqrt{1+k}\}$, and the subscripts R and G indicate frequencies of rational and gravity waves, respectively. The eigenvectors corresponding to the eigenvalues above can be obtained by substituting each value of σ in (7.41). In this way, we can build a matrix $\hat{\mathbf{V}}$ whose columns correspond to the eigenvectors $\hat{\mathbf{v}}_\ell$ of the problem of $\hat{\mathbf{L}}$. That is,

$$\hat{\mathbf{V}} = \frac{1}{\sqrt{2(1+k)}} \begin{pmatrix} 1 & \sqrt{2k} & 1 \\ -\sqrt{1+k} & 0 & \sqrt{1+k} \\ -\sqrt{k} & \sqrt{2} & -\sqrt{k} \end{pmatrix}, \tag{7.43}$$

where the first and third columns of $\hat{\mathbf{V}}$ correspond to the eigenvalues $\sigma_G^\pm$ and the middle column corresponds to the eigenvalue σ_R. It is easy to verify that the column vectors form a complete orthonormal set of eigenvectors and therefore the matrix $\hat{\mathbf{V}}$ is unitary: $\hat{\mathbf{V}}^T = \hat{\mathbf{V}}^{-1}$. Moreover, the matrix $\hat{\mathbf{V}}$ is the one that diagonalizes the matrix $\hat{\mathbf{L}}$:

$$\hat{\mathbf{V}}^{-1}\hat{\mathbf{L}}\hat{\mathbf{V}} = \mathbf{\Lambda}\,, \tag{7.44}$$

where $\mathbf{\Lambda}$ is a diagonal matrix whose elements are the respective eigenvalues: $\mathbf{\Lambda} = diag(\sigma_G^-, \sigma_R, \sigma_G^+)$.

Returning to the solution, (7.40), we write the expansion of the initial vector by utilizing the eigenvectors of $\hat{\mathbf{L}}$ as

$$\hat{\mathbf{w}}(0) = \sum_\ell \hat{c}_\ell \hat{\mathbf{v}}_\ell \tag{7.45}$$

and noticing that the eigenvectors of $e^{-if_0\hat{\mathbf{L}}t}$ are the same as those of $\hat{\mathbf{L}}$, with eigenvalues $e^{i\sigma_\ell t}$, we have

$$\hat{\mathbf{w}}(t) = \sum_\ell \hat{c}_\ell \hat{\mathbf{v}}_\ell e^{i\sigma_\ell t}\,. \tag{7.46}$$

Once the initial condition is known $\hat{\mathbf{w}}(0) = (\psi(0), \xi(0), \phi(0))^T$, the expression (7.45) can be inverted to obtain the expansion coefficients $\hat{c}_\ell$. Therefore, using the fact that the matrix $\hat{\mathbf{V}}$ is unitary we have

$$\hat{\mathbf{c}} = \hat{\mathbf{V}}^T \hat{\mathbf{w}}(0) \tag{7.47}$$

where we define $\hat{\mathbf{c}} \equiv (\hat{c}_-, \hat{c}_0, \hat{c}_+)^T$.

7.6 Numerical Solution: A Finite Difference Method

In general, there are no analytic solutions for the system of governing equations, including the thermodynamic processes. Therefore, these equations are solved numerically in some way (e.g., Haltiner & Williams [70]) with computer assistance. In this section we will exemplify a practical way of solving the governing equations by means of applying a finite difference scheme to a simple set of equations.

Consider the system of two–dimensional shallow–water equations, on a β–plane, linearized about a basic state with zero meridional wind $v \equiv 0$, and constant zonal wind $u = U$ (see Exercise 6.2, for a guide to the derivation of these equations):

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial x} - (f - U_y)v = 0\,, \tag{7.48a}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial y} + fu = 0\,, \tag{7.48b}$$

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + \Phi(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \Phi_y v = 0\,, \tag{7.48c}$$

where u, v are the perturbations in the velocity field, ϕ is the perturbation in the geopotential field, $f = f_0 + \beta y$ is the Coriolis parameter, and the basic state satisfies:

$$fU + \frac{d\Phi}{dy} = 0\,, \tag{7.49}$$

and the equations are applied to a doubly periodic domain.

The system of equations (7.48) can be written in vector form as

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial}{\partial x}(\mathbf{A}\,\mathbf{w}) + \frac{\partial}{\partial y}(\mathbf{B}\,\mathbf{w}) + \mathbf{C}\,\mathbf{w} = \mathbf{0}\,, \tag{7.50}$$

where $\mathbf{w} = (u, v, \phi)^T$, as in the previous section, and the matrices $\mathbf{A}$, $\mathbf{B}$ e $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} U & 0 & 1 \\ 0 & U & 0 \\ \Phi & 0 & U \end{pmatrix}, \tag{7.51a}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \Phi & 0 \end{pmatrix}, \tag{7.51b}$$

$$\mathbf{C} = \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.51c}$$

Notice that these matrices depend on the variable y, since the geopotential function of the basic state Φ and the Coriolis parameter f are functions of the latitude.

Let us apply the Richtmyer two step–version of the Lax–Wendroff finite difference scheme (see Richtmyer & Morton [117]; Ghil et al. [66]; and Parrish & Cohn [113]). For that we consider the 3–vector $\mathbf{w}(x, y, t)$ to be in a two–dimensional uniform grid $I \times J$ whose approximate value at a point is given by

$$\mathbf{w}(x_i, y_j, t_k) = \mathbf{w}_{ij}^k \approx \mathbf{w}[(i-1)\Delta x, (j-1)\Delta y, k\Delta t]\,, \tag{7.52}$$

with $i = 1, \ldots, I$, $j = 1, \ldots, J$, $k = 0, 1, \ldots$, and $\Delta x = L_x/I$, $\Delta y = L_y/(J-1)$, for L_x, L_y representing the extension of the rectangular domain in the zonal and meridional directions, respectively.

The Richtmyer version of the Lax–Wendroff scheme has the form of a predictor–corrector scheme for which the first step, the predictor, can be written as:

$$\mathbf{w}_{i+1/2,j+1/2}^{k+1/2} = \mu_x \mu_y \mathbf{w}_{i+1/2,j+1/2}^k - \frac{1}{2}\mathbf{L}_{j+1/2}\mathbf{w}_{i+1/2,j+1/2}^k \tag{7.53}$$

for $i = 1, 2, \ldots, I$ e $j = 1, 2, \ldots, J-1$, where the operator $\mathbf{L}_j$ is defined by

$$\mathbf{L}_j \equiv \lambda_x \mu_y \delta_x \mathbf{A}_j + \lambda_y \mu_x \delta_y \mathbf{B}_j + \Delta t \mu_x \mu_y \mathbf{C}_j, \tag{7.54}$$

with $\lambda_x \equiv \Delta t/\Delta x$, $\lambda_y \equiv \Delta t/\Delta y$, and we notice that $\mathbf{C}$ does not depend on y. The second order operators of spatial mean and difference are defined as:

$$\delta_x \mathbf{w}_{ij} \equiv \mathbf{w}_{i+1/2,j} - \mathbf{w}_{i-1/2,j}\,, \tag{7.55a}$$

$$\mu_x \mathbf{w}_{ij} \equiv \frac{1}{2}(\mathbf{w}_{i+1/2,j} + \mathbf{w}_{i-1/2,j})\,, \tag{7.55b}$$

and analogously for δ_y and μ_y:

$$\delta_y \mathbf{w}_{ij} \equiv \mathbf{w}_{i,j+1/2} - \mathbf{w}_{i,j-1/2}\,, \tag{7.56a}$$

$$\mu_y \mathbf{w}_{ij} \equiv \frac{1}{2}(\mathbf{w}_{i,j+1/2} + \mathbf{w}_{i,j-1/2})\,. \tag{7.56b}$$

The second step of this finite difference scheme, the corrector step, serves to propagate the state from the half–grid intermediate points to the full–grid points, that is,

$$\mathbf{w}_{ij}^{k+1} = \mathbf{w}_{ij}^{k} - \mathbf{L}_j \mathbf{w}_{ij}^{k+1/2} \tag{7.57}$$

for $i = 1, 2, \ldots, I$ and $j = 1, 2, \ldots, J$.

The periodic boundary conditions in the East-West direction as well as in the North-South direction can be taken into consideration by doing:

$$\mathbf{w}_{i,J+1} = \mathbf{w}_{i,1} \tag{7.58a}$$

$$\mathbf{w}_{i,0} = \mathbf{w}_{i,J} \tag{7.58b}$$

for $i = 1, 2, \cdots, I$, and

$$\mathbf{w}_{I+1,j} = \mathbf{w}_{1,j} \tag{7.59a}$$

$$\mathbf{w}_{0,j} = \mathbf{w}_{I,j} \tag{7.59b}$$

for $j = 1, 2, \cdots, J$.

Although it is not necessary, and in general cannot be done in implementation of numeric methods for practical problems in meteorology, we can combine the expressions (7.53) and (7.57) to write the finite difference system of equations in the following, more compact, form

$$\mathbf{w}^{k+1} \;=\; \mathbf{\Psi}\mathbf{w}^{k}, \tag{7.60}$$

where $\mathbf{\Psi}$ is the transition matrix of the system, also called the dynamics matrix. By writing the system of equations in this form it becomes easy to understand the connection between the problems studied in the previous lectures and the problem of assimilation of meteorological data to be studied below.

In any event, we can illustrate the morphology of the transition matrix by considering an idealized grid with resolution 4×5. The two stages (7.53) and (7.57) of the finite difference scheme can be combined as

$$\begin{aligned} \mathbf{w}_{ij}^{k+1} \;=\; & \mathbf{Q}_j^1 \mathbf{w}_{i-1,j-1}^k + \mathbf{Q}_j^2 \mathbf{w}_{i,j-1}^k + \mathbf{Q}_j^3 \mathbf{w}_{i+1,j-1}^k + \\ & \mathbf{Q}_j^4 \mathbf{w}_{i-1,j}^k + \mathbf{Q}_j^5 \mathbf{w}_{i,j}^k + \mathbf{Q}_j^6 \mathbf{w}_{i+1,j}^k + \\ & \mathbf{Q}_j^7 \mathbf{w}_{i-1,j+1}^k + \mathbf{Q}_j^8 \mathbf{w}_{i,j+1}^k + \mathbf{Q}_j^9 \mathbf{w}_{i+1,j+1}^k\,, \end{aligned} \tag{7.61}$$

where the matrices $\mathbf{Q}$ have dimension 3×3, and consist of linear combinations of the matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, calculated at a specific grid points. Explicit form for the auxiliary matrices $\mathbf{Q}$ can be found in Parrish & Cohn [113], with an appropriate modification due to different boundary conditions. A simplified version of (7.61) is treated here in the exercises.

$$
\mathbf{\Psi} = \left(\begin{array}{cccccccccccccccccccc}
\mathbf{Q}_1^5 & \mathbf{Q}_1^6 & 0 & \mathbf{Q}_1^4 & \mathbf{Q}_1^8 & \mathbf{Q}_1^9 & 0 & \mathbf{Q}_1^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_1^2 & \mathbf{Q}_1^3 & 0 & \mathbf{Q}_1^1 \\
\mathbf{Q}_1^4 & \mathbf{Q}_1^5 & \mathbf{Q}_1^6 & 0 & \mathbf{Q}_1^7 & \mathbf{Q}_1^8 & \mathbf{Q}_1^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_1^1 & \mathbf{Q}_1^2 & \mathbf{Q}_1^3 & 0 \\
0 & \mathbf{Q}_1^4 & \mathbf{Q}_1^5 & \mathbf{Q}_1^6 & 0 & \mathbf{Q}_1^7 & \mathbf{Q}_1^8 & \mathbf{Q}_1^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_1^1 & \mathbf{Q}_1^2 & \mathbf{Q}_1^3 \\
\mathbf{Q}_1^6 & 0 & \mathbf{Q}_1^4 & \mathbf{Q}_1^5 & \mathbf{Q}_1^9 & 0 & \mathbf{Q}_1^7 & \mathbf{Q}_1^8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_1^3 & 0 & \mathbf{Q}_1^1 & \mathbf{Q}_1^2 \\
\mathbf{Q}_2^2 & \mathbf{Q}_2^3 & 0 & \mathbf{Q}_2^1 & \mathbf{Q}_2^5 & \mathbf{Q}_2^6 & 0 & \mathbf{Q}_2^4 & \mathbf{Q}_2^8 & \mathbf{Q}_2^9 & 0 & \mathbf{Q}_2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{Q}_2^1 & \mathbf{Q}_2^2 & \mathbf{Q}_3^2 & 0 & \mathbf{Q}_2^4 & \mathbf{Q}_2^5 & \mathbf{Q}_2^6 & 0 & \mathbf{Q}_2^7 & \mathbf{Q}_2^8 & \mathbf{Q}_2^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{Q}_2^1 & \mathbf{Q}_2^2 & \mathbf{Q}_3^2 & 0 & \mathbf{Q}_2^4 & \mathbf{Q}_2^5 & \mathbf{Q}_2^6 & 0 & \mathbf{Q}_2^7 & \mathbf{Q}_2^8 & \mathbf{Q}_2^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{Q}_2^3 & 0 & \mathbf{Q}_2^1 & \mathbf{Q}_2^2 & \mathbf{Q}_2^6 & 0 & \mathbf{Q}_2^4 & \mathbf{Q}_2^5 & \mathbf{Q}_2^9 & 0 & \mathbf{Q}_2^7 & \mathbf{Q}_2^8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{Q}_3^2 & \mathbf{Q}_3^3 & 0 & \mathbf{Q}_3^1 & \mathbf{Q}_3^5 & \mathbf{Q}_3^6 & 0 & \mathbf{Q}_3^4 & \mathbf{Q}_3^8 & \mathbf{Q}_3^9 & 0 & \mathbf{Q}_3^7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{Q}_3^1 & \mathbf{Q}_3^2 & \mathbf{Q}_3^3 & 0 & \mathbf{Q}_3^4 & \mathbf{Q}_3^5 & \mathbf{Q}_3^6 & 0 & \mathbf{Q}_3^7 & \mathbf{Q}_3^8 & \mathbf{Q}_3^9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{Q}_3^1 & \mathbf{Q}_3^2 & \mathbf{Q}_3^3 & 0 & \mathbf{Q}_3^4 & \mathbf{Q}_3^5 & \mathbf{Q}_3^6 & 0 & \mathbf{Q}_3^7 & \mathbf{Q}_3^8 & \mathbf{Q}_3^9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{Q}_3^3 & 0 & \mathbf{Q}_3^1 & \mathbf{Q}_3^2 & \mathbf{Q}_3^6 & 0 & \mathbf{Q}_3^4 & \mathbf{Q}_3^5 & \mathbf{Q}_3^9 & 0 & \mathbf{Q}_3^7 & \mathbf{Q}_3^8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_4^2 & \mathbf{Q}_4^3 & 0 & \mathbf{Q}_4^1 & \mathbf{Q}_4^5 & \mathbf{Q}_4^6 & 0 & \mathbf{Q}_4^4 & \mathbf{Q}_4^8 & \mathbf{Q}_4^9 & 0 & \mathbf{Q}_4^7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_4^1 & \mathbf{Q}_4^2 & \mathbf{Q}_4^3 & 0 & \mathbf{Q}_4^4 & \mathbf{Q}_4^5 & \mathbf{Q}_4^6 & 0 & \mathbf{Q}_4^7 & \mathbf{Q}_4^8 & \mathbf{Q}_4^9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_4^1 & \mathbf{Q}_4^2 & \mathbf{Q}_3^4 & 0 & \mathbf{Q}_4^4 & \mathbf{Q}_4^5 & \mathbf{Q}_4^6 & 0 & \mathbf{Q}_4^7 & \mathbf{Q}_4^8 & \mathbf{Q}_4^9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_4^3 & 0 & \mathbf{Q}_4^1 & \mathbf{Q}_4^2 & \mathbf{Q}_4^6 & 0 & \mathbf{Q}_4^4 & \mathbf{Q}_4^5 & \mathbf{Q}_4^9 & 0 & \mathbf{Q}_4^7 & \mathbf{Q}_4^8 \\
\mathbf{Q}_5^8 & \mathbf{Q}_5^9 & 0 & \mathbf{Q}_5^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_5^2 & \mathbf{Q}_5^3 & 0 & \mathbf{Q}_5^1 & \mathbf{Q}_5^5 & \mathbf{Q}_5^6 & 0 & \mathbf{Q}_5^4 \\
\mathbf{Q}_5^7 & \mathbf{Q}_5^8 & \mathbf{Q}_5^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_5^1 & \mathbf{Q}_5^2 & \mathbf{Q}_3^5 & 0 & \mathbf{Q}_5^4 & \mathbf{Q}_5^5 & \mathbf{Q}_5^6 & 0 \\
0 & \mathbf{Q}_5^7 & \mathbf{Q}_5^8 & \mathbf{Q}_5^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_5^1 & \mathbf{Q}_5^2 & \mathbf{Q}_3^5 & 0 & \mathbf{Q}_5^4 & \mathbf{Q}_5^5 & \mathbf{Q}_5^6 \\
\mathbf{Q}_5^9 & 0 & \mathbf{Q}_5^7 & \mathbf{Q}_5^8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}_5^3 & 0 & \mathbf{Q}_5^1 & \mathbf{Q}_5^2 & \mathbf{Q}_5^6 & 0 & \mathbf{Q}_5^4 & \mathbf{Q}_5^5
\end{array}\right)
$$

Figure 7.1: Morphology of the dynamics for the discretized system of o shallow–water equations, with the Richtmyer version of the Lax–Wendroff finite difference scheme on a 4×5 resolution grid.

From the expression (7.61) we see that the state vector at a grid point is determined by a combination of the values at 8 adjacent grid points and the value at the same point in the previous step. The matrices $\mathbf{Q}$ are then blocks within the dynamics matrix $\mathbf{\Psi}$ which, for a 4×5 grid has the form displayed in Fig. 7.1 .

Exercises

1. (Daley [39], Problem 6.1) Show that the eigenvectors of the operator $\hat{\mathbf{L}}$ in Section 6.5, given by the columns of the matrix $\hat{\mathbf{V}}$, are orthogonal. Show also that the matrix $\hat{\mathbf{V}}$ is unitary, that is, it satisfies $\hat{\mathbf{V}}^T = \hat{\mathbf{V}}^{-1}$.

2. Derive the shallow–water systems of equations. Starting from Newton's equations (7.1) without external forcing, that is, for $\mathbf{f} = \mathbf{0}$, and considering a cartesian coordinate system, show that the explicit form of (7.1) is

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - g
\end{aligned}
$$

where x, y and z indicate the zonal, meridional, and vertical directions, respectively, $\mathbf{v} = (u, v, w)^T$, and $\mathbf{g} = g\mathbf{k}$ for $\mathbf{k}$ representing a unit vector in the vertical direction.

   (a) Assuming hydrostatic balance, and a homogeneous atmosphere, in which $\bar{p} = p_0 = const.$, $\bar{\rho} = \rho_0 = const.$, and $\bar{z} = h(x, y, t)$, show that the horizontal pressure gradient is independent of the vertical coordinate z.

   (b) Performing a scale analysis in the equations for u and v above, show that these can be reduced to

$$
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}
$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu \quad = \quad -g\frac{\partial h}{\partial y} \tag{7.62}$$

(c) Considering now the equation for the vertical velocity w, and remembering that we are assuming hydrostatic balance, show by scale analysis considerations that this equation can be reduced to

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = 0$$

(d) Noticing that, due to the results of item (b), the horizontal quantities do not depend on z, integrate the equation for w, obtained in the previous item, imposing the following boundary conditions:

$$\begin{aligned} w(x,y,z,t) = 0 &\qquad \text{na superficie, onde } z = h_t(x,y) \\ w(x,y,z,t) = 0 &\quad \text{no topo da atmosfera, onde } z = h(x,y,t) \end{aligned}$$

Hence, show that the vertical equation reduces to

$$\frac{\partial h}{\partial t} + \frac{\partial [u(h-h_t)]}{\partial x} + \frac{\partial [v(h-h_t)]}{\partial y} = 0$$

for the height of the atmosphere.

3. Assuming the absence of topography, show that the shallow–water system of equations obtained in the previous problem, linearized about the following basic state:

$$\begin{aligned} u &= U(y) + u' \\ v &= 0 + v' \\ h &= H(y) + h' \end{aligned}$$

and with $f = f_0 + \beta y$, reduces to:

$$\begin{aligned} \frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \frac{\partial \phi'}{\partial x} - (f - U_y)v' &= 0 \\ \frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + \frac{\partial \phi'}{\partial y} + fu' &= 0 \\ \frac{\partial \phi'}{\partial t} + + U\frac{\partial \phi'}{\partial x} + \Phi(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}) + \Phi_y v' &= 0 \end{aligned}$$

where we introduced the geopotential height for the basic state as $\Phi = gH$, its corresponding perturbation as $\phi' = gh'$ and

$$fU + \Phi_y = 0$$

Here, the subscript y indicates derivation with respect to the variable y.

4. Defining the total energy of the system governed by the linear shallow–water equations, obtained in the previous problem, as

$$E = \frac{1}{2}\int\int \Phi(u^2 + v^2) + \phi^2]\, dx\, dy$$

where u, v and ϕ refer to perturbation fields, show that

$$\frac{dE}{dt} = -\int\int \Phi U_y uv \, dx \, dy$$

where the integrals extend through the whole (x, y) plane. Interpret the case $U(y) = U_0 = const.$.

5. (Cohn [30], Ghil et al. [66]) Let us apply the finite difference scheme of Section 6.6 to the one dimensional shallow–water system of equations:

$$\begin{aligned} \frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \frac{\partial \phi'}{\partial x} - fv' &= 0 \\ \frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + fu' &= 0 \\ \frac{\partial \phi'}{\partial t} + U\frac{\partial \phi'}{\partial x} + \Phi_0\frac{\partial u'}{\partial x} - f_0 U v' &= 0 \end{aligned}$$

where U, f_0, e Φ_0 are constants. In this case:

(a) Write the system's equations in flux form, that is,

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{w}}{\partial x} + \mathbf{C}\mathbf{w} = 0$$

obtaining explicit expressions for $\mathbf{A}$ and $\mathbf{C}$.

(b) Show that the second step of the Lax–Wendroff scheme can be written as:

$$\mathbf{w}_i^{k+1} = \mathbf{w}_i^k - \lambda\mathbf{A}(\mathbf{w}_{i+1/2}^{k+1/2} - \mathbf{w}_{i-1/2}^{k+1/2}) - \frac{\Delta t}{2}\mathbf{C}(\mathbf{w}_{i-1/2}^{k+1/2} + \mathbf{w}_{i+1/2}^{k+1/2})$$

for $i = 1, 2, \cdots, I$, and $\lambda = \Delta t/\Delta x$.

(c) Show that the first step of the Lax–Wendroff scheme can be written as:

$$\mathbf{w}_{i+1/2}^{k+1/2} = \frac{1}{2}(\mathbf{I} - \frac{\Delta t}{2}\mathbf{C})(\mathbf{w}_i^k + \mathbf{w}_{i+1}^k) - \frac{\lambda}{2}\mathbf{A}(\mathbf{w}_{i+1}^k - \mathbf{w}_i^k)$$

for $i = 1, 2, \cdots, I$.

(d) Substituting this result into item (b), as well as the result obtained via the transformation $i \to i-1$ in item (c), show that

$$\mathbf{w}_i^{k+1} = \mathbf{Q}_{-1}\mathbf{w}_{i-1}^{k+1} + \mathbf{Q}_0\mathbf{w}_i^{k+1} + \mathbf{Q}_{+1}\mathbf{w}_{i+1}^{k+1}$$

where

$$\mathbf{Q}_0 = \mathbf{I} - \lambda^2\mathbf{A}^2 - \frac{\Delta t}{2}\mathbf{C}(\mathbf{I} - \frac{\Delta t}{2}\mathbf{C})$$

and

$$\mathbf{Q}_{\pm 1} = \mp\frac{\lambda}{2}\mathbf{A} + \frac{\lambda^2}{2}\mathbf{A}^2 \pm \frac{\lambda\Delta t}{4}(\mathbf{AC} + \mathbf{CA}) - \frac{\Delta t}{4}\mathbf{C}(\mathbf{I} - \frac{\Delta t}{2}\mathbf{C})$$

(e) Indicate the morphology of the one-time step transition matrix.