Chapter 5

The Linear Kalman Filter

In this lecture we derive and study the Kalman filter and its properties for the case of time–discrete dynamics and time–discrete observations. The case of time–continuous dynamics with time–continuous observations is mentioned without many details, and the case of time–continuous dynamics with time–discrete observations is not considered is this course. The content of this lecture can be found in classic books of stochastic processes and estimation theory, such as, Anderson & Moore [1], Gelb [60], Jazwinski [84], Meditch [103], and Sage & Melsa [121]. In this lecture, we also introduce a convenient notation to treat the assimilation problem of meteorological and oceanographic data, to be discussed in lectures that follow.

5.1 Derivation of the Linear Kalman Filter

5.1.1 Estimation Problem in Linear Systems

We derive the Kalman filter using the estimation approach of minimum variance, following the derivation of Todling & Cohn [129], which deals with the problem of atmospheric data assimilation to be studied later.

Consider a time–discrete, linear stochastic dynamical system written in matrix–vector notation as

$$\mathbf{w}_k^t = \mathbf{\Psi}_{k-1}\mathbf{w}_{k-1}^t + \mathbf{b}_{k-1}^t \,, \tag{5.1}$$

for the discrete times t_k, with $k = 1, 2, \ldots$, and where $\mathbf{w}_k^t$ is an n–vector representing the *true* state of the system at time t_k, $\mathbf{\Psi}_k$ is an $n \times n$ matrix that represents the dynamics, and the n–vector $\mathbf{b}_k^t$ is an additive random noise, which we refer to as the model error. The process $\mathbf{b}_k^t$ is assumed to be white in time, with mean zero and (co)variance $\mathbf{Q}_k$, that is,

$$\mathcal{E}\{\mathbf{b}_k^t\} = \mathbf{0} \,, \;\; \mathcal{E}\{\mathbf{b}_k^t (\mathbf{b}_{k'}^t)^T\} = \mathbf{Q}_k \delta_{kk'} \,. \tag{5.2}$$

Consider also a linear observation process described by

$$\mathbf{w}_k^o = \mathbf{H}_k \mathbf{w}_k^t + \mathbf{b}_k^o \,, \tag{5.3}$$

where k now is a multiple of ℓ, the number of time steps of between two consecutive observations in time. The m_k–vector $\mathbf{w}_k^o$ is the vector of observations, the matrix $m_k \times n$ represents a linear transformation between the true variables into the observed ones, and the m_k–vector $\mathbf{b}_k^o$ is an additive noise, representing error in the observational process, as for example, error due to instrument accuracy. We assume that the random noise $\mathbf{v}_k$ is white in time, with mean zero and (co)variance $\mathbf{R}_k$, that is,

$$\mathcal{E}\{\mathbf{b}_k^o\} = \mathbf{0}\,, \;\; \mathcal{E}\{\mathbf{b}_k^o)(\mathbf{b}_{k'}^o)^T\} = \mathbf{R}_k \delta_{kk'}\,, \tag{5.4}$$

We also assume that the observation noise $\mathbf{v}_k$ and the model error are uncorrelated, that is,

$$\mathcal{E}\{\mathbf{b}_k^t(\mathbf{b}_{k'}^o)^T\} = \mathbf{0}\,. \tag{5.5}$$

The problem treated in the previous lecture was that of estimating $\mathbf{w}_k^t$ given the observation process (5.3) alone. In this lecture, we add to the estimation problem the constraint that the variable of interest comes from the linear stochastic dynamical system (5.1). However, since the dynamical system in (5.1) involves the stochastic noise $\mathbf{b}_k^t$ and an unknown initial state, we replace that model by what we refer to as a forecast model that we write as

$$\mathbf{w}_k^f = \mathbf{\Psi}_{k,k-\ell}\mathbf{w}_{k-\ell}^a\,, \tag{5.6}$$

where the symbol f stands for forecast and the symbol a stands for the "initial" condition at time $t_{k-\ell}$, from which we start a forecast, and referred to as the analysis. The forecast model represents another way we have of estimating the state of the system at a particular time. The matrix $\mathbf{\Psi}_{k,k-\ell}$ is the propagator, or transition matrix, between times $t_{k-\ell}$ and t_k, and is given by

$$\mathbf{\Psi}_{k,k-\ell} \equiv \mathbf{\Psi}_{k-1}\mathbf{\Psi}_{k-2}\cdots\mathbf{\Psi}_{k-\ell}\,, \tag{5.7}$$

where here we make a distinction between the propagator and the one–time step dynamics through the double subscripts to indicate the propagator.

An estimate of the state of the system at time t_k can be obtained by means of a linear combination between the observation at time t_k and the forecast at the same time. Therefore, we can write for the estimate $\mathbf{w}_k^a$ at time t_k,

$$\mathbf{w}_k^a = \tilde{\mathbf{L}}_k\mathbf{w}_k^f + \tilde{\mathbf{K}}_k\mathbf{w}_k^o\,, \tag{5.8}$$

where $\tilde{\mathbf{L}}_k$ and $\tilde{\mathbf{K}}_k$ are weighting matrices still to be determined.

Let us define the forecast and (estimate) analysis errors as

$$\mathbf{e}_k^f \equiv \mathbf{w}_k^f - \mathbf{w}_k^t\,, \tag{5.9a}$$

$$\mathbf{e}_k^a \equiv \mathbf{w}_k^a - \mathbf{w}_k^t\,. \tag{5.9b}$$

In analogy to what we saw in Lecture 4, we would like to have an estimate that is unbiased. In this way, subtracting $\mathbf{w}_k^t$ from both sides of (5.8), as well as from $\mathbf{w}_k^f$ in that expression, and using (5.3) it follows that

$$\mathbf{e}_k^a = \tilde{\mathbf{L}}_k\mathbf{e}_k^f + \tilde{\mathbf{K}}_k\mathbf{b}_k^o + (\tilde{\mathbf{L}}_k + \tilde{\mathbf{K}}_k\mathbf{H}_k - \mathbf{I})\mathbf{w}_k^t \tag{5.10}$$

Now assuming that the forecast error, at time t_k, is unbiased, that is, $\mathcal{E}\{\mathbf{e}_k^f\} = \mathbf{0}$, we should satisfy

$$(\tilde{\mathbf{L}}_k + \tilde{\mathbf{K}}_k\mathbf{H}_k - \mathbf{I})\mathcal{E}\{\mathbf{w}_k^t\} = \mathbf{0} \tag{5.11}$$

to obtain an unbiased estimate (analysis), i.e., $\mathcal{E}\{\mathbf{e}_k^a\} = \mathbf{0}$. As in general $\mathcal{E}\{\mathbf{w}_k^t\} \neq \mathbf{0}$, we have that

$$\tilde{\mathbf{L}}_k = \mathbf{I} - \tilde{\mathbf{K}}_k\mathbf{H}_k \tag{5.12}$$

is the condition for an unbiased $\mathbf{w}_k^a$.

Substituting result (5.12) in (5.8) we can write for the estimate of the state of the system

$$\mathbf{w}_k^a = \mathbf{w}_k^f + \tilde{\mathbf{K}}_k(\mathbf{w}_k^o - \mathbf{H}_k\mathbf{w}_k^f)\,, \tag{5.13}$$

and for the estimate error

$$\mathbf{e}_k^a = \left(\mathbf{I} - \tilde{\mathbf{K}}_k\mathbf{H}_k\right)\mathbf{e}_k^f + \tilde{\mathbf{K}}_k\mathbf{b}_k^o\,. \tag{5.14}$$

The weight matrix $\tilde{\mathbf{K}}_k$, or gain matrix as it is more commonly known, represents the weights given to the difference between the observation vector and the forecast transformed by the observation matrix $\mathbf{H}_k$. We have seen in Lecture 4, that different procedures come up with a formula for the estimate that resembles (5.13), however they use distinct gain matrices, e.g., recall the comparison between minimum variance estimation and least squares estimation.

Using (5.1) and (5.6) it follows that

$$\mathbf{e}_k^f = \mathbf{\Psi}_{k,k-l}\mathbf{e}_{k-l}^a - \sum_{j=0}^{l-1}\mathbf{\Psi}_{k,k-j}\mathbf{b}_{k-j-1}^t\,, \tag{5.15}$$

which is an equation for the evolution of forecast error.

Introducing the forecast and analysis error covariance matrices

$$\mathbf{P}_k^f \equiv \mathcal{E}\{(\mathbf{w}_k^t - \mathbf{w}_k^f)(\mathbf{w}_k^t - \mathbf{w}_k^f)^T\} \tag{5.16a}$$

$$\mathbf{P}_k^a \equiv \mathcal{E}\{(\mathbf{w}_k^t - \mathbf{w}_k^a)(\mathbf{w}_k^t - \mathbf{w}_k^a)^T\}\,, \tag{5.16b}$$

we can proceed as in Section 3.2.2 to obtain an expression for the evolution of the forecast error covariance:

$$\mathbf{P}_k^f = \mathbf{\Psi}_{k,k-l}\mathbf{P}_{k-l}^a\mathbf{\Psi}_{k,k-l}^T + \sum_{j=0}^{l-1}\mathbf{\Psi}_{k,k-j}\mathbf{Q}_{k-j-1}\mathbf{\Psi}_{k,k-j}^T\,, \tag{5.17}$$

which is a form equivalent (iterated) to the discrete Lyapunov equation (3.40).

An expression for the (estimated) analysis error covariance $\mathbf{P}_k^a$ can be determined by multiplying (5.14) by its transpose and applying the ensemble average operator to the resulting expression. Therefore, we have

$$\mathbf{P}_k^a = (\mathbf{I} - \tilde{\mathbf{K}}_k\mathbf{H}_k)\mathbf{P}_k^f(\mathbf{I} - \tilde{\mathbf{K}}_k\mathbf{H}_k)^T + \tilde{\mathbf{K}}_k\mathbf{R}_k\tilde{\mathbf{K}}_k^T\,, \tag{5.18}$$

which is referred to as Joseph's formula. Equations (5.17) and (5.18) completely describe the evolution of errors in the forecast and analysis. An interesting property of the equations for

the error (co)variances is that they are independent of the estimates (analysis and forecast), and also from the observations. The only necessary quantities to predict the evolution of the error (co)variances are the noise (co)variance matrices, $\mathbf{Q}_k$ and $\mathbf{R}_k$, the initial error (co)variance matrix $\mathbf{P}_0^a$, and matrices $\mathbf{H}_k$ and $\tilde{\mathbf{K}}_k$, at each time t_k. In principle, all these matrices are known, except for the gain matrix $\tilde{\mathbf{K}}_k$ which is to be determined by means of an optimization procedure that requires minimum error variance.

5.1.2 The Kalman Filter

To treat the problem stated above in the lights of minimum variance estimation we introduce an estimator that serves as a measure of reliability of the analysis. That is, a quantity measuring the distance between the estimate and the true value of the state of the system at time t_k,

$$\begin{aligned} \mathcal{J}_k^a &\equiv \mathcal{E}\{||\mathbf{e}_k^a||^2_{\mathbf{E}_k}\} \\ &= \mathcal{E}\left\{ (\mathbf{e}_k^a)^T \mathbf{E}_k \, \mathbf{e}_k^a \right\} \\ &= \mathcal{E}\left\{ \mathrm{Tr}\left[\mathbf{E}_k \, \mathbf{e}_k^a (\mathbf{e}_k^a)^T \right] \right\} \\ &= \mathrm{Tr}\,(\mathbf{E}_k \mathbf{P}_k^a) \ . \end{aligned} \tag{5.19}$$

As in Lecture 4, we want this measure of error to be minimum with respect to the elements of the gain matrix $\tilde{\mathbf{K}}_k$. The matrix $n \times n$ matrix $\mathbf{E}_k$ introduced in the functional above is a scaling matrix, which we assume to be positive definite and deterministic, which in many cases can be substituted by the identity matrix. As we will see below, the solution of the minimization $\mathcal{J}_k^a$ is in fact independent of $\mathbf{E}_k$.

Substituting the expression (5.18) for $\mathbf{P}_k^a$ in (5.19), differentiating with respect to $\tilde{\mathbf{K}}_k$, (using the differentiation rules of Exercise 4.4), and equating the result to zero we obtain

$$\mathbf{E}_k \left\{ \mathbf{H}_k \mathbf{P}_k^f (\mathbf{I} - \tilde{\mathbf{K}}_k \mathbf{H}_k)^T \ - \ \mathbf{R}_k \tilde{\mathbf{K}}_k^T \right\} = \mathbf{0} \tag{5.20}$$

Therefore, independently of $\mathbf{E}_k$, the quantity between curly brackets becomes zero for

$$\tilde{\mathbf{K}}_k = \mathbf{K}_k \equiv \mathbf{P}_k^f \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k^f \mathbf{H}_k^T + \mathbf{R}_k)^{-1} , \tag{5.21}$$

which corresponds to the minimum of $\mathcal{J}_k^a$. The matrix $\mathbf{K}_k$ is the optimal weighting matrix, known as the Kalman gain matrix, since this estimation problem was solved by Kalman [87]. Although estimation problems date back from the times of Gauss [57], it was Kalman who solved the problem in the dynamical systems context, using the state–space approach. As a matter of fact, Kalman derived the result obtained above in a much more elegant way based on the orthogonal projections theorem. The solution obtained by Kalman has practical consequences that go much beyond previous results in estimation theory. Kalman & Bucy [90] extended the Kalman filter to the case of time–continuous dynamics and observation process. An excellent review of filtering theory can be found in Kailath [86], and the influence of Kalman's work in several theoretical and applied areas is collected in Antoulas [3].

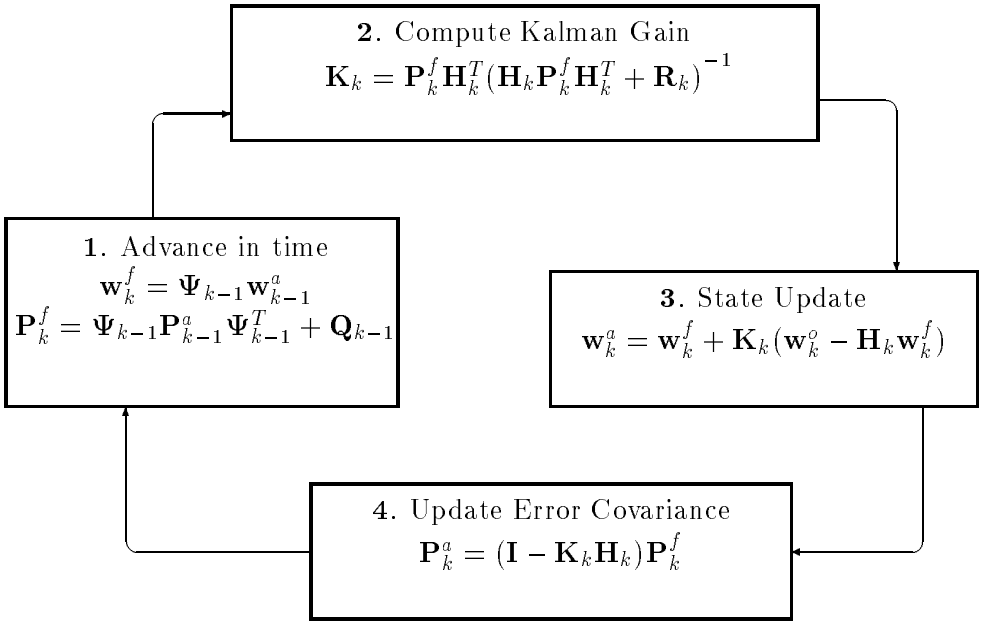

Figure 5.1: Schematic diagram of the linear Kalman filter.

Substituting the Kalman gain matrix in the expression for the analysis error covariance (5.18), it is simple to show that this equation reduces to

$$\mathbf{P}_k^a = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_k^f \,, \tag{5.22}$$

which is a simpler expression. The optimal estimate of the state of the system at time t_k is given by (5.13) with a general gain matrix $\tilde{\mathbf{K}}_k$ replaced by its optimal value, that is,

$$\mathbf{w}_k^a = \mathbf{w}_k^f + \mathbf{K}_k (\mathbf{w}_k^o - \mathbf{H}_k \mathbf{w}_k^f) \,. \tag{5.23}$$

Fig. 5.1 shows schematically the steps involved in the execution of the linear Kalman filter for the case $\ell = 1$, that is, when the observations are available at each time step. The case $\ell = 1$ will be considered from this point on to keep the notation simple.

5.1.3 Comments: Minimum Variance and Conditional Mean

We saw in the previous lecture that in Bayes estimation theory the estimate of minimum variance is given by the conditional mean. Let us now establish the connection between the derivation of the Kalman filter given above and the example discussed in the previous lecture, of estimation of a constant vector. What we will see is that, that the example corresponds to the analysis step of the Kalman filter.

Let us indicate by $\mathbf{W}_k^o = \{\mathbf{w}_k^o, \mathbf{w}_{k-1}^o, \cdots, \mathbf{w}_1^o\}$, the set of all observations (5.3) up and including time t_k. Similarly to the case in the previous lecture, the problem of estimating the state of the system at time t_k, based on the observations $\mathbf{W}_k^o$ can be placed as the problem of determining the conditional probability density $p(\mathbf{w}_k^t|\mathbf{W}_k^o)$, where, to simplify notation we omit the subscript in p referring to the stochastic process in question. By the result of Section 4.2.1, we know that

$$\begin{aligned} \mathbf{w}_k^a &= \mathcal{E}\{\mathbf{w}_k^t|\mathbf{W}_k^o\} \\ &= \int_{-\infty}^{+\infty} \mathbf{w}_k^t p(\mathbf{w}_k^t|\mathbf{W}_k^o)\, d\mathbf{w}_k^t \end{aligned} \tag{5.24}$$

and therefore, knowledge of $p(\mathbf{w}_k^t|\mathbf{W}_k^o)$ is fundamental to determine the estimate.

In fact, using repeatedly the definition of conditional probability density we can write

$$\begin{aligned} p(\mathbf{w}_k^t|\mathbf{W}_k^o) &= p(\mathbf{w}_k^t|\mathbf{w}_k^o, \mathbf{W}_{k-1}^o) \\ &= \frac{p(\mathbf{w}_k^t, \mathbf{w}_k^o, \mathbf{W}_{k-1}^o)}{p(\mathbf{w}_k^o, \mathbf{W}_{k-1}^o)} \\ &= \frac{p(\mathbf{w}_k^o|\mathbf{w}_k^t, \mathbf{W}_{k-1}^o)p(\mathbf{w}_k^t, \mathbf{W}_{k-1}^o)}{p(\mathbf{w}_k^o, \mathbf{W}_{k-1}^o)} \\ &= \frac{p(\mathbf{w}_k^o|\mathbf{w}_k^t, \mathbf{W}_{k-1}^o)p(\mathbf{w}_k^t|\mathbf{W}_{k-1}^o)p(\mathbf{W}_{k-1}^o)}{p(\mathbf{w}_k^o|\mathbf{W}_{k-1}^o)p(\mathbf{W}_{k-1}^o)} \\ &= \frac{p(\mathbf{w}_k^o|\mathbf{w}_k^t, \mathbf{W}_{k-1}^o)p(\mathbf{w}_k^t|\mathbf{W}_{k-1}^o)}{p(\mathbf{w}_k^o|\mathbf{W}_{k-1}^o)}, \end{aligned} \tag{5.25}$$

which related the transition probability of interest with transition probabilities that can be calculated more promptly.

Since the sequence of observational noise $\{\mathbf{b}_k^o\}$ is white, the following simplification applies:

$$p(\mathbf{w}_k^o|\mathbf{w}_k^t, \mathbf{W}_{k-1}^o) = p(\mathbf{w}_k^o|\mathbf{w}_k^t) \tag{5.26}$$

and therefore,

$$p(\mathbf{w}_k^t|\mathbf{W}_k^o) = \frac{p(\mathbf{w}_k^o|\mathbf{w}_k^t)p(\mathbf{w}_k^t|\mathbf{W}_{k-1}^o)}{p(\mathbf{w}_k^o|\mathbf{W}_{k-1}^o)} \tag{5.27}$$

It remains for us to determine each one of the transition probability densities in this expression.

Assuming the probability distributions of $\mathbf{w}_0^t$, $\mathbf{b}_k^t$ and $\mathbf{v}_k$ are Gaussian, we can draw a straight relationship among the variables here and those in Section 4.3. Specifically, we can identify $\mathbf{z}$ with $\mathbf{w}_k^o$ and $\mathbf{w}$ with $\mathbf{w}_k^t$, therefore, the probability densities $p_{\mathbf{z}}(\mathbf{z})$ and $p_{\mathbf{z}|\mathbf{w}}(\mathbf{z}|\mathbf{w})$ can be identified with the probability densities $p(\mathbf{w}_k^o)$ and $p(\mathbf{w}_k^o|\mathbf{w}_k^t)$, respectively. Consequently, we can write for $p(\mathbf{w}_k^o|\mathbf{w}_k^t)$,

$$p(\mathbf{w}_k^o|\mathbf{w}_k^t) = \frac{1}{(2\pi)^{m_k/2}|\mathbf{R}_k|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{w}_k^o - \mathbf{H}_k\mathbf{w}_k^t)^T\mathbf{R}_k^{-1}(\mathbf{w}_k^o - \mathbf{H}_k\mathbf{w}_k^t)\right] \tag{5.28}$$

where we notice that

$$\mathcal{E}\{\mathbf{w}_k^o|\mathbf{w}_k^t\} = \mathcal{E}\{(\mathbf{H}_k\mathbf{w}_k^t + \mathbf{b}_k^o)|\mathbf{w}_k^t\} = \mathbf{H}_k\mathbf{w}_k^t \tag{5.29}$$

and

$$\begin{aligned} cov\{\mathbf{w}_k^o, \mathbf{w}_k^o|\mathbf{w}_k^t\} &\equiv \mathcal{E}\{[\mathbf{w}_k^o - \mathcal{E}\{\mathbf{w}_k^o|\mathbf{w}_k^t\}][\mathbf{w}_k^o - \mathcal{E}\{\mathbf{w}_k^o|\mathbf{w}_k^t\}]^T|\mathbf{w}_k^t\} \\ &= \mathbf{R}_k . \end{aligned} \tag{5.30}$$

Analogously, we have

$$p(\mathbf{w}_k^o|\mathbf{W}_{k-1}^o) = \frac{1}{(2\pi)^{m_k/2}|\mathbf{\Gamma}_k|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{w}_k^o - \mathbf{H}_k\mathbf{w}_k^f)^T\mathbf{\Gamma}_k^{-1}(\mathbf{w}_k^o - \mathbf{H}_k\mathbf{w}_k^f)\right] \tag{5.31}$$

where we define $\mathbf{w}_k^f$ as

$$\mathbf{w}_k^f \equiv \mathcal{E}\{\mathbf{w}_k^t|\mathbf{W}_{k-1}^o\}, \tag{5.32}$$

the matrix $m_k \times m_k$ matrix $\mathbf{\Gamma}_k$ as

$$\mathbf{\Gamma}_k \equiv \mathbf{H}_k\mathbf{P}_k^f\mathbf{H}_k^T + \mathbf{R}_k , \tag{5.33}$$

and the $n \times n$ matrix $\mathbf{P}_k^f$ as

$$\begin{aligned} \mathbf{P}_k^f &\equiv \mathcal{E}\{[\mathbf{w}_k^t - \mathcal{E}\{\mathbf{w}_k^t|\mathbf{W}_{k-1}^o\}][\mathbf{w}_k^t - \mathcal{E}\{\mathbf{w}_k^t|\mathbf{W}_{k-1}^o\}]^T|\mathbf{W}_{k-1}^o\} \\ &= \mathcal{E}\{[\mathbf{w}_k^t - \mathbf{w}_k^f][\mathbf{w}_k^t - \mathbf{w}_k^f]^T|\mathbf{W}_{k-1}^o\} . \end{aligned} \tag{5.34}$$

To fully determine the *a posteriori* conditional probability density $p(\mathbf{w}_k^t|\mathbf{W}_k^o)$, it remains for us to find the *a priori* conditional probability density $p(\mathbf{w}_k^t|\mathbf{W}_{k-1}^o)$. Since we are assuming that $\mathbf{w}_0^t$ and $\mathbf{b}_k^t$ are Gaussian distributed, $p(\mathbf{w}_{k-1}^t|\mathbf{W}_{k-1}^o)$ is Gaussian, and it follows from the linearity of (5.1) that $p(\mathbf{w}_k^t|\mathbf{W}_{k-1}^o)$ is also Gaussian. Therefore, all that remains for us to determine are the mean $\mathcal{E}\{\mathbf{w}_k^t|\mathbf{W}_{k-1}^o\}$ and the (co)variance $cov\{\mathbf{w}_k^t, \mathbf{w}_k^t|\mathbf{W}_{k-1}^o\}$.

From the definition (5.32) of $\mathbf{w}_k^f$ we have

$$\begin{aligned} \mathbf{w}_k^f &= \mathcal{E}\{\mathbf{w}_k^t|\mathbf{W}_{k-1}^o\} \\ &= \mathbf{\Psi}_{k-1}\mathcal{E}\{\mathbf{w}_{k-1}^t|\mathbf{W}_{k-1}^o\} + \mathcal{E}\{\mathbf{b}_{k-1}^t|\mathbf{W}_{k-1}^o\} \\ &= \mathbf{\Psi}_{k-1}\mathcal{E}\{\mathbf{w}_{k-1}^t|\mathbf{W}_{k-1}^o\} + \mathcal{E}\{\mathbf{b}_{k-1}^t\} \\ &= \mathbf{\Psi}_{k-1}\mathbf{w}_{k-1}^a \end{aligned} \tag{5.35}$$

where the last equality is obtained by observing that $\mathbf{b}_{k-1}^t$ has mean zero, and by using the definition of the estimate $\mathbf{w}_{k-1}^a$, as the conditional mean at time t_{k-1}. This expression represents the time evolution of the estimate, and it justifies the somewhat *ad hoc* forecast model that appeared in (5.6). The expression above is also identical to that found in (3.33) for the evolution of the mean.

The expression for the (co)variance matrix $cov\{\mathbf{w}_k^t, \mathbf{w}_k^t|\mathbf{W}_{k-1}^o\}$ can be easily shown to be

$$\begin{aligned} cov\{\mathbf{w}_k^t, \mathbf{w}_k^t|\mathbf{W}_{k-1}^o\} &= \mathbf{\Psi}_{k-1}\mathbf{P}_{k-1}^a\mathbf{\Psi}_{k-1}^T + \mathbf{Q}_{k-1} \\ &= \mathbf{P}_k^f, \end{aligned} \tag{5.36}$$

where we recall that to simplify notation we are assuming that observations are available at all times, that is, the expression above corresponds to that in (5.17) with $\ell = 1$. Furthermore, (5.36) is identical to the time–discrete Lyapunov equation (3.40).

From the result (5.36) and the definition (5.32), we can write

$$p(\mathbf{w}_k^t|\mathbf{W}_{k-1}^o) \;=\; \frac{1}{(2\pi)^{n/2}|\mathbf{P}_k^f|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{w}_k^t - \mathbf{w}_k^f)^T(\mathbf{P}_k^f)^{-1}(\mathbf{w}_k^t - \mathbf{w}_k^f)\right] \tag{5.37}$$

so that, proceeding as in Section 4.3, the conditional probability density (5.27) of interest becomes

$$p(\mathbf{w}_k^t|\mathbf{W}_k^o) \;=\; \frac{1}{(2\pi)^{n/2}|\mathbf{P}_k^a|^{1/2}} \exp\left[-\frac{1}{2}J_k^a\right] \tag{5.38}$$

where J_k^a is cost function defined as

$$J_k^a \;\equiv\; (\mathbf{e}_k^a)^T(\mathbf{P}_k^a)^{-1}\mathbf{e}_k^a \tag{5.39}$$

where $\mathbf{e}_k^a \equiv (\mathbf{w}_k^a - \mathbf{w}_k^t)$ as in (5.9). We can now identify the quantities $\hat{\mathbf{w}}_{\rm MV}$ and $\mathbf{P}_{\tilde{\mathbf{w}}}$ of Section 4.3 with $\mathbf{w}_k^a$ and $\mathbf{P}_k^a$, respectively. Consequently, it follows from this correspondence that

$$(\mathbf{P}_k^a)^{-1} \;=\; (\mathbf{P}_k^f)^{-1} \;+\; \mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k\,. \tag{5.40}$$

Since in Section 4.3 we showed that $\hat{\mathbf{w}}_{\rm MV}$ was the minimum variance estimate (4.51) for the problem dealt in that section, it follows immediately that $\mathbf{w}_k^a$ is the minimum variance estimate of the problem we are studying in this section.

To complete the correspondence between the treatment of this section and that of the previous section, we notice that the most remarkable difference between these two treatments is that the ensemble average operator of the previous section was the *unconditional* ensemble average. On the other hand, in this section, the ensemble average operators are the *conditional* ones, that is, conditioned on the observations. As a matter of fact, during the derivation performed in the previous section we advanced the result obtained in this section that the forecast and analysis error covariance matrices $\mathbf{P}_k^f$ and $\mathbf{P}_k^a$ are in fact independent from the observations, see (5.36) and (5.40), that is,

$$\begin{aligned}
\mathbf{P}_k^f &= \mathcal{E}\{[\mathbf{w}_k^t - \mathcal{E}\{\mathbf{w}_k^t|\mathbf{W}_{k-1}^o\}][\mathbf{w}_k^t - \mathcal{E}\{\mathbf{w}_k^t|\mathbf{W}_{k-1}^o\}]^T|\mathbf{W}_{k-1}^o\} \\
&= \mathcal{E}\{[\mathbf{w}_k^t - \mathbf{w}_k^f][\mathbf{w}_k^t - \mathbf{w}_k^f]^T|\mathbf{W}_{k-1}^o\} \\
&= \mathcal{E}\{[\mathbf{w}_k^t - \mathbf{w}_k^f][\mathbf{w}_k^t - \mathbf{w}_k^f]^T\}\,,
\end{aligned} \tag{5.41}$$

and

$$\begin{aligned}
\mathbf{P}_k^a &= \mathcal{E}\{[\mathbf{w}_k^t - \mathcal{E}\{\mathbf{w}_k^t|\mathbf{W}_k^o\}][\mathbf{w}_k^t - \mathcal{E}\{\mathbf{w}_k^t|\mathbf{W}_k^o\}]^T|\mathbf{W}_k^o\} \\
&= \mathcal{E}\{[\mathbf{w}_k^t - \mathbf{w}_k^a][\mathbf{w}_k^t - \mathbf{w}_k^a]^T|\mathbf{W}_k^o\} \\
&= \mathcal{E}\{[\mathbf{w}_k^t - \mathbf{w}_k^a][\mathbf{w}_k^t - \mathbf{w}_k^a]^T\}\,.
\end{aligned} \tag{5.42}$$

Consequently we can replace the conditional error (co)variances by the unconditional error (co)variances.

Following some remarks in the previous chapter, we see that an equivalent cost function to that in (5.39), associated to the maximum *a posteriori* estimate, is

$$J_{\rm 3dVar}(\mathbf{w}_k^t) \;\equiv\; (\mathbf{w}_k^o - \mathbf{H}_k\mathbf{w}_k^t)^T\mathbf{R}_k^{-1}(\mathbf{w}_k^o - \mathbf{H}_k\mathbf{w}_k^t) \;+\; (\mathbf{w}_k^t - \mathbf{w}_k^f)^T(\mathbf{P}_k^f)^{-1}(\mathbf{w}_k^t - \mathbf{w}_k^f)\,. \tag{5.43}$$

This cost function can also be written in its 3–dimensional variational form (e.g., Courtier [36]), as

$$J_{\rm 3dVar}(\delta \mathbf{w}_k) \equiv (\mathbf{v}_k - \mathbf{H}_k \delta \mathbf{w}_k)^T \mathbf{R}_k^{-1} (\mathbf{v}_k - \mathbf{H}_k \delta \mathbf{w}_k) + \delta \mathbf{w}_k^T (\mathbf{P}_k^f)^{-1} \delta \mathbf{w}_k \tag{5.44}$$

where $\delta \mathbf{w}_k \equiv \mathbf{w}_k^t - \mathbf{w}_k^f = -\mathbf{e}_k^f$, and we notice that

$$\begin{aligned} \mathbf{w}_k^o - \mathbf{H}_k \mathbf{w}_k^t &= \mathbf{w}_k^o - \mathbf{H}_k \mathbf{w}_k^f + \mathbf{H}_k \mathbf{w}_k^f - \mathbf{H}_k \mathbf{w}_k^t \\ &= \mathbf{v}_k - \mathbf{H}_k \delta \mathbf{w}_k \end{aligned} \tag{5.45}$$

where $\mathbf{v}_k$ is the innovation vector, $\mathbf{v}_k \equiv \mathbf{w}_k^o - \mathbf{H}_k \mathbf{w}_k^f$. And from the same discussion presented before, the minimization of (5.44) produces to the same solution as that found from the minimum variance approach.

5.2 Properties of the Kalman Filter

5.2.1 Whiteness of the Innovation Process

The behavior, or more adequately the performance of the Kalman filter is reflected in the statistical properties of the so called innovation sequence, where the innovation vector at time t_k is defined as

$$\mathbf{v}_k \equiv \mathbf{w}_k^o - \mathbf{H}_k \mathbf{w}_k^f \,. \tag{5.46}$$

Adding and subtracting $\mathbf{w}_k^t$ on the right hand side of this expression, and using the equation for the observation process (5.3), we can re–write the innovation vector as

$$\mathbf{v}_k = \mathbf{v}_k - \mathbf{H}_k \mathbf{e}_k^f \tag{5.47}$$

from where it follows that $\mathcal{E}\{\mathbf{v}_k\} = \mathbf{0}$, that is, the innovation sequence has mean zero.

In this section we are interested in investigate the behavior of the cross–, or lagged–innovation covariance matrix, between times t_k and t_{k-j}, defined as

$$\begin{aligned} \mathbf{\Gamma}_{k,k-j} &\equiv \mathcal{E}\{(\mathbf{v}_k - \mathcal{E}\{\mathbf{v}_k\})(\mathbf{v}_{k-j} - \mathcal{E}\{\mathbf{v}_{k-j}\})^T\} \\ &= \mathcal{E}\{\mathbf{v}_k \mathbf{v}_{k-j}^T\} \end{aligned} \tag{5.48}$$

using that the innovation sequence has mean zero. From (5.47) we can write

$$\begin{aligned} \mathbf{\Gamma}_{k,k-j} &= \mathcal{E}\{[\mathbf{v}_k - \mathbf{H}_k \mathbf{e}_k^f][\mathbf{v}_{k-j} - \mathbf{H}_{k-j} \mathbf{e}_{k-j}^f]^T\} \\ &= \mathbf{H}_k \mathcal{E}\{\mathbf{e}_k^f (\mathbf{e}_{k-j}^f)^T\} \mathbf{H}_{k-j}^T + \mathcal{E}\{\mathbf{v}_k (\mathbf{v}_{k-j})^T\} \\ &\quad - \mathbf{H}_k \mathcal{E}\{\mathbf{e}_k^f (\mathbf{v}_{k-j})^T\} - \mathcal{E}\{\mathbf{v}_k (\mathbf{e}_{k-j}^f)^T\} \mathbf{H}_{k-j}^T \end{aligned} \tag{5.49}$$

For the particular case of $j = 0$, the innovation covariance takes the form:

$$\mathbf{\Gamma}_k = \mathbf{H}_k \mathbf{P}_k^f \mathbf{H}_k^T + \mathbf{R}_k \tag{5.50}$$

where we used (5.4) and (5.16).

To investigate the case with $j \geq 1$, it helps to derive a general expression for the forecast error $\mathbf{e}_k^f$. In this regard, let us combine (5.14) and (5.15) to get

$$\mathbf{e}_k^f = \mathbf{\Psi}_{k-1}\left[\mathbf{I} - \tilde{\mathbf{K}}_{k-1}\mathbf{H}_{k-1}\right]\mathbf{e}_{k-1}^f + \mathbf{\Psi}_{k-1}\tilde{\mathbf{K}}_{k-1}\mathbf{b}_{k-1}^o - \mathbf{b}_{k-1}^t \tag{5.51}$$

for any gain matrix $\tilde{\mathbf{K}}_{k-1}$, and reminding the reader that we are considering the case $\ell = 1$. Making the transformation $k \rightarrow k-1$ in the expression above, we have

$$\mathbf{e}_{k-1}^f = \mathbf{\Psi}_{k-2}\left[\mathbf{I} - \tilde{\mathbf{K}}_{k-2}\mathbf{H}_{k-2}\right]\mathbf{e}_{k-2}^f + \mathbf{\Psi}_{k-2}\tilde{\mathbf{K}}_{k-2}\mathbf{b}_{k-2}^o - \mathbf{b}_{k-2}^t \tag{5.52}$$

and substituting this back in (5.51) it follows that

$$\begin{aligned}\mathbf{e}_k^f &= \mathbf{\Psi}_{k-1}\left[\mathbf{I} - \tilde{\mathbf{K}}_{k-1}\mathbf{H}_{k-1}\right]\mathbf{\Psi}_{k-2}\left[\mathbf{I} - \tilde{\mathbf{K}}_{k-2}\mathbf{H}_{k-2}\right]\mathbf{e}_{k-2}^f \\ &\quad + \mathbf{\Psi}_{k-1}\left[\mathbf{I} - \tilde{\mathbf{K}}_{k-1}\mathbf{H}_{k-1}\right]\mathbf{\Psi}_{k-2}\tilde{\mathbf{K}}_{k-2}\mathbf{b}_{k-2}^o + \mathbf{\Psi}_{k-1}\tilde{\mathbf{K}}_{k-1}\mathbf{b}_{k-1}^o \\ &\quad - \mathbf{\Psi}_{k-1}\left[\mathbf{I} - \tilde{\mathbf{K}}_{k-1}\mathbf{H}_{k-1}\right]\mathbf{b}_{k-2}^t - \mathbf{b}_{k-1}^t\,.\end{aligned} \tag{5.53}$$

We can continue this iterative procedure by making the transformation $k \rightarrow k-2$ in (5.51), substitute the result back in the expression above, and so on, so that after j iterations we get

$$\mathbf{e}_k^f = \mathbf{\Phi}_{k,k-j}\mathbf{e}_{k-j}^f + \sum_{i=k-j}^{k-1}\mathbf{\Phi}_{k,i+1}\left[\mathbf{\Psi}_i\tilde{\mathbf{K}}_i\mathbf{b}_i^o - \mathbf{b}_i^t\right] \tag{5.54}$$

where we define the transition matrix $\mathbf{\Phi}_{k,k-j}$ as

$$\begin{aligned}\mathbf{\Phi}_{k,k-j} &\equiv \mathbf{\Psi}_{k-1}\left[\mathbf{I} - \tilde{\mathbf{K}}_{k-1}\mathbf{H}_{k-1}\right]\mathbf{\Psi}_{k-2}\left[\mathbf{I} - \tilde{\mathbf{K}}_{k-2}\mathbf{H}_{k-2}\right] \\ &\quad \ldots\mathbf{\Psi}_{k-j}\left[\mathbf{I} - \tilde{\mathbf{K}}_{k-j}\mathbf{H}_{k-j}\right]\end{aligned} \tag{5.55}$$

and also $\mathbf{\Phi}_{k,k} \equiv \mathbf{I}$.

Substituting the result (5.54) in the general expression for the innovation covariance matrix (5.49) we have

$$\begin{aligned}\mathbf{\Gamma}_{k,k-j} &= \mathbf{H}_k\mathbf{\Phi}_{k,k-j}\mathbf{P}_{k-j}^f\mathbf{H}_{k-j}^T + \mathcal{E}\{\mathbf{v}_k(\mathbf{v}_{k-j})^T\} \\ &\quad - \mathbf{H}_k\sum_{i=k-j}^{k-1}\mathbf{\Phi}_{k,i+1}\left[\mathbf{\Psi}_i\tilde{\mathbf{K}}_i\mathcal{E}\{\mathbf{b}_i^o(\mathbf{v}_{k-j})^T\} - \mathcal{E}\{\mathbf{b}_i^t(\mathbf{v}_{k-j})^T\}\right]\mathbf{H}_{k-j}^T\end{aligned} \tag{5.56}$$

where we notice that, by causality, the term containing $\mathcal{E}\{\mathbf{v}_k(\mathbf{e}_{k-j}^f)^T\}$ in (5.49) is zero. Using the fact that the sequence of observation noise is white (5.4), and also that the model error $\mathbf{b}_k^t$ are uncorrelated with the observation error $\mathbf{v}_{k'}$ (5.5), for all k and k', it follows that

$$\begin{aligned}\mathbf{\Gamma}_{k,k-j} &= \mathbf{H}_k\mathbf{\Phi}_{k,k-j}\mathbf{P}_{k-j}^f\mathbf{H}_{k-j}^T - \mathbf{H}_k\mathbf{\Phi}_{k,k-j+1}\mathbf{\Psi}_{k-j}\tilde{\mathbf{K}}_{k-j}\mathcal{E}\{\mathbf{b}_{k-j}^o(\mathbf{v}_{k-j})^T\} \\ &= \mathbf{H}_k\mathbf{\Phi}_{k,k-j}\mathbf{P}_{k-j}^f\mathbf{H}_{k-j}^T - \mathbf{H}_k\mathbf{\Phi}_{k,k-j+1}\mathbf{\Psi}_{k-j}\tilde{\mathbf{K}}_{k-j}\mathbf{R}_{k-j}\end{aligned} \tag{5.57}$$

We can write this expression in a more convenient form, by noticing that

$$\mathbf{\Phi}_{k,k-j} = \mathbf{\Phi}_{k,k-j+1}\mathbf{\Psi}_{k-j}\left[\mathbf{I} - \tilde{\mathbf{K}}_{k-j}\mathbf{H}_{k-j}\right] \tag{5.58}$$

and making use of the optimal Kalman gain matrix $\mathbf{K}_k$, that is,

$$\begin{aligned}
\mathbf{\Gamma}_{k,k-j} &= \mathbf{H}_k\mathbf{\Phi}_{k,k-j+1}\mathbf{\Psi}_{k-j}\left[\mathbf{I} - \tilde{\mathbf{K}}_{k-j}\mathbf{H}_{k-j}\right]\mathbf{P}^f_{k-j}\mathbf{H}^T_{k-j} \\
&\quad - \mathbf{H}_k\mathbf{\Phi}_{k,k-j+1}\mathbf{\Psi}_{k-j}\tilde{\mathbf{K}}_{k-j}\mathbf{R}_{k-j} \\
&= \mathbf{H}_k\mathbf{\Phi}_{k,k-j+1}\mathbf{\Psi}_{k-j}\left[\mathbf{P}^f_{k-j}\mathbf{H}^T_{k-j} - \tilde{\mathbf{K}}_{k-j}\mathbf{H}_{k-j}\mathbf{P}^f_{k-j}\mathbf{H}^T_{k-j} - \tilde{\mathbf{K}}_{k-j}\mathbf{R}_{k-j}\right] \\
&= \mathbf{H}_k\mathbf{\Phi}_{k,k-j+1}\mathbf{\Psi}_{k-j}\left[\mathbf{P}^f_{k-j}\mathbf{H}^T_{k-j} - \tilde{\mathbf{K}}_{k-j}(\mathbf{H}_{k-j}\mathbf{P}^f_{k-j}\mathbf{H}^T_{k-j} + \mathbf{R}_{k-j})\right] \\
&= \mathbf{H}_k\mathbf{\Phi}_{k,k-j+1}\mathbf{\Psi}_{k-j}\left[\mathbf{K}_{k-j} - \tilde{\mathbf{K}}_{k-j}\right]\left(\mathbf{H}_{k-j}\mathbf{P}^f_{k-j}\mathbf{H}^T_{k-j} + \mathbf{R}_{k-j}\right) \\
&= \mathbf{H}_k\mathbf{\Phi}_{k,k-j+1}\mathbf{\Psi}_{k-j}\left[\mathbf{K}_{k-j} - \tilde{\mathbf{K}}_{k-j}\right]\mathbf{\Gamma}_{k-j}
\end{aligned} \tag{5.59}$$

where the second to last equality is obtained by noticing that (5.21) can be written as

$$\mathbf{P}^f_{k-j}\mathbf{H}^T_{k-j} = \mathbf{K}_{k-j}\left(\mathbf{H}_{k-j}\mathbf{P}^f_{k-j}\mathbf{H}^T_{k-j} + \mathbf{R}_{k-j}\right) \tag{5.60}$$

making $k \to k-j$. Consequently, for the optimal filter, when $\tilde{\mathbf{K}}_{k-j} = \mathbf{K}_{k-j}$, we see that the innovation covariance is zero, that is,

$$\mathbf{\Gamma}_{k,k-j} = \mathbf{0} \quad \text{for all } k, \text{ and for all } j > 0 \quad . \tag{5.61}$$

In other words, the innovation sequence is white in time when filter is optimal. This property stimulates the monitoring of the innovation sequence to determine the performance of a general sub–optimal filter.

5.2.2 Orthogonality between the Estimate and the Estimation Error

The estimate produced by the Kalman filter, $\mathbf{w}^a_k$, at any given time t_k, and its correspondent error $\mathbf{e}^a_k$ are orthogonal. Mathematically, this is expressed as

$$\mathcal{E}\{\mathbf{w}^a_k(\mathbf{e}^a_k)^T\} = \mathbf{0}\,, \tag{5.62}$$

which is only true in the optimal case, that is, when $\tilde{\mathbf{K}}_k = \mathbf{K}_k$. A path to demonstrate this property is indicated in Exercise 5.4.

5.2.3 Observability and Controllability

The concepts of observability and controllability are independent of the Kalman filter theory being considered in this lecture. These concepts are related to dynamic systems in general. However, they are of fundamental importance when studying stability properties of the Kalman filter, and for that reason we introduce these concepts in what follows.

Observability is a concept introduced to express our ability to construct the states $\mathbf{w}^t_0, \mathbf{w}^t_1, \cdots, \mathbf{w}^t_k$ of a system, given a sequence of observations $\mathbf{w}^o_0, \mathbf{w}^o_1, \cdots, \mathbf{w}^o_k$. To exemplify observability

(cf. Gelb [60]), consider the evolution equation for the true state of the system for the case in which there is no stochastic forcing and in which the dynamics is independent of time, that is,

$$\mathbf{w}_k^t \;=\; \mathbf{\Psi}^\ell \mathbf{w}_{k-\ell}^t \tag{5.63}$$

represented the n–vector of the state of the system at time t_k obtained from the state at time $t_{k-\ell}$. Furthermore, consider a perfect observation process, for which the observation matrix $\mathbf{H}$ is a vector $\mathbf{h}^T$ of dimension $1 \times n$, and independent of time. In this way, we can write

$$\begin{aligned} w_0^o &= \mathbf{h}^T \mathbf{w}_0^t \\ w_1^o &= \mathbf{h}^T \mathbf{\Psi} \mathbf{w}_0^t \\ w_2^o &= \mathbf{h}^T \mathbf{\Psi}^2 \mathbf{w}_0^t \\ &\vdots \\ w_{n-1}^o &= \mathbf{h}^T \mathbf{\Psi}^{n-1} \mathbf{w}_0^t \end{aligned} \tag{5.64}$$

or yet, using vector notation,

$$\begin{pmatrix} w_0^o \\ w_1^o \\ \vdots \\ w_{k-1}^o \end{pmatrix} \;=\; \mathbf{Z} \mathbf{w}_0^t \tag{5.65}$$

Therefore the question of observability reduces to the ability of reconstructing the initial state of the system by means of the observations $w_0^o, w_1^o, \cdots, w_{n-1}^o$. Whether we can recovering the initial condition $\mathbf{w}_0^t$ of the system from the observations or not, can be assessed by considering the matrix $\mathbf{Z} = \mathbf{Z}_n$, of dimension $n \times n$, defined as

$$\mathbf{Z}_n \;\equiv\; \begin{pmatrix} \mathbf{h} & \mathbf{\Psi}^T \mathbf{h} & \cdots & (\mathbf{\Psi}^T)^{n-1} \mathbf{h} \end{pmatrix}^T \tag{5.66}$$

and whether this matrix is invertible or not. The matrix $\mathbf{Z}_n$ is invertible if it is of rank n. We say that a system is *observational* in a time $t_k > t_0$, if it is possible to construct an initial state $\mathbf{w}_0^t$ from observations $\mathbf{w}_k^o$ in the time interval (t_0, t_k). The system is said to be *completely observational* if the states $\mathbf{w}_k^t$ can be obtained from all of the observations $\mathbf{w}_k^o$.

In the general case, when the matrix $\mathbf{H}$ is of dimension $m \times n$, where m is the number of available observations, the observability matrix $\mathbf{Z}_n$ is redefined as:

$$\mathbf{Z}_n \;\equiv\; \begin{pmatrix} \mathbf{H}^T & \mathbf{\Psi}^T \mathbf{H}^T & \cdots & (\mathbf{\Psi}^T)^{n-1} \mathbf{H}^T \end{pmatrix}^T \tag{5.67}$$

and it is a matrix of dimension $nm \times n$, which should be of rank n for the system to be completely observable.

The concept of observability can be made more precise by introducing the so called *information matrix* $\mathcal{I}$,

$$\mathcal{I}(k, k-N) \equiv \sum_{i=k-N}^{k} \mathbf{\Psi}_{i,k}^T \mathbf{H}_i^T \mathbf{R}_i^{-1} \mathbf{H}_i \mathbf{\Psi}_{i,k} \tag{5.68}$$

which occurs in several recursive forms in least squares problems (or in the Kalman filter; see Jazwinski [84] pp. 205–207). According to Kalman [88] the dynamic system (5.1) and (5.3) is said to be *completely observable* if, and only if,

$$\mathcal{I}(k,0) > \mathbf{0} \tag{5.69}$$

for all $k > 0$. Moreover, the system is said to be *uniformly completely observable* if there is an integer N, and positive constants α and β, such that

$$\mathbf{0} < \alpha\mathbf{I} \leq \mathcal{I}(k, k-N) \leq \beta\mathbf{I} \tag{5.70}$$

for all $k \geq N$. It is interesting to notice that observability depends on the properties of the dynamics $\mathbf{\Psi}_{k,k-1}$ and the observation matrix $\mathbf{H}_k$, but not explicitly on the observations $\mathbf{w}_k^o$.

Analogously, we can introduce the concept of *controllability*. This concept comes from the idea of introducing a deterministic forcing term in the evolution equation to drive the system toward a pre–specified state, within a certain period of time. This subject is, in itself, the motivation for the development of a theory called optimal control. Analogously to what is done in estimation theory, in optimal control a performance index [similar to the cost function $\mathcal{J}$ in (5.19)] serves as a measure of the proximity of the solution to the specified state. The minimization of the performance index determines the optimal forcing term, in the least squares sense, necessary to drive the state of the system to the specified state. The problem of *linear* optimal control is said to be the *dual* of the *linear*, estimation problem, in the sense that results from estimation theory have equivalent counterparts in control theory. In particular, the concept of observability, briefly introduced above, is the dual of the concept of controllability. As a consequence, we can study controllability by means of the *controllability matrix*, defined as

$$\mathcal{C}(k, k-N) \equiv \sum_{i=k-N}^{k} \mathbf{\Psi}_{i,k}\mathbf{Q}_i^{-1}\mathbf{\Psi}_{i,k}^T \tag{5.71}$$

which is the dual analogous of the observability matrix. Consequently, we say that the dynamic system (5.1) and (5.3) is *completely controllable* if, and only if,

$$\mathcal{C}(k,0) > \mathbf{0} \tag{5.72}$$

for all k. Furthermore, we say that the system is *uniformly completely controllable* if there exists an integer N, and positive constants α and β such that

$$\mathbf{0} < \alpha\mathbf{I} \leq \mathcal{C}(k, k-N) \leq \beta\mathbf{I} \tag{5.73}$$

for all $k \geq N$. More details about this duality can be found in Kalman's original work [87], as well as in textbooks such as Gelb [60], Bryson & Ho [20], and also in the atmospheric sciences literature Ghil & Malanotte–Rizzoli [64].

The concepts of observability and controllability mentioned above are fundamental to establish *stability* results for the Kalman filter. In what follows, we summarize these results, following Dee's summary [44], which is based on the discussion Jazwinski's Section 7.6 [84].

When we inquire about system stability in the context of the Kalman filter, we are referring to the stability of the stochastic system described by the analysis equation

$$\mathbf{w}_k^a = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{\Psi}_{k,k-1}\mathbf{w}_{k-1}^a + \mathbf{K}_k\mathbf{w}_k^o \tag{5.74}$$

where $\mathbf{K}_k$ is a Kalman gain matrix (5.21). The dynamics $\mathbf{\Psi}_{k,k-1}$ is assumed to be *stable*, that is ,

$$||\mathbf{\Psi}_{k,0}|| \leq c_1 \tag{5.75}$$

for all $k \geq 0$. Here $||.||$ is an appropriate matrix norm, such as the spectral norm. In fact, the homogeneous system corresponding to (5.1) is said to be *asymptotically stable* if

$$||\mathbf{\Psi}_{k,0}|| \to 0 \tag{5.76}$$

for $k \to \infty$. Furthermore, the homogeneous system corresponding to (5.1) is said to be *uniformly asymptotically stable* if

$$||\mathbf{\Psi}_{k,0}|| \to c_2 \exp(-c_3 k) \tag{5.77}$$

for all $k \geq 0$.

For this stable dynamics, the following results can be obtained, for the system governed by (5.74):

1. The analysis error covariance matrix $\mathbf{P}_k^a$ is uniformly bounded from above and below:

$$[\mathcal{I}(k,k-N) + \mathcal{C}^{-1}(k,k-N)]^{-1} \leq \mathbf{P}_k^a \leq [\mathcal{I}^{-1}(k,k-N) + \mathcal{C}(k,k-N)]^{-1} \tag{5.78}$$

   for all $k \geq N$.

2. If $\mathbf{P}_0^a \geq \mathbf{0}$, the Kalman filter is *uniformly asymptotically stable*, that is, there are constants c_4 and c_5 such that

$$||\mathbf{\Phi}_{k,0}|| \leq c_4 \exp(-c_5 k) \tag{5.79}$$

   for all $k \geq 0$, where $\mathbf{\Phi}_{k,0}$ is the transition matrix introduced in (5.55).

3. If $\mathbf{P}_k^a$ and $\mathbf{S}_k^a$ are two solutions of the Kalman filter equations for two initial conditions $\mathbf{P}_0^a \geq \mathbf{0}$ and $\mathbf{S}_0^a \geq \mathbf{0}$, then

$$||\mathbf{P}_k^a - \mathbf{S}_k^a|| \leq c_4 \exp(-2c_5 k)||\mathbf{P}_0^a - \mathbf{S}_0^a|| \tag{5.80}$$

   which means that the error estimates of the Kalman filter are stable with respect to the errors of the initial state. In other words, the linear Kalman filter eventually — as data is processed in time —"forgets" about the uncertainty in the initial error covariance.

The notions of observability and controllability were initially introduced for systems governed by ordinary differential equations (see Ghil & Ide [63] for an application of interest to atmospheric sciences). These concepts can be extended to the case of systems governed by partial differential equations. A series of articles on this subject can be found in the Stavroulakis [125]. The problem of observability for discrete partial differential equations was investigated by Cohn & Dee [31].

Table 5.1: Computational requirements of the Kalman filter ($m_k = m$).

"Brute–force" implementation of the Kalman filter				
Ref.	Variable	Equation	Calculation	Flops
F1	$\mathbf{w}_k^f$	$\mathbf{\Psi}_{k-1}\mathbf{w}_{k-1}^a$	$\mathbf{\Psi w}$	$2n^2 - n$
F2	$\mathbf{P}_k^f$	$\mathbf{\Psi}_{k-1}\mathbf{P}_{k-1}^a\mathbf{\Psi}_{k-1}^T + \mathbf{Q}_{k-1}$	$\mathbf{P\Psi}^T$	$2n^3 - n^2$
			$\mathbf{\Psi(P\Psi}^T)$	$2n^3 - n^2$
			$(\mathbf{\Psi P\Psi}^T) + \mathbf{Q}$	n^2
F3	$\mathbf{K}_k$	$\mathbf{P}_k^f\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_k^f\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$	$\mathbf{HP}$	$2n^2m - nm$
			$(\mathbf{HP})\mathbf{H}^T$	$2nm^2 - m^2$
			$(\mathbf{HPH}^T) + \mathbf{R}$	m^2
			$(\mathbf{HPH}^T + \mathbf{R})^{-1}$	$2m^3$
			$(\mathbf{PH}^T)(\mathbf{HPH}^T + \mathbf{R})^{-1}$	$2nm^2 - nm$
F4	$\mathbf{w}_k^a$	$\mathbf{w}_k^f + \mathbf{K}_k(\mathbf{w}_k^o - \mathbf{H}_k\mathbf{w}_k^f)$	$\mathbf{Hw}^f$	$2nm - m$
			$\mathbf{w}^o - \mathbf{Hw}^f$	m
			$\mathbf{K}(\mathbf{w}^o - \mathbf{Hw}^f)$	$2nm - n$
			$\mathbf{w}^f + [\mathbf{K}(\mathbf{w}^o - \mathbf{Hw}^f)]$	n
F5	$\mathbf{P}_k^a$	$(\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_k^f$	$\mathbf{K(HP)}$	$2nm^2 - m^2$
			$\mathbf{P} - \mathbf{K(HP)}$	n^2

5.3 Computational Aspects of the Kalman Filter

5.3.1 Generalities

We show in Table 5.3.1 the equations involved in the implementation of the Kalman filter. Although these equations are used for the case of linear systems, many approximations for the nonlinear case involve similar equations with equivalent computational cost — some computational burden is added to nonlinear systems due to the calculation of the Jacobian matrices (see the following lecture). The table displays computational cost measured in units of flops – floating point operations (multiplications and additions) — related to "brute-force" implementation of these equations. By "brute-force" we mean implementations following the operations in the table neither taking into account storage savings of certain quantities nor preventing repetitive calculations of other quantities. A detailed treatment of various implementations of the Kalman filter equations is given in Mendel [106], however for atmospheric data assimilation applications the description here should suffice. In these applications, many of the matrices in the formulas in Table 5.3.1 are not explicitly invoked due to their complexity, and are rather treated in operator form.

The following are factors that may be exploited to render computation costs more acceptable:

- the symmetry of the error covariance matrices can be used to reduce storage requirement.

- the analysis $\mathbf{P}_k^a$ and forecast $\mathbf{P}_k^f$ error covariance matrices can share the same space in memory.

- in applications to atmospheric data assimilation, the dynamics $\mathbf{\Psi}_k$ is a sparse matrix due relatively small finite–difference stencils, and only its non–zero elements need to be stored in memory. As a matter of fact, in this case, the operations corresponding to the application of $\mathbf{\Psi}_k$ to an n–vector is of order n, instead of n^2, as indicated in the table for the general case. Moreover, $\mathbf{\Psi}$ never really exists as a matrix, but rather as an operator.

The Kalman filter is subject to computational instabilities due to different possible ways to program its equations. A simple case is discussed below showing that Joseph's formula (5.18) for calculating the analysis error covariance matrix is computationally more stable than the expression (5.22), with respect to errors in calculating the gain matrix $\mathbf{K}_k$ (see next section). Even the ordering of the factors in the multiplication among matrices in the algorithm is relevant and may be responsible for numerical instability as discussed in details by Verhaegen & Van Dooren [134].

Assuming that $n \gg m$, or else, that the number of degrees of freedom n of the system is much greater than the number of observations $m_k = m$, at any given time, it is clear from Table 5.3.1 that equation F2 is responsible for the major part of the computational cost in the Kalman filter algorithm. In general, the cost of propagating the analysis error covariance matrix, to get the forecast error covariance matrix, is of the order of n^3; in the particular case of sparse dynamics, the cost gets reduced to n^2. For problems governed by *partial* differential equations, as in the case of atmospheric data assimilation, the number of degrees of freedom n reaches levels as high as 10^6–10^7, with great potential for increase as resolution of atmospheric models increase. This large number of degrees of freedom for problems in assimilation data assimilation prohibits "brute–force" implementation of the Kalman filter, even when the factors for cost reduction mentioned above are taken into account. Consequently, we are required to develop approximations to equation F2, and in some cases even to the analysis error covariance update equation F5. A lot of the research in applying the Kalman filter to atmospheric data assimilation has been done with relation to this topic (see Todling & Cohn [129], and references therein).

5.3.2 Sensitivity of the Filter to the Gains

The asymptotic stability concept for the Kalman filter discussed previously in this lecture is relatively strong, and not always the conditions for uniform asymptotic stability are satisfied. In practice, however, instability in the Kalman filter algorithm, or in suboptimal implementations of the algorithm, can be associated to lack of knowledge of model errors, observation errors, and even to specific problems due to numerical implementation of the algorithm. In this section, we look at more closely to this last aspect of instability, that is, that due to numerical implementation. We show that certain formulas are in fact more

prone to numerical errors and can be, sometimes, the cause of eventual divergence of the filter.

In order to simplify notation, we momentarily omit the index referring time in the filter equations. In this manner the error covariance matrix update equation can be written using Joseph's formula as

$$\mathbf{P}^a = (\mathbf{I} - \mathbf{KH})\mathbf{P}^f(\mathbf{I} - \mathbf{KH})^T + \mathbf{KRK}^T \tag{5.81}$$

where the Kalman gain matrix is given by

$$\mathbf{K} = \mathbf{P}^f\mathbf{H}^T(\mathbf{HP}^f\mathbf{H}^T + \mathbf{R})^{-1} \tag{5.82}$$

Alternatively, as we have seen above, the simpler formula for the analysis error covariance matrix can be obtained by substituting the optimal gain (5.82) in (5.81), that is,

$$\mathbf{P}^a = (\mathbf{I} - \mathbf{KH})\mathbf{P}^f \tag{5.83}$$

Numerical implementation of the Kalman filter generates numerical errors, even when the optimal filter is utilized — e.g., due to randoff error. In this regard, we want to investigate the effect in $\mathbf{P}^a$ caused by small errors in calculating $\mathbf{K}$ numerically. For that, assume that the gain $\mathbf{K}$ undergoes a modification $\delta\mathbf{K}$ after numerically solving (5.82), so that from (5.83) it follows that,

$$\mathbf{P}^a + \delta\mathbf{P}^a = (\mathbf{I} - \mathbf{KH})\mathbf{P}^f + \delta\mathbf{KHP}^f \tag{5.84}$$

and therefore, the instantaneous error in $\mathbf{P}^a$ is given by

$$\delta\mathbf{P}^a = \delta\mathbf{KHP}^f \tag{5.85}$$

which is of first order in $\delta\mathbf{K}$.

Instead, using Joseph's formula (5.81) for the modified gain we have

$$\begin{aligned}
\mathbf{P}^a + \delta\mathbf{P}^a_{Joe} &= (\mathbf{I} - \mathbf{KH} - \delta\mathbf{KH})\mathbf{P}^f(\mathbf{I} - \mathbf{KH} - \delta\mathbf{KH})^T \\
&\quad + (\mathbf{K} + \delta\mathbf{K})\mathbf{R}(\mathbf{K} + \delta\mathbf{K})^T \\
&= (\mathbf{I} - \mathbf{KH})\mathbf{P}^f(\mathbf{I} - \mathbf{KH})^T + \mathbf{KRK}^T \\
&\quad - (\mathbf{I} - \mathbf{KH})\mathbf{P}^f\mathbf{H}^T\delta\mathbf{K}^T - \delta\mathbf{KHP}^f(\mathbf{I} - \mathbf{KH})^T \\
&\quad + \delta\mathbf{KHP}^f\mathbf{H}^T\delta\mathbf{K}^T + \mathbf{KR}\delta\mathbf{K}^T + \delta\mathbf{KRK}^T + \delta\mathbf{KR}\delta\mathbf{K}^T \\
&= (\mathbf{I} - \mathbf{KH})\mathbf{P}^f(\mathbf{I} - \mathbf{KH})^T + \mathbf{KRK}^T + \delta\mathbf{K}(\mathbf{HP}^f\mathbf{H}^T + \mathbf{R})\delta\mathbf{K}^T \\
&\quad + [\mathbf{K}(\mathbf{HP}^f\mathbf{H}^T + \mathbf{R}) - \mathbf{P}^f\mathbf{H}^T]\delta\mathbf{K}^T \\
&\quad + \delta\mathbf{K}[\mathbf{K}(\mathbf{HP}^f\mathbf{H}^T + \mathbf{R}) - \mathbf{P}^f\mathbf{H}^T]^T
\end{aligned} \tag{5.86}$$

and therefore, using (5.82) and (5.81) it follows that

$$\delta\mathbf{P}^a_{Joe} = \delta\mathbf{K}(\mathbf{HP}^f\mathbf{H}^T + \mathbf{R})\delta\mathbf{K}^T \tag{5.87}$$

This shows that Joseph's formula is of second order in errors made when calculating the gain matrix, and therefore it is numerically more stable. Consequently, in many engineering implementations of the Kalman filter Joseph's formula is preferably used.

5.3.3 Serial Processing of Observations

Serial processing of observations was introduced in the literature by Bierman [12], and discussed in Parrish & Cohn [113] in the context of atmospheric data assimilation. In this section, we assume for simplicity that all the available observations are uncorrelated at all times t_k. We have in mind the uncorrelatedness not only in time, but also among variables at a fixed time.

When m observations are available at time t_k, to say these observations are uncorrelated among themselves is to say that the matrix $\mathbf{R}_k$ is diagonal, for all k, that is

$$\mathbf{R}_k = diag(\sigma_1^2, ..., \sigma_p^2) \tag{5.88}$$

where σ_i, $i = 1, 2, ..., m$, are the observation error standard deviations. Following the treatment of Parrish & Cohn [113], let us omit the index k in this section to simplify notation.

In this case, the observation process in (5.3) can be decomposed as

$$w_j^o = \mathbf{h}_j^T \mathbf{w}^t + b_j^o \tag{5.89}$$

for $j = 1, 2, ..., p$, where w_j^o is a single scalar observation, the vector $\mathbf{h}_j^T$ is the j–th row of the observation matrix $\mathbf{H}$, and b_j^o is a random number that satisfies

$$\mathcal{E}\{(b_j^o)^2\} = \sigma_j^2 \,, \tag{5.90}$$

for each j.

The assumption that the m observations, available at any given time, are uncorrelated of each another means that these observations can be processed (or assimilated) as if they became available at infinitesimally small time intervals apart. Consequently, we can iterate the equations (5.21), (5.18) and (5.23) over the observations so that we get, for each observation j:

$$\mathbf{k}_j = \mathbf{P}_{j-1}\mathbf{h}_j(\mathbf{h}_j^T\mathbf{P}_{j-1}\mathbf{h}_j + \sigma_j^2)^{-1} \tag{5.91a}$$

$$\mathbf{P}_j = (\mathbf{I} - \mathbf{k}_j\mathbf{h}_j^T)\mathbf{P}_{j-1} \tag{5.91b}$$

$$\mathbf{w}_j = \mathbf{w}_{j-1} + \mathbf{k}_j(w_j^o - \mathbf{h}_j^T\mathbf{w}_j) \tag{5.91c}$$

which resembles the algorithm derived in Section 4.4 for processing a newly available observation vector with the least squares algorithm. In that case, we have also assumed uncorrelatedness among observations, which was explicitly seen when writing (4.82).

Since the quantities in parenthesis in (5.91a) and in (5.91c) are scalars, and the vector $\mathbf{P}_{j-1}\mathbf{h}_j$ is used many times in different places, we can introduce an auxiliary vector $\mathbf{v}_j$ (which should not be confused with the innovation vector introduced in earlier in this lecture),

$$\mathbf{v}_j = \mathbf{P}_{j-1}\mathbf{h}_j \,, \tag{5.92}$$

so that the observation process gets reduced to the following algorithm: initialize with the forecast error covariance matrix and the forecast state vector,

$$\mathbf{P}_0 = \mathbf{P}^f \,, \tag{5.93}$$

and

$$\mathbf{w}_0 = \mathbf{w}^f \; ; \tag{5.94}$$

respectively, and iterate the following set of equations:

$$\begin{aligned}
\alpha_j &= \mathbf{h}_j^T \mathbf{v}_j + \sigma_j^2 \; , && \text{(5.95a)} \\
\mathbf{k}_j &= \frac{1}{\alpha_j} \mathbf{v}_j \; , && \text{(5.95b)} \\
\bar{\mathbf{P}}_j &= \mathbf{P}_{j-1} - \mathbf{k}_j \mathbf{v}_j^T \; , && \text{(5.95c)} \\
\bar{\mathbf{v}}_j &= \bar{\mathbf{P}}_j \mathbf{h}_j \; , && \text{(5.95d)} \\
\mathbf{P}_j &= \bar{\mathbf{P}}_j - \bar{\mathbf{v}}_j \mathbf{k}_j^T + \sigma_j^2 \mathbf{k}_j \mathbf{k}_j^T \; , && \text{(5.95e)} \\
\beta_j &= w_j^o - \mathbf{h}_j^T \mathbf{w}_{j-1} \; , && \text{(5.95f)} \\
\mathbf{w}_j &= \mathbf{w}_{j-1} + \beta_j \mathbf{k}_j \; , && \text{(5.95g)}
\end{aligned}$$

for each $j = 1, 2, ..., m$, so that at the last iteration we have

$$\mathbf{P}_m = \mathbf{P}^a \; , \tag{5.96}$$

for the analysis error covariance matrix, and

$$\mathbf{w}_m = \mathbf{w}^a \; , \tag{5.97}$$

for the analysis state vector. The computational advantage of this algorithm is that it avoids the need to invert the $m \times m$ innovation error covariance matrix in (5.21), to calculate Kalman gain matrix $\mathbf{K}_k$. In the serial processing procedure, the inversion of this matrix is replaced by the inversion of the m scalar quantities in (5.95a). The demonstration of consistence between the serial algorithm above and the standard algorithm can be done by following an analogous procedure to that of Section 4.4, to process a newly available observation with the least squares algorithm.

The use of Joseph's formula and the consequent use of $\bar{\mathbf{P}}_j$ may suggest the need to define an auxiliary matrix of the size of the forecast error covariance matrix. However, this is only apparent, due to the notation used in writing the algorithm above. When programming these equations, the matrix $\mathbf{P}_j$ is the only one required, that is, matrices $\bar{\mathbf{P}}_j$ and $\mathbf{P}_j$ can share the same storage space. Also notice that when the elements of the state vector are directly observed, that is, when there are no linear combinations between the elements of the state vector in order to produce the observations, the elements of the vector $\mathbf{h}_j$ are all zeros except for one of them, which is in fact the unity. Consequently, the operations in (5.92) and (5.95d) are equivalent to extracting a column of the matrices $\mathbf{P}_j$.

One disadvantage of the serial processing is that we do not have access to the complete gain matrix $\mathbf{K}$, but rather only to the arrays $\mathbf{k}_j$. If we are only interested in the final result of the analysis, there is no need to obtain $\mathbf{K}$ explicitly; however, if we are particularly interested in investigating the influence of a certain observation on to distinct elements of the state vector (e.g., Ghil et al. [66]), it is necessary to calculate the complete gain matrix. The simplest way to recover the gain matrix, when using serial processing, is to do so after having obtained the analysis error covariance matrix by making use of the alternative expression for the gain matrix,

$$\mathbf{K}_k = \mathbf{P}_k^a \mathbf{H}_k^T \mathbf{R}_k^{-1} \; , \tag{5.98}$$

where, in writing the expression above we restored the time subscript k, to emphasize the fact that this should be done at the end of each analysis time t_k.

Exercises

1. Show that (5.18) reduces to (5.22) for the optimal Kalman filter gain.

2. (Gelb [60], Problem 4.8). Consider the following continuous–time dynamical system, and corresponding continuous–time observation process:

$$\dot{\mathbf{x}} = \mathbf{F}(t)\mathbf{x} + \mathbf{G}(t)\mathbf{w}$$

$$\mathbf{z} = \mathbf{H}(t)\mathbf{x} + \mathbf{v}$$

where the noises $\mathbf{w}$ e $\mathbf{v}$ are considered $\mathcal{N}(\mathbf{0}, \mathbf{Q}(t))$ and $\mathcal{N}(\mathbf{0}, \mathbf{R}(t))$, respectively, and are also decorrelated. Assume that the state estimate evolves according to the following expression:

$$\dot{\hat{\mathbf{x}}} = \tilde{\mathbf{L}}\hat{\mathbf{x}} + \tilde{\mathbf{K}}\mathbf{z}$$

where the matrices $\tilde{\mathbf{L}}$ and $\tilde{\mathbf{K}}$ are to be determined following estimation and optimization arguments. Imposing the restriction that the estimate be unbiased, show that $\tilde{\mathbf{L}} = \mathbf{F} - \tilde{\mathbf{K}}\mathbf{H}$, and obtain the following simplified form for the estimate evolution equation:

$$\dot{\hat{\mathbf{x}}} = \mathbf{F}(t)\hat{\mathbf{x}} + \tilde{\mathbf{K}}(\mathbf{z} - \mathbf{H}\hat{\mathbf{x}})$$

Next, show that the error estimate covariance matrix evolves according to the following expression:

$$\dot{\mathbf{P}} = (\mathbf{F} - \tilde{\mathbf{K}}\mathbf{H})\mathbf{P} + \mathbf{P}(\mathbf{F} - \tilde{\mathbf{K}}\mathbf{H})^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T + \tilde{\mathbf{K}}\mathbf{R}\tilde{\mathbf{K}}^T;$$

notice that this is a general expression, in the sense that it is valid for any matrix $\tilde{\mathbf{K}}$. This expression is continuum equivalent of the Joseph formula (5.18) for the discrete–time case. As a matter of fact, we can show through a limiting procedure equivalent to that of Section 3.1.3, that the expression for the discrete case reduces to the expression above as time approaches zero (e.g., see Gelb [60]). Defining a cost function as a measure of the ratio of error change, that is, $J = \mathrm{Tr}(\dot{\mathbf{P}})$, show that its minimization leads to the following expression for the optimal gain matrix $\tilde{\mathbf{K}} = \mathbf{K}$:

$$\mathbf{K} = \mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}$$

Using this formula for $\mathbf{K}$, show that the evolution equation for the error covariance is transformed to

$$\dot{\mathbf{P}} = \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T - \mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{P} + \mathbf{G}\mathbf{Q}\mathbf{G}^T,$$

which is known as the Riccati equation (e.g., Bittanti et al. [13])

3. (Gelb [60], Problem 4.11). Consider the following dynamical system and measurement processes:

$$\dot{x} = ax + w$$

$$z = bx + v$$

where the noises w and v are white in time, and normal, with mean zero and variances $q = const.$ and $r = const.$, respectively, for constants a and b. Assuming the initial error variance is p_0, show that the optimal filter error variance is given by

$$p(t) = \frac{(ap_0 + q)\sinh \beta t \; + \; \beta p_0 \cosh \beta t}{\left(\frac{b^2}{r} p_0 - a\right)\sinh \beta t \; + \; \beta \cosh \beta t}$$

where

$$\beta = a\sqrt{1 + \frac{b^2 q}{a^2 r}}$$

Furthermore, show that the steady–state ($t \to \infty$) variance is given by

$$p_\infty = \frac{ar}{b^2}(1 + \frac{\beta}{a})$$

which is independent of the initial variance p_0. Obtain p_∞ for a perfect model, that is, when $q = 0$. Give an interpretation to this result.

4. Show that the Kalman filter estimate $\mathbf{w}_k^a$ is orthogonal to its error $\mathbf{e}_k^a$, for all k. Using finite induction, start by showing that

$$\mathcal{E}\{\mathbf{w}_1^a (\mathbf{e}_1^a)^T\} \; = \; \mathbf{0}$$

and that

$$\mathcal{E}\{\mathbf{w}_2^a (\mathbf{e}_2^a)^T\} \; = \; \mathbf{0}$$

Then, assume that $\mathcal{E}\{\mathbf{w}_k^a (\mathbf{e}_k^a)^T\} \; = \; \mathbf{0}$ is true, and show that

$$\mathcal{E}\{\mathbf{w}_{k+1}^a (\mathbf{e}_{k+1}^a)^T\} \; = \; \mathbf{0}$$

is satisfied.

5. (Ghil et al. [66]) Consider the Kalman filter applied to the scalar, discrete–time system:

$$x_k = ax_{k-1} \; + \; w_k$$

$$z_k = x_k \; + \; v_k$$

where the noises w_k and v_k are white, normal with mean zero and variances $q = const.$ and $r = const.$, respectively. In this case, the Kalman filter reduces to the following system of equations:

$$p_k^f = Ap_{k-\ell}^a \; + \; Bq$$

$$p_k^a = \begin{cases} rp_k^f/(p_k^f + r)\,, & \text{para } k = j\ell,\, j = 1, 2, \cdots \\ p_k^f & \text{de outro modo} \end{cases}$$

where

$$A = a^{2\ell}\,, \quad B = \textstyle\sum_{m=1}^{\ell-1} a^{2m}$$

Defining $s_j = p_{j\ell}^a$, for $j = 0, 1, , 2, \cdots$ show that

$$s_j = \frac{(As_{j-1} + Bq)r}{As_{j-1} + Bq + r}$$

Consider now the perfect model case, that is, when $q = 0$, with initial error variance $p_0 = s_0$. Show that for $|a| \neq 1$,

$$s_j = \frac{A^j(A-1)s_0 r}{A(A^j-1)s_0 + (A-1)r}$$

and that for $|a| = 1$,

$$s_j = \frac{s_0 r}{j s_0 + r}$$

Finally, show that when $j \to \infty$ we have

$$\begin{array}{ll} s_j \to 0 & \text{para } |a| \leq 1 \quad , \\ s_j \to (1 - \frac{1}{A})r \, , & \text{para } |a| > 1 \, . \end{array}$$

Interpret the asymptotic results obtained above.

6. (Chui & Chen [25], Problema 2.14) Some typical engineering applications are classified under the designation ARMA (autoregressive moving–average), and can be written as:

$$\mathbf{v}_k = \sum_{i=1}^{N} \mathbf{B}_i \mathbf{v}_{k-i} + \sum_{i=0}^{M} \mathbf{A}_i \mathbf{u}_{k-i} \, ,$$

where the matrices $\mathbf{B}_1, \ldots, \mathbf{B}_N$ are $n \times n$ dimensional, and the matrices $\mathbf{A}_0, \ldots, \mathbf{A}_M$, are $n \times q$, and are independent of the time variable k. Considering $M \leq N$, show that this process can be written in the following vector form:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k$$

$$\mathbf{v}_k = \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k$$

for a vector $\mathbf{x}_k$ of dimension nN, with $\mathbf{x}_0 = \mathbf{0}$, and where

$$\mathbf{A} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & & & \vdots \\ \mathbf{B}_{N-1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \\ \mathbf{B}_N & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \, , \qquad \mathbf{B}^T = \begin{pmatrix} \mathbf{A}_1 + \mathbf{B}_1\mathbf{A}_0 \\ \mathbf{A}_2 + \mathbf{B}_2\mathbf{A}_0 \\ \vdots \\ \mathbf{A}_M + \mathbf{B}_M\mathbf{A}_0 \\ \mathbf{B}_{M+1}\mathbf{A}_0 \\ \vdots \\ \mathbf{B}_N\mathbf{A}_0 \end{pmatrix} \, ,$$

$$\mathbf{C} = [\mathbf{I}\,\mathbf{0}\,\cdots\,\mathbf{0}] \quad \text{e} \quad \mathbf{D} = \mathbf{A}_0 \, .$$

7. *Multiple choice.* (from Bryson & Ho [20]) Consider the scalar estimation problem

$$x_{i+1} = x_i + w_i$$

$$z_i = x_i + v_i$$

where $w_i \sim \mathcal{N}(0, q)$ and white; $v_i \sim \mathcal{N}(0, 1)$ and white; w_i and v_j are uncorrelated for all i and j; and there is no initial knowledge of x_0. If $0 < q < \infty$, then the optimal estimate $\hat{x}_i$ is given by

(a) $\hat{x}_i = \frac{1}{i}\sum_{j=1}^{i} z_j$

(b) $\hat{x}_i = z_i$

(c) $\hat{x}_{i+1} = \hat{x}_i + k_i(z_{i+1} - \hat{x}_i)$, $1/(i+1) < k_i < 1$

(d) $\hat{x}_{i+1} = \hat{x}_i + k_i(z_{i+1} - \hat{x}_i)$, $1 < k_i < \infty$

Justify your answer.

8. *Multiple choice.* (from Bryson & Ho [20]) A static estimate of $\mathbf{x}$ is made from a measurement $\mathbf{z}$:

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}$$

with $\mathbf{v} \sim \mathcal{N}(\bar{\mathbf{v}}, \mathbf{R})$ and $\mathbf{x} \sim \mathcal{N}(\bar{\mathbf{x}}, \mathbf{P})$. The estimate is

$$\hat{\mathbf{x}} = \bar{\mathbf{x}} + \mathbf{K}(\mathbf{z} - \mathbf{H}\bar{\mathbf{x}})$$

where $\mathbf{K}$ is some constant matrix. The estimate is

(a) unbiased

(b) biased with a bias of $(\mathbf{K}\mathbf{H}\bar{\mathbf{x}})$

(c) biased with a bias of $(\mathbf{K}\bar{\mathbf{v}})$

(d) biased with a bias of $[\mathbf{K}(\bar{\mathbf{v}} - \mathbf{H}\bar{\mathbf{x}})]$

Justify your answer.

9. *Computer Assignment.* (Partially based on Lewis (1986) [1], Example 2.5-2.) *Computer Assignment.* Consider the following linear dynamical process[2]

$$\mathbf{x}_k \equiv \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = \begin{pmatrix} 1 & T \\ -\omega^2 T & 1 - 2\alpha T \end{pmatrix} \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \end{pmatrix} + \begin{pmatrix} w_1(k-1) \\ w_2(k-1) \end{pmatrix}$$

and the following observation process

$$z(k) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + v(k)$$

for $\mathbf{w}(k) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$, $v(k) \sim \mathcal{N}(0, r/T)$ and both uncorrelated from each other at all times. Here the (co)variance $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}$$

For the choice of parameters: $\omega = 0$, $\alpha = -0.1$,$r = 0.02$, and $T = 0.02$, address the following questions:

(a) Is the dynamical system stable or unstable?

[1] Lewis, F.L., 1986: *Optimal Estimation with an Introduction to Stochastic Control Theory.* John Wiley & Sons, 376 pp.

[2] This dynamical system arises from an Euler discretization of "damped" harmonic oscillator given by

$$\ddot{y}(t) + 2\alpha\dot{y} + \omega^2 = 0$$

where stochastic forcing is applied after discretization.

(b) Using Matlab, simulate the stochastic dynamical system from $k = 0$ to $k = 500$, starting from $\mathbf{x}_0 = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}$. Plot the state $\mathbf{x}_k$ against k.

(c) Using the linear Kalman filter, simulate the evolution of the error (co)variance matrix, starting from the initial condition $\mathbf{P}_0^a = \mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix. Plot the analysis error variance, in both variables, for the same time interval as in the previous item.

(d) Is the filter stable or unstable? Explain.

(e) Are your answers to questions (a) and (d) incompatible? Explain.

(f) Plot the true state evolution together with the analysis estimate[3] for both variables and for the time interval in item (b).

(g) *Suboptimal filters:* Let us now study the behavior of two suboptimal filters. Before starting, however, replace the analysis error (co)variance equation in your Matlab program by Joseph formula (if you are now already using it). We mentioned in this lecture that Joseph formula is valid for any gain matrix $\tilde{\mathbf{K}}_k$, thus we can use it to evaluate the performance of suboptimal filters.

i. Assuming the calculation of the forecast error (co)variance is computationally too costly for the present problem, we want to construct a suboptimal filter that somehow replaces the calculation of $\mathbf{P}_k^f$ by a simpler equation. Let us think on replacing the equation for $\mathbf{P}_k^f$ by the simple expression $\mathbf{P}_k^f = \mathbf{I}$. With this choice of forecast error (co)variance, it is simple to see that the gain matrix becomes

$$\begin{aligned} \tilde{\mathbf{K}}_k &= \mathbf{H}^T(\mathbf{H}\mathbf{H}^T + r/T)^{-1} \\ &= \tfrac{1}{1+r/T}\mathbf{H}^T \end{aligned}$$

where we used explicitly that $\mathbf{H} = (1 \;\; 0)$ for the system under consideration. Keeping the equation for $\mathbf{P}_k^f$, in your Matlab code as dictated by the Kalman filter, replace the expression for the optimal gain by the one given above. This turns the state estimate in a suboptimal estimate. Also, since you have kept the original expression for the forecast error (co)variance evolution, and your are using Joseph formula for the analysis error (co)variance, these two quantities provide now filter performance information due to suboptimal choices of gains. With the "approximate" gain matrix above, is the resulting filter stable or unstable? Explain. If this is not a successful choice of gain matrix, can you explain why that is?

ii. Let us know build another suboptimal filter that replaces the gain by the asymptotic gain obtained from the optimal run in item (b). To obtain the optimal asymptotic gain, you need to run the experiment in item (b) again, output the gain matrix at the last time step from that run, and use it as a suboptimal choice for the gain matrix in this item. You should actually

[3] Remember that your initial estimate should be sampled from the initial state where the initial error is $\mathcal{N}(0, \mathbf{P}_0^a)$, that is,

$$\mathbf{x}_0^a = \mathbf{x}_0 + \boxed{\text{chol}(\mathbf{P}_0^a)} * \boxed{\text{randn}(:)}$$

writing is a very symbolic manner.

make sure that the gain has asymptote by looking at its value for a few time steps before the last time step, and verifying that these values are indeed the same. Now run a similar experiment than that of the previous item, but using the asymptotic gain for the suboptimal gains at all time steps. Is the resulting filter stable or unstable? (Note: This choice of gain corresponds to using the so called Wiener filter.)