Chapter 3

Stochastic Differential Equations

In this lecture we will present a simple discussion of systems governed by stochastic differential equations. To maintain the most pleasant possible notation, we will not make explicit distinction between the random variable $\mathbf{u}$ and its corresponding value $\boldsymbol{u}$, the first being utilized to represent the stochastic process of interest. This notation will be utilized through the end of this course.

3.1 Linear Dynamical Systems

Linear transformations of an r.v. consist of one of the most fundamental transformations in stochastic processes. A linear transformation of special interest is found in dynamical systems. In this case, a certain initial condition evolves according to the dynamics of a *linear operator*. In case the distribution function of the r.v.'s in question is a Gaussian, a linear transformation of this variable produces an r.v. with the Gaussian distribution (see Exercise 1.4). This occurs in problems in linear dynamical systems: given an initial condition with the Gaussian distribution, the final result will also be Gaussian distributed. In this lecture, we concentrate in calculating mean and (co)variances of stochastic processes, since in the normally distributed case these quantities define the process completely. Higher moments are necessary when either the process is not Gaussian or the process is nonlinear. The treatment in this lecture is general and independent of the distribution under consideration, meaning that any moments can in principle be calculated according to the procedures given below. Both time–continuous and time–discrete stochastic processes are discussed here. A brief introduction to the case of systems governed by stochastic random fields is discussed in the end of this lecture.

3.1.1 Continuous Processes

Consider the following linear dynamics of first order in time for the random n–vector $\mathbf{u}$:

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}(t)}{dt} = \mathbf{F}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t)\,, \tag{3.1}$$

where the mean and the (co)variance of the initial state $\mathbf{u}(t_0)$ and of the m–vector o noise $\mathbf{w}(t)$ are given by:

$$\boldsymbol{\mu}_{\mathbf{w}}(t) = \mathcal{E}\{\mathbf{w}(t)\} \qquad \mathbf{P}_{\mathbf{w}}(t_1,t_2) = cov\{\mathbf{w}(t_1),\mathbf{w}(t_2)\} \tag{3.2}$$

$$\boldsymbol{\mu}_{\mathbf{u}}(t_0) = \mathcal{E}\{\mathbf{u}(t_0)\} \qquad \mathbf{P}_{\mathbf{u}}(t_0) = var\{\mathbf{u}(t_0)\} \tag{3.3}$$

where the matrices $\mathbf{P}_{\mathbf{w}}$ and $\mathbf{P}_{\mathbf{u}}$ are of dimension $m \times m$ and $n \times n$, respectively. Moreover, we consider the process $\mathbf{w}(t)$ to be uncorrelated with the initial process $\mathbf{u}(t_0)$, that is,

$$cov\{\mathbf{u}(t_0),\mathbf{w}(t)\} = \mathbf{0}\,, \tag{3.4}$$

for $t \geq t_0$. The problem we want to approach is to find the mean $\boldsymbol{\mu}_{\mathbf{u}}(t)$ and the covariance $\mathbf{P}_{\mathbf{u}}(t_1,t_2)$, for any $t, t_1, t_2 > t_0$.

The general solution of (3.1) is given by

$$\mathbf{u}(t) = \boldsymbol{\Psi}(t,t_0)\mathbf{u}(t_0) + \int_{t_0}^{t} \boldsymbol{\Psi}(t,\tau)\mathbf{G}(\tau)\mathbf{w}(\tau)\,d\tau\,, \tag{3.5}$$

where $\boldsymbol{\Psi}(t,t_0)$ is the transition matrix of the system, the solution of the homogeneous linear differential equation:

$$\dot{\boldsymbol{\Psi}}(t,t_0) = \frac{d\boldsymbol{\Psi}(t,t_0)}{dt} = \mathbf{F}(t)\boldsymbol{\Psi}(t,t_0)\,, \tag{3.6}$$

with initial condition

$$\boldsymbol{\Psi}(t_0,t_0) = \mathbf{I}\,, \tag{3.7}$$

where $\mathbf{I}$ is the identity matrix.

We can determine the mean of $\mathbf{u}(t)$ by applying the ensemble mean operator to (3.5):

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{u}}(t) \equiv \mathcal{E}\{\mathbf{u}(t)\} &= \mathcal{E}\{\boldsymbol{\Psi}(t,t_0)\mathbf{u}(t_0)\} + \mathcal{E}\{\int_{t_0}^{t} \boldsymbol{\Psi}(t,\tau)\mathbf{G}(\tau)\mathbf{w}(\tau)\,d\tau\} \\ &= \boldsymbol{\Psi}(t,t_0)\boldsymbol{\mu}_{\mathbf{u}}(t_0) + \int_{t_0}^{t} \boldsymbol{\Psi}(t,\tau)\mathbf{G}(\tau)\boldsymbol{\mu}_{\mathbf{w}}(\tau)\,d\tau \end{aligned} \tag{3.8}$$

where the last equality is obtained by exchanging the ensemble mean operator with the integration operator, since they act on different variables. In this way, the mean of the process $\mathbf{u}(t)$ satisfies an expression analogous to the solution of the equation (3.1), except that the processes $\mathbf{u}$ and $\mathbf{w}$ are substituted by their respective means.

The integration in (3.8) is complicated in the majority of cases, therefore it is convenient that we determine an auxiliary expression to obtain the mean of $\mathbf{u}(t)$. This can be done by applying the ensemble mean operator directly to the equation (3.1). In this case we obtain:

$$\dot{\boldsymbol{\mu}}_{\mathbf{u}}(t) = \mathbf{F}(t)\boldsymbol{\mu}_{\mathbf{u}}(t) + \mathbf{G}(t)\boldsymbol{\mu}_{\mathbf{w}}(t)\,, \tag{3.9}$$

subject to the initial condition $\boldsymbol{\mu}_{\mathbf{u}}(t_0)$. The solution of this equation is obviously given by (3.8); however, computationally the equation above has a much simpler solution.

Before deriving the equation for the (co)variance $\mathbf{P}_{\mathbf{u}}(t_1, t_2)$ it is helpful to notice that from the definition of cross–covariance $\mathbf{P}_{\mathbf{uv}}(t_1, t_2) = cov\{\mathbf{u}(t_1), \mathbf{v}(t_2)\}$, for general n–vectors $\mathbf{u}(t_1)$ and m–vectors $\mathbf{v}(t_2)$, it follows that if we change $\mathbf{u}(t_1) \rightarrow \mathbf{A}(t)\mathbf{u}(t_1)$, for any non–stochastic $n \times n$ matrix $\mathbf{A}$(t), we have

$$\begin{aligned}
cov\{\mathbf{A}(t)\mathbf{u}(t_1), \mathbf{v}(t_2)\} &= \mathcal{E}\{[\mathbf{A}(t)\mathbf{u}(t_1) - \mathcal{E}\{\mathbf{A}(t)\mathbf{u}(t_1)\}][\mathbf{v}(t_2) - \mathcal{E}\{\mathbf{v}(t_2)\}]^T\} \\
&= \mathcal{E}\{[\mathbf{A}(t)\mathbf{u}(t_1) - \mathbf{A}(t)\mathcal{E}\{\mathbf{u}(t_1)\}][\mathbf{v}(t_2) - \mathcal{E}\{\mathbf{v}(t_2)\}]^T\} \\
&= \mathcal{E}\{[\mathbf{A}(t)[\mathbf{u}(t_1) - \mathcal{E}\{\mathbf{u}(t_1)\}]][\mathbf{v}(t_2) - \mathcal{E}\{\mathbf{v}(t_2)\}]^T\} \\
&= \mathbf{A}(t)\mathcal{E}\{[\mathbf{u}(t_1) - \mathcal{E}\{\mathbf{u}(t_1)\}][\mathbf{v}(t_2) - \mathcal{E}\{\mathbf{v}(t_2)\}]^T\} \\
&= \mathbf{A}(t)cov\{\mathbf{u}(t_1), \mathbf{v}(t_2)\} \\
&= \mathbf{A}(t)\mathbf{P}_{\mathbf{uv}}(t_1, t_2) \qquad (3.10)
\end{aligned}$$

Analogously, if instead of changing $\mathbf{u}(t_1)$, we had changed $\mathbf{v}(t_2) \rightarrow \mathbf{B}(\tau)\mathbf{v}(t_2)$, for an arbitrary non–stochastic $m \times m$ matrix $\mathbf{B}(\tau)$, we would have

$$\begin{aligned}
cov\{\mathbf{u}(t_1), \mathbf{B}(\tau)\mathbf{v}(t_2)\} &= \mathcal{E}\{[\mathbf{u}(t_1) - \mathcal{E}\{\mathbf{u}(t_1)\}][\mathbf{B}(\tau)\mathbf{v}(t_2) - \mathcal{E}\{\mathbf{B}(\tau)\mathbf{v}(t_2)\}]^T\} \\
&= \mathcal{E}\{[\mathbf{u}(t_1) - \mathcal{E}\{\mathbf{u}(t_1)\}][\mathbf{B}(\tau)[\mathbf{v}(t_2) - \mathcal{E}\{\mathbf{v}(t_2)\}]^T]\} \\
&= \mathcal{E}\{[\mathbf{u}(t_1) - \mathcal{E}\{\mathbf{u}(t_1)\}][\mathbf{v}(t_2) - \mathcal{E}\{\mathbf{v}(t_2)\}]^T\mathbf{B}^T(\tau)\} \\
&= \mathcal{E}\{[\mathbf{u}(t_1) - \mathcal{E}\{\mathbf{u}(t_1)\}][\mathbf{v}(t_2) - \mathcal{E}\{\mathbf{v}(t_2)\}]^T\}\mathbf{B}^T(\tau) \\
&= cov\{\mathbf{u}(t_1), \mathbf{v}(t_2)\}\mathbf{B}^T(\tau) \\
&= \mathbf{P}_{\mathbf{uv}}(t_1, t_2)\mathbf{B}^T(\tau) \qquad (3.11)
\end{aligned}$$

It is left for the reader to show that if both transformations on the vectors $\mathbf{u}(t_1)$ and $\mathbf{v}(t_2)$ are performed simultaneously, it follows that,

$$cov\{\mathbf{A}(t)\mathbf{u}(t_1), \mathbf{B}(\tau)\mathbf{v}(t_2)\} = \mathbf{A}(t)\mathbf{P}_{\mathbf{uv}}(t_1, t_2)\mathbf{B}^T(\tau) \qquad (3.12)$$

Using the relationships above and the solution (3.5) we can derive the expression for the covariance, that is,

$$\begin{aligned}
\mathbf{P}_{\mathbf{u}}(t_1, t_2) &\equiv cov\{\mathbf{u}(t_1), \mathbf{u}(t_2)\} \\
&= \boldsymbol{\Psi}(t_1, t_0)cov\{\mathbf{u}(t_0), \mathbf{u}(t_0)\}\boldsymbol{\Psi}^T(t_2, t_0) \\
&\quad + \boldsymbol{\Psi}(t_1, t_0)cov\left\{\mathbf{u}(t_0), \int_{t_0}^{t_2} \boldsymbol{\Psi}(t_2, \tau)\mathbf{G}(\tau)\mathbf{w}(\tau)\, d\tau\right\} \\
&\quad + cov\left\{\int_{t_0}^{t_1} \boldsymbol{\Psi}(t_1, \tau)\mathbf{G}(\tau)\mathbf{w}(\tau)\, d\tau, \boldsymbol{\Psi}(t_2, t_0)\mathbf{u}(t_0)\right\} \\
&\quad + cov\left\{\int_{t_0}^{t_1} \boldsymbol{\Psi}(t_1, \tau_1)\mathbf{G}(\tau_1)\mathbf{w}(\tau_1)\, d\tau_1,\right. \\
&\qquad \left.\int_{t_0}^{t_2} \boldsymbol{\Psi}(t_2, \tau_2)\mathbf{G}(\tau_2)\mathbf{w}(\tau_2)\, d\tau_2\right\} \qquad (3.13)
\end{aligned}$$

or also,

$$\mathbf{P}_{\mathbf{u}}(t_1, t_2) = \boldsymbol{\Psi}(t_1, t_0)\mathbf{P}_{\mathbf{u}}(t_0)\boldsymbol{\Psi}^T(t_2, t_0)$$

$$
\begin{aligned}
&+ \boldsymbol{\Psi}(t_1,t_0)\int_{t_0}^{t_2} cov\{\mathbf{u}(t_0),\mathbf{w}(\tau)\}\mathbf{G}^T(\tau)\boldsymbol{\Psi}^T(t_2,\tau)\,d\tau \\
&+ \int_{t_0}^{t_1} \boldsymbol{\Psi}(t_1,\tau)\mathbf{G}(\tau)cov\{\mathbf{w}(\tau),\mathbf{u}(t_0)\}\boldsymbol{\Psi}^T(t_2,t_0)\,d\tau \\
&+ \int_{t_0}^{t_1} d\tau_1 \int_{t_0}^{t_2} d\tau_2 \boldsymbol{\Psi}(t_1,\tau_1)\mathbf{G}(\tau_1)\mathbf{P}_{\mathbf{w}}(\tau_1,\tau_2)\mathbf{G}^T(\tau_2)\boldsymbol{\Psi}^T(t_2,\tau_2)\,.
\end{aligned}
\tag{3.14}
$$

Utilizing the condition that $\mathbf{w}(t)$ and $\mathbf{u}(t_0)$ are uncorrelated (3.4), the expression above can be written as:

$$
\begin{aligned}
\mathbf{P}_{\mathbf{u}}(t_1,t_2) &= \boldsymbol{\Psi}(t_1,t_0)\mathbf{P}_{\mathbf{u}}(t_0)\boldsymbol{\Psi}^T(t_2,t_0) \\
&\quad + \int_{t_0}^{t_1} d\tau_1 \int_{t_0}^{t_2} d\tau_2 \boldsymbol{\Psi}(t_1,\tau_1)\mathbf{G}(\tau_1)\mathbf{P}_{\mathbf{w}}(\tau_1,\tau_2)\mathbf{G}^T(\tau_2)\boldsymbol{\Psi}^T(t_2,\tau_2)\,,
\end{aligned}
\tag{3.15}
$$

for $t_1, t_2 > t_0$.This expression is of little utility in this form. Therefore, let us suppose that the process $\mathbf{w}(t)$ is a white noise, which means that the covariance $\mathbf{P}_{\mathbf{w}}(\tau_1,\tau_2)$ in this case is given by:

$$
\mathbf{P}_{\mathbf{w}}(\tau_1,\tau_2) = \mathbf{Q}(\tau_1)\delta(\tau_1-\tau_2)\,, \tag{3.16}
$$

where $\delta(\tau)$ is the symmetric Dirac delta (distribution) function, defined by:

$$
\int_a^b f(\tau)\delta(\tau-t)\,d\tau = \begin{cases} 0\,, & \text{if } t<a \text{ or } t>b \\ \frac{f(a)}{2}\,, & \text{if } t=a \\ \frac{f(b)}{2}\,, & \text{if } t=b \\ f(t)\,, & \text{if } a<t<b \end{cases} \quad . \tag{3.17}
$$

Supposing, for the moment, that $t_1 < t_2$, and using the fact that $\mathbf{w}(t)$ is a white noise, the expression (3.15) can be decomposed as:

$$
\begin{aligned}
&\int_{t_0}^{t_1} d\tau_1 \int_{t_0}^{t_2} d\tau_2 \boldsymbol{\Psi}(t_1,\tau_1)\mathbf{G}(\tau_1)\mathbf{P}_{\mathbf{w}}(\tau_1,\tau_2)\mathbf{G}^T(\tau_2)\boldsymbol{\Psi}^T(t_2,\tau_2) = \\
&\int_{t_0}^{t_1} d\tau_1 \left\{ \int_{t_0}^{t_1} d\tau_2 \boldsymbol{\Psi}(t_1,\tau_1)\mathbf{G}(\tau_1)\mathbf{P}_{\mathbf{w}}(\tau_1,\tau_2)\mathbf{G}^T(\tau_2)\boldsymbol{\Psi}^T(t_2,\tau_2) \right. \\
&\qquad \left. + \int_{t_1}^{t_2} d\tau_2 \boldsymbol{\Psi}(t_1,\tau_1)\mathbf{G}(\tau_1)\mathbf{P}_{\mathbf{w}}(\tau_1,\tau_2)\mathbf{G}^T(\tau_2)\boldsymbol{\Psi}^T(t_2,\tau_2) \right\} = \\
&\int_{t_0}^{t_1} d\tau_1 \boldsymbol{\Psi}(t_1,\tau_1)\mathbf{G}(\tau_1)\mathbf{Q}(\tau_1) \left\{ \int_{t_0}^{t_1} d\tau_2\, \delta(\tau_1-\tau_2)\mathbf{G}^T(\tau_2)\boldsymbol{\Psi}^T(t_2,\tau_2) \right. \\
&\qquad \left. + \int_{t_1}^{t_2} d\tau_2\, \delta(\tau_1-\tau_2)\mathbf{G}^T(\tau_2)\boldsymbol{\Psi}^T(t_2,\tau_2) \right\} = \\
&\int_{t_0}^{t_1} d\tau_1 \boldsymbol{\Psi}(t_1,\tau_1)\mathbf{G}(\tau_1)\mathbf{Q}(\tau_1)\mathbf{G}^T(\tau_1)\boldsymbol{\Psi}^T(t_2,\tau_1)
\end{aligned}
\tag{3.18}
$$

where the last equality is obtained by using the definition of the symmetric Dirac function (3.17): the second integral within the brackets gives no contribution. An equivalent argument applies when we take $t_1 > t_2$, however, for this case the integration over the interval $[t_0, t_1$ should be carried over first.

Substituting this result in (3.15) we have that

$$\begin{aligned}\mathbf{P_u}(t_1,t_2) &= \boldsymbol{\Psi}(t_1,t_0)\mathbf{P_u}(t_0)\boldsymbol{\Psi}^T(t_2,t_0) \\ &\quad + \int_{t_0}^{\min(t_1,t_2)} \boldsymbol{\Psi}(t_1,\tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^T(\tau)\boldsymbol{\Psi}^T(t_2,\tau)\,, \end{aligned} \tag{3.19}$$

which is a much simpler expression. The equation for the variance $\mathbf{P_u}(t)$ can be found by letting $t_1 = t_2 = t$:

$$\begin{aligned}\mathbf{P_u}(t) &= \boldsymbol{\Psi}(t,t_0)\mathbf{P_u}(t_0)\boldsymbol{\Psi}^T(t,t_0) \\ &\quad + \int_{t_0}^{t} \boldsymbol{\Psi}(t,\tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^T(\tau)\boldsymbol{\Psi}^T(t,\tau)\,. \end{aligned} \tag{3.20}$$

An analogous comment to the one made for the solution of the equation for the mean is applicable for the variance and covariance expressions, that is, the integrations in (3.19) and (3.20) are very complicated and it is worthwhile to search for simpler expressions to work with. Direct differentiation of the solution (3.20) leads to a differential equation for the variance (see Exercise 3.1):

$$\dot{\mathbf{P}}_\mathbf{u}(t) = \mathbf{F}(t)\mathbf{P_u}(t) + \mathbf{P_u}(t)\mathbf{F}^T(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t)\,, \tag{3.21}$$

which can be obtained in an even simpler way by means of applying the definition of variance to the quantity $\dot{\mathbf{P}}_\mathbf{u}(t)$. The equation (3.21) is known as the Lyapunov equation. Given the initial condition of the variance $\mathbf{P_u}(t_0)$, the Lyapunov equation determines the dynamic evolution of the variance for any $t > t_0$. Notice that the Lyapunov equation does not require knowledge of the transition matrix $\boldsymbol{\Psi}$.

It is possible to show that for the case of white noise the covariance $\mathbf{P_u}(t_1,t_2)$ can be obtained by means of the expressions:

$$\mathbf{P_u}(t_1,t_2) = \begin{cases} \boldsymbol{\Psi}(t_1,t_2)\mathbf{P_u}(t_2)\,, & \text{if } t_1 > t_2 \\ \mathbf{P_u}(t_1)\boldsymbol{\Psi}^T(t_2,t_1)\,, & \text{if } t_1 < t_2 \end{cases}\,, \tag{3.22}$$

(see Exercise 3.2).

3.1.2 Discrete Processes

Consider now the first-order discrete n–dimensional dynamical system:

$$\mathbf{u}(k+1) = \boldsymbol{\Psi}(k+1,k)\mathbf{u}(k) + \boldsymbol{\Gamma}(k)\mathbf{w}(k)\,, \tag{3.23}$$

where the notation $\mathbf{u}(k)$ stands for $\mathbf{u}(t_k)$, $T = t_{k+1} - t_k$ is the sampling interval, and the noise $\mathbf{w}(k)$ is a vector of dimension m.

Analogously to the continuous case, we define the mean and the covariance of the discrete processes $\mathbf{u}(0)$ and $\mathbf{w}(k)$ as:

$$\boldsymbol{\mu}_\mathbf{w}(k) = \mathcal{E}\{\mathbf{w}(k)\} \qquad \mathbf{P_w}(k,j) = cov\{\mathbf{w}(k),\mathbf{w}(j)\} \tag{3.24}$$

$$\boldsymbol{\mu}_\mathbf{u}(0) = \mathcal{E}\{\mathbf{u}(0)\} \qquad \mathbf{P_u}(0) = var\{\mathbf{u}(0)\} \tag{3.25}$$

also, the processes $\mathbf{u}(0)$ and $\mathbf{w}(k)$ are considered to be uncorrelated,

$$cov\{\mathbf{u}(0), \mathbf{w}(k)\} = \mathbf{0}\,, \tag{3.26}$$

for all $k \geq 0$, and the matrices $\mathbf{P_w}$ and $\mathbf{P_u}$ are $m \times m$ and $n \times n$ dimensional, respectively.

To obtain the general solution of (3.23) let us write its corresponding expression for $\mathbf{u}(1)$ and $\mathbf{u}(2)$, that is,

$$\mathbf{u}(1) = \mathbf{\Psi}(1,0)\mathbf{u}(0) + \mathbf{\Gamma}(0)\mathbf{w}(0) \tag{3.27}$$

and

$$\mathbf{u}(2) = \mathbf{\Psi}(2,1)\mathbf{u}(1) + \mathbf{\Gamma}(1)\mathbf{w}(1) \tag{3.28}$$

respectively. Substituting the first equation into the second we get

$$\begin{aligned} \mathbf{u}(2) &= \mathbf{\Psi}(2,1)[\mathbf{\Psi}(1,0)\mathbf{u}(0) + \mathbf{\Gamma}(0)\mathbf{w}(0)] + \mathbf{\Gamma}(1)\mathbf{w}(1) \\ &= \mathbf{\Psi}(2,0)\mathbf{u}(0) + \sum_{j=0}^{1} \mathbf{\Psi}(2,j+1)\mathbf{\Gamma}(j)\mathbf{w}(j)\,, \end{aligned} \tag{3.29}$$

where we noticed that $\mathbf{\Psi}(2,0) = \mathbf{\Psi}(2,1)\mathbf{\Psi}(1,0)$, and $\mathbf{\Psi}(2,2) = \mathbf{I}$. Continuing this procedure, we can show that the solution of the equation (3.23) can be written as

$$\mathbf{u}(k) = \mathbf{\Psi}(k,0)\mathbf{u}(0) + \sum_{j=0}^{k-1} \mathbf{\Psi}(k,j+1)\mathbf{\Gamma}(j)\mathbf{w}(j)\,, \tag{3.30}$$

for $k > 0$,

$$\mathbf{\Psi}(k,0) = \mathbf{\Psi}(k,k-1)\mathbf{\Psi}(k-1,k-2)\ldots\mathbf{\Psi}(2,1)\mathbf{\Psi}(1,0) \tag{3.31}$$

and $\mathbf{\Psi}(k,k) = \mathbf{I}$ for all k.

The expression for the mean is calculated by applying the ensemble mean operator to the solution written above, that is,

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{u}}(k) &= \mathcal{E}\{\mathbf{u}(k)\} \\ &= \mathcal{E}\{\mathbf{\Psi}(k,0)\mathbf{u}(0)\} + \mathcal{E}\{\sum_{j=0}^{k-1} \mathbf{\Psi}(k,j+1)\mathbf{\Gamma}(j)\mathbf{w}(j)\} \\ &= \mathbf{\Psi}(k,0)\boldsymbol{\mu}_{\mathbf{u}}(0) + \sum_{j=0}^{k-1} \mathbf{\Psi}(k,j+1)\mathbf{\Gamma}(j)\boldsymbol{\mu}_{\mathbf{w}}(j)\,, \end{aligned} \tag{3.32}$$

since the ensemble mean operator acts only on the stochastic quantities $\mathbf{u}(0)$ and $\mathbf{w}(j)$. An alternative, recursive equation for the mean can be obtained by applying the ensemble mean operator directly to (3.23), that is,

$$\boldsymbol{\mu}_{\mathbf{u}}(k+1) = \mathbf{\Psi}(k+1,k)\boldsymbol{\mu}_{\mathbf{u}}(k) + \mathbf{\Gamma}(k)\boldsymbol{\mu}_{\mathbf{w}}(k)\,. \tag{3.33}$$

Before we determine the covariance $\mathbf{P_u}(k,j) = cov\{\mathbf{u}(k), \mathbf{u}(j)\}$, it is useful to recognize that a general cross–covariance $\mathbf{P_{uv}}(k,j) = cov\{\mathbf{u}(k), \mathbf{v}(j)\}$, for arbitrary n–vectors $\mathbf{u}(k)$, and

m–vectors $\mathbf{v}(j)$, and arbitrary non–stochastic $n \times n$ matrices $\mathbf{A}(k')$, and $m \times m$ matrices $\mathbf{B}(j')$, we can write

$$
\begin{aligned}
cov\{\mathbf{A}(k')\mathbf{u}(k), \mathbf{v}(j)\} &= \mathbf{A}(k')\mathbf{P}_{\mathbf{uv}}(k,j) && \text{(3.34a)}\\
cov\{\mathbf{u}(k), \mathbf{B}(j')\mathbf{v}(j)\} &= \mathbf{P}_{\mathbf{uv}}(k,j)\mathbf{B}^T(j') && \text{(3.34b)}\\
cov\{\mathbf{A}(k')\mathbf{u}(k), \mathbf{B}(j')\mathbf{v}(j)\} &= \mathbf{A}(k')\mathbf{P}_{\mathbf{uv}}(k,j)\mathbf{B}^T(j') && \text{(3.34c)}
\end{aligned}
$$

which can be demonstrated in an analogous way as done for the continuous–time case.

Using the relations above and (3.30, the covariance $\mathbf{P}_{\mathbf{u}}(k,j)$ can be calculated by

$$
\begin{aligned}
\mathbf{P}_{\mathbf{u}}(k,j) &\equiv cov\{\mathbf{u}(k)\mathbf{u}(j)\}\\
&= \mathbf{\Psi}(k,0)\mathbf{P}_{\mathbf{u}}(0)\mathbf{\Psi}^T(j,0)\\
&\quad + \mathbf{\Psi}(k,0)\sum_{\ell=0}^{j-1} cov\{\mathbf{u}(0), \mathbf{w}(\ell)\}\mathbf{\Gamma}^T(\ell)\mathbf{\Psi}^T(j,\ell+1)\\
&\quad + \sum_{i=0}^{k-1} \mathbf{\Psi}(k,i+1)\mathbf{\Gamma}(i)cov\{\mathbf{w}(i), \mathbf{u}(0)\}\mathbf{\Psi}^T(j,0)\\
&\quad + \sum_{i=0}^{k-1}\sum_{\ell=0}^{j-1} \mathbf{\Psi}(k,i+1)\mathbf{\Gamma}(i)\mathbf{P}_{\mathbf{w}}(i,\ell)\mathbf{\Gamma}^T(\ell)\mathbf{\Psi}^T(j,\ell+1)\,, \qquad \text{(3.35)}
\end{aligned}
$$

where $\mathbf{P}_{\mathbf{w}}(k,j) = cov\{\mathbf{w}(k), \mathbf{w}(j)\}$. In this way, the assumption of decorrelation (3.26) leads us to write

$$
\begin{aligned}
\mathbf{P}_{\mathbf{u}}(k,j) &= \mathbf{\Psi}(k,0)\mathbf{P}_{\mathbf{u}}(0)\mathbf{\Psi}^T(j,0)\\
&\quad + \sum_{i=0}^{k-1}\sum_{\ell=0}^{j-1} \mathbf{\Psi}(k,i+1)\mathbf{\Gamma}(i)\mathbf{P}_{\mathbf{w}}(i,\ell)\mathbf{\Gamma}^T(\ell)\mathbf{\Psi}^T(j,\ell+1)\,, \qquad \text{(3.36)}
\end{aligned}
$$

which is the general expression for the covariance of the discrete process (3.23).

As for the continuous case, we can obtain a simpler expression for the covariance if we consider the sequence $\{\mathbf{w}(k)\}$ to be white. Therefore, for the case in which

$$
\mathbf{P}_{\mathbf{w}}(k,j) = \mathbf{Q}_k\delta_{k,j}\,, \qquad \text{(3.37)}
$$

where $\delta_{k,j}$ is the Kronecker delta, the equation for the covariance is reduced to:

$$
\begin{aligned}
\mathbf{P}_{\mathbf{u}}(k,j) &= \mathbf{\Psi}(k,0)\mathbf{P}_{\mathbf{u}}(0)\mathbf{\Psi}^T(j,0)\\
&\quad + \sum_{i=0}^{\min(j-1,k-1)} \mathbf{\Psi}(k,i+1)\mathbf{\Gamma}(i)\mathbf{Q}_i\mathbf{\Gamma}^T(i)\mathbf{\Psi}^T(j,i+1)\,. \qquad \text{(3.38)}
\end{aligned}
$$

A recursive expression for the variance $\mathbf{P}_{\mathbf{u}}(k+1)$ can be obtained directly from the definition of variance and from (3.23). That is, by forming the outer product of (3.23) with itself and by applying the ensemble mean operator it follows that

$$
\begin{aligned}
\mathbf{P}_{\mathbf{u}}(k+1) &= \mathbf{\Psi}(k+1,k)\mathbf{P}_{\mathbf{u}}(k)\mathbf{\Psi}^T(k+1,k)\\
&\quad + \mathbf{\Psi}(k+1,k)cov\{\mathbf{u}(k), \mathbf{w}(k)\}\mathbf{\Gamma}^T(k)\\
&\quad + \mathbf{\Gamma}(k)cov\{\mathbf{w}(k), \mathbf{u}(k)\}\mathbf{\Psi}^T(k+1,k)\\
&\quad + \mathbf{\Gamma}(k)\mathbf{P}_{\mathbf{w}}(k)\mathbf{\Gamma}^T(k)\,. \qquad \text{(3.39)}
\end{aligned}
$$

Since $\mathbf{u}(k)$ depends only on $\mathbf{w}(j)$, for $j < k$, the second and third terms of the expression above vanish, so that the variance can be written as

$$\mathbf{P_u}(k+1) = \Psi(k+1,k)\mathbf{P_u}(k)\Psi^T(k+1,k) + \mathbf{\Gamma}(k)\mathbf{P_w}(k)\mathbf{\Gamma}^T(k)\,, \tag{3.40}$$

which the corresponding discrete Lyapunov equation.

For a white noise sequence $\{\mathbf{w}(k)\}$ we have $\mathbf{P_w}(k) = \mathbf{P_w}(k,k) = \mathbf{Q}_k$, and also

$$\mathbf{P_u}(k,j) = \begin{cases} \Psi(k,j)\mathbf{P_u}(j) & k \geq j \\ \mathbf{P_u}(k)\Psi^T(j,k) & k \leq j \end{cases}\,, \tag{3.41}$$

which can be verified by substitution of (3.40) into (3.38). That is, consider the case $k \geq j$, then:

$$\begin{aligned}
\mathbf{P_u}(k,j) &= \Psi(k,0)\mathbf{P_u}(0)\Psi^T(j,0) \\
&\quad + \sum_{i=0}^{j-1} \Psi(k,i+1)\left[\mathbf{P_u}(i+1)\right. \\
&\qquad \left. -\Psi(i+1,i)\mathbf{P_u}(i)\Psi^T(i+1,i)\right]\Psi^T(j,i+1) \\
&= \Psi(k,0)\mathbf{P_u}(0)\Psi^T(j,0) \\
&\quad + \sum_{i=0}^{j-1} \Psi(k,i+1)\mathbf{P_u}(i+1)\Psi^T(j,i+1) \\
&\quad - \sum_{i=0}^{j-1} \Psi(k,i)\mathbf{P_u}(i)\Psi^T(j,i) \\
&= \sum_{i=0}^{j-1} \Psi(k,i+1)\mathbf{P_u}(i+1)\Psi^T(j,i+1) - \sum_{i=1}^{j-1} \Psi(k,i)\mathbf{P_u}(i)\Psi^T(j,i) \\
&= \sum_{i=1}^{j-1} \Psi(k,i)\mathbf{P_u}(i)\Psi^T(j,i) + \Psi(k,j)\mathbf{P_u}(j)\Psi^T(j,j) \\
&\quad - \sum_{i=1}^{j-1} \Psi(k,i)\mathbf{P_u}(i)\Psi^T(j,i)
\end{aligned} \tag{3.42}$$

where, the two sums in the last equality cancel, and we used $\Psi(j,j) = \mathbf{I}$ to obtain the desired result. The case $k \leq j$ can be obtained in an analogous manner.

3.1.3 Relation between the Continuous and Discrete Cases

A fundamental relation between continuous white noise and discrete white noise is that, as the sample of the discrete stochastic process becomes dense, the covariance of the discrete white process:

$$cov\{\mathbf{w}(kT), \mathbf{w}(jT)\} = \mathbf{Q}_k\delta_{k,j} \tag{3.43}$$

becomes the covariance of the continuous process:

$$cov\{\mathbf{w}(t), \mathbf{w}(\tau)\} = \lim_{k,j\to\infty} cov\{\mathbf{w}(kT), \mathbf{w}(jT)\} = \mathbf{Q}(t)\delta(t-\tau) \tag{3.44}$$

where the limit is also taken for $kT \to t$, $jT \to \tau$, and also for $T \to 0$. The variances in the expressions above are related by

$$\mathbf{Q}(t = kT) = T\,\mathbf{Q}_k \tag{3.45}$$

where some care should be taken with the notation used here: in spite of the fact that the variance matrices for the discrete and continuous processes above are represented by the same letter, they are in fact distinct matrices; the distinction is made by using subscripts in the discrete case $\mathbf{Q}_k$, in contrast to the explicit functional time dependence $\mathbf{Q}(t)$ for the continuous case.

The transition matrix of the continuous system can be written formally as

$$\mathbf{\Psi}(t, t_0) = \exp\{\int_{t_0}^{t} \mathbf{F}(s)\, ds\} \tag{3.46}$$

so that, for $t = (k+1)T$ and $t_0 = kT$, we can write

$$\begin{aligned} \mathbf{\Psi}((k+1)T, kT) &= \mathbf{I} + \int_{kT}^{(k+1)T} \mathbf{F}(s)\, ds \\ &\approx \mathbf{I} + \mathbf{F}(kT)T\,, \end{aligned} \tag{3.47}$$

where we made a gross approximation of the integral — which becomes reasonable as the sample becomes dense.

Substituting (3.45) and (3.47) in (3.40) we have that

$$\mathbf{P_u}((k+1)T) = [\mathbf{I} + T\mathbf{F}(kT)]\mathbf{P_u}(kT)[\mathbf{I} + T\mathbf{F}(kT)]^T + T\mathbf{G}(kT)\mathbf{Q}(kT)\mathbf{G}^T(kT)\,, \tag{3.48}$$

where we made the correspondence: $\mathbf{G}(t = kT) = \mathbf{\Gamma}(kT)/T$. The expression above can be also written as

$$\begin{aligned} \mathbf{P_u}((k+1)T) &= \mathbf{P_u}(kT) + T\mathbf{F}(kT)\mathbf{P_u}(kT) + T\mathbf{P_u}(kT)\mathbf{F}^T(kT) \\ &\quad + T\mathbf{G}(kT)\mathbf{Q}(kT)\mathbf{G}^T(kT) + o(T^2)\,, \end{aligned} \tag{3.49}$$

so that in the limit $T \to 0$, and $kT \to t$, we have

$$\begin{aligned} \dot{\mathbf{P}}_\mathbf{u}(t) &= \lim_{T\to 0} \frac{\mathbf{P_u}((k+1)T) - \mathbf{P_u}(kT)}{T} \\ &= \mathbf{F}(t)\mathbf{P_u}(t) + \mathbf{P_u}(t)\mathbf{F}^T(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t)\,, \end{aligned} \tag{3.50}$$

where we retained only the terms of lower order in T. This means that the limit for the discrete variance evolution equation as the sample time becomes dense is given by the Lyapunov equation.

3.2 Nonlinear Dynamical Systems

3.2.1 Continuous Processes

Consider now the system of nonlinear differential equations for the random n–vector $\mathbf{u}(t)$,

$$\frac{d\mathbf{u}(t)}{dt} = \mathbf{f}[\mathbf{u}(t), t] + \mathbf{G}[\mathbf{u}(t), t]\mathbf{w}(t)\,, \tag{3.51}$$

where $\mathbf{w}(t)$ is a random m–vector white in time, with mean zero and (co)variance $\mathbf{Q}(t)$, that is,

$$\mathcal{E}\{\mathbf{w}(t)\} = \mathbf{0} \tag{3.52a}$$

$$cov\{\mathbf{w}(t), \mathbf{w}(\tau)\} = \mathbf{Q}(t)\delta(t-\tau) \tag{3.52b}$$

moreover, we also assume $\mathbf{w}(t)$ to be Gaussian. The function $\mathbf{G}[\mathbf{u}(t), t]$ is an $n \times m$ matrix.

Formally, the solution of the equation above can be written in the form

$$\mathbf{u}(t) = \mathbf{u}(t_0) + \int_{t_0}^{t} \mathbf{f}[\mathbf{u}(s), s]\, ds + \int_{t_0}^{t} \mathbf{G}[\mathbf{u}(s), s]\, d\mathbf{b}(s)\,, \tag{3.53}$$

where $\{\mathbf{b}(t)\}$ is the Wiener process defined in Section 2.8, since we assumed $\{\mathbf{w}(t)\}$ to be white and Gaussian. The first integral in the solution (3.53) is an ordinary integral (in the sense of Riemann); however, the second integral is questionable, since it involves increments $d\mathbf{b}$ of a function which is not necessarily finite. This last integration can be accomplished by employing possible generalizations of the concept of Lebesgue–Stieltjes integrals to the stochastic realm. One of these generalizations is due to Ito, and it defines what is called stochastic integral calculus.

The treatment of stochastic integrals is beyond the scope of what we intend to cover in this course. It is worth saying that for the case in which the matrix function $\mathbf{G}$ is independent of the process $\mathbf{u}(t)$, there is no difference between the stochastic integral calculus and ordinary calculus, in reference to solving the last integral in (3.53). Therefore, from this point on let us consider a simplified version of (3.51) given by

$$\frac{d\mathbf{u}(t)}{dt} = \mathbf{f}[\mathbf{u}(t), t] + \mathbf{G}(t)\mathbf{w}(t)\,. \tag{3.54}$$

As in the previous sections we are interested in determining the moments of the statistics of the process $\{\mathbf{u}(t)\}$. We know that in the *linear* case, the assumption of a Gaussian driving forcing $\{\mathbf{w}(t)\}$ implies that the process $\{\mathbf{u}(t)\}$ is Gaussian as well, for Gaussian $\{\mathbf{u}(0)\}$. For this reason we concentrated on deriving equations for the first two moments in the previous sections. In the *nonlinear* case, the Gaussian assumptions on $\{\mathbf{u}(0)\}$ and $\{\mathbf{w}(t)\}$ do not guarantee the process $\{\mathbf{u}(t)\}$ to be Gaussian, consequently, even for Gaussian initial condition and driving forcing all moments are required in principle to describe the statistics of the process $\{\mathbf{u}(t)\}$ completely. However, to keep the calculations simple, we are still only going to concentrate in deriving equations for the first two moments of $\{\mathbf{u}(t)\}$.

The easiest way to obtain an expression for the mean is to apply the ensemble mean operator to equation (3.54). Proceeding this way, it follows that

$$\frac{d\boldsymbol{\mu}_{\mathbf{u}}(t)}{dt} = \mathcal{E}\{\mathbf{f}[\mathbf{u}(t), t]\}\,, \tag{3.55}$$

where we used the fact that the process $\{\mathbf{w}(t)\}$ has mean zero. To determine an explicit expression for the right-hand side of the equation above, we expand the function $\mathbf{f}$ about the mean $\boldsymbol{\mu}_{\mathbf{u}}$. In this way, a Taylor expansion up to the second order yields,

$$\begin{aligned} \mathbf{f}[\mathbf{u}(t), t] \approx\ & \mathbf{f}[\boldsymbol{\mu}_{\mathbf{u}}(t), t] + \mathcal{F}'[\boldsymbol{\mu}_{\mathbf{u}}(t), t](\mathbf{u}(t) - \boldsymbol{\mu}_{\mathbf{u}}(t)) \\ & + \frac{1}{2}\mathcal{F}''[\boldsymbol{\mu}_{\mathbf{u}}(t), t]\,(\mathbf{u}(t) - \boldsymbol{\mu}_{\mathbf{u}}(t)) \otimes (\mathbf{u}(t) - \boldsymbol{\mu}_{\mathbf{u}}(t)) \end{aligned} \tag{3.56}$$

where $\mathcal{F}'$ is the $n \times n$ gradient (Jacobian) matrix of $\mathbf{f}$ given by

$$\begin{aligned}\mathcal{F}'[\boldsymbol{\mu}_{\mathbf{u}}(t), t] &= \left.\frac{\partial \mathbf{f}[\mathbf{u}(t), t]}{\partial \mathbf{u}^T(t)}\right|_{\mathbf{u}(t)=\boldsymbol{\mu}_{\mathbf{u}}(t)} \\ &= \left[\frac{\partial \mathbf{f}[\mathbf{u}(t), t]}{\partial u_1} \vdots \frac{\partial \mathbf{f}[\mathbf{u}(t), t]}{\partial u_2} \vdots \cdots \vdots \frac{\partial \mathbf{f}[\mathbf{u}(t), t]}{\partial u_n}\right]\Bigg|_{\mathbf{u}(t)=\boldsymbol{\mu}_{\mathbf{u}}(t)} \end{aligned} \tag{3.57}$$

and $\mathcal{F}''$ is the $n \times n^2$ Hessian matrix given by

$$\begin{aligned}\mathcal{F}''[\boldsymbol{\mu}_{\mathbf{u}}(t), t] &= \left.\frac{\partial^2 \mathbf{f}[\mathbf{u}(t), t]}{\partial \mathbf{u}^T(t) \partial \mathbf{u}^T(t)}\right|_{\mathbf{u}(t)=\boldsymbol{\mu}_{\mathbf{u}}(t)} \\ &= \left.\frac{\partial \mathcal{F}'[\mathbf{u}(t), t]}{\partial \mathbf{u}^T(t)}\right|_{\mathbf{u}(t)=\boldsymbol{\mu}_{\mathbf{u}}(t)} \\ &= \left[\frac{\partial \mathcal{F}'[\mathbf{u}(t), t]}{\partial u_1} \vdots \frac{\partial \mathcal{F}'[\mathbf{u}(t), t]}{\partial u_2} \vdots \cdots \vdots \frac{\partial \mathcal{F}'[\mathbf{u}(t), t]}{\partial u_n}\right]\Bigg|_{\mathbf{u}(t)=\boldsymbol{\mu}_{\mathbf{u}}(t)} \end{aligned} \tag{3.58}$$

Here we are using Vetter's notation [136] and [137] for the calculus of matrices (see also Brewer [18] for an overview of matrix calculus). The operation $\otimes$ represents the Kronecker product for matrices, which for any $n \times m$ matrix $\mathbf{A}$ and $p \times q$ matrix $\mathbf{B}$ is defined as

$$\mathbf{A} \otimes \mathbf{B} \equiv \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1m}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2m}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\mathbf{B} & a_{n2}\mathbf{B} & \dots & a_{nm}\mathbf{B} \end{pmatrix} \tag{3.59}$$

where a_{ij} is the (i, j)–th element of $\mathbf{A}$. The result $\mathbf{A} \otimes \mathbf{B}$ is a matrix of dimension $np \times mq$.

Using this definition, the Kronecker product of a general n–vector $\mathbf{v}$ with itself can be written as

$$\mathbf{v} \otimes \mathbf{v} = \begin{pmatrix} v_1\mathbf{v} \\ v_2\mathbf{v} \\ \vdots \\ v_n\mathbf{v} \end{pmatrix} \tag{3.60}$$

which is a column vector of dimension $n^2 = n^2 \times 1$. Furthermore, let us introduce the notation $vec(.)$ to represent the vector (column string) constructed from the columns of a general $n \times m$ matrix $\mathbf{A}$ as

$$\begin{aligned} vec(\mathbf{A}) &\equiv \left[\mathbf{a}_1^T \, \mathbf{a}_2^T \, \cdots \mathbf{a}_m^T\right]^T \\ &= \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} \end{aligned} \tag{3.61}$$

where $\mathbf{a}_i$, for $i = 1, \cdots, m$, is the i–th n–dimensional column of the matrix $\mathbf{A}$. According to this definition, $vec(\mathbf{A})$ is a vector of dimension $nm = nm \times 1$. Using this notation, it follows that (3.60) we be written as

$$\mathbf{v} \otimes \mathbf{v} = vec[\mathbf{v}\mathbf{v}^T] \tag{3.62}$$

Hence, referring back to (3.56), we see that

$$[\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)] \otimes [\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)] = vec[(\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t))(\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t))^T] \tag{3.63}$$

It is relatively simple to verify that the second-order term in (3.56) can be written explicitly as

$$\begin{aligned} & \mathcal{F}''[\boldsymbol{\mu}_\mathbf{u}(t), t][\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)] \otimes [\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)] \\ = & \sum_{i=1}^{n} \sum_{j=1}^{n} [u_i(t) - \mu_i(t)][u_j(t) - \mu_j(t)] \left. \frac{\partial^2 \mathbf{f}[\mathbf{u}(t), t]}{\partial u_i(t) \partial u_j(t)} \right|_{\mathbf{u}(t)=\boldsymbol{\mu}_\mathbf{u}(t)} \end{aligned} \tag{3.64}$$

where, to simplify the notation, the subscript $\mathbf{u}$ was neglected when we wrote the i–th element μ_i of the mean of $\mathbf{u}(t)$, in the expression above.

Consequently, the equation for the mean, after application of the ensemble mean operator, reduces to

$$\frac{d\boldsymbol{\mu}_\mathbf{u}(t)}{dt} = \mathbf{f}[\boldsymbol{\mu}_\mathbf{u}(t), t] + \frac{1}{2} \mathcal{F}''[\boldsymbol{\mu}_\mathbf{u}(t), t] \, vec[\mathbf{P}_\mathbf{u}(t)] \,, \tag{3.65}$$

where we notice that the first-order term in the Taylor expansion of $\mathbf{f}[\mathbf{u}(t), t]$ is automatically canceled. Therefore, the evolution of the mean depends on the variance $\mathbf{P}_\mathbf{u}(t)$. This is an unpleasant property of nonlinear systems: the evolution of moments of a given order depends on moments of higher order. To solve the equation above, it is necessary to determine an expression for $\mathbf{P}_\mathbf{u}(t)$. As we will observe below, this expression also depends on still higher-order moments, and so on. Consequently, depending on the nonlinearities in $\mathbf{f}[\mathbf{u}(t), t]$, it may be impossible to obtain a closed system of equations that determines completely the statistics of the process. In practice, we seek approximations, known as closures, for the equations of the desired moments, in order to obtain a solvable (closed) system of equations.

A simple approximation for the mean equation is to use only up to the first-order term in the expansion of the function $\mathbf{f}[\mathbf{u}(t), t]$. This leads us to the following expression for the evolution of the mean:

$$\frac{d\boldsymbol{\mu}_\mathbf{u}(t)}{dt} = \mathbf{f}[\boldsymbol{\mu}_\mathbf{u}(t), t] \,, \tag{3.66}$$

which is a closed equation — it is not coupled to any other equation.

To find an equation for the variance we can differentiate its definition, that is,

$$\begin{aligned} \frac{d\mathbf{P}_\mathbf{u}(t)}{dt} &= \frac{d\mathcal{E}\{[\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)][\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)]^T\}}{dt} \\ &= \mathcal{E}\left\{[\dot{\mathbf{u}}(t) - \dot{\boldsymbol{\mu}}_\mathbf{u}(t)][\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)]^T + [\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)][\dot{\mathbf{u}}(t) - \dot{\boldsymbol{\mu}}_\mathbf{u}(t)]^T\right\} \\ &= \mathcal{E}\Big\{[\mathbf{f}[\mathbf{u}(t), t] - \dot{\boldsymbol{\mu}}_\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t)]\,[\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)]^T \\ &\quad + [\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)]\,[\mathbf{f}[\mathbf{u}(t), t] - \dot{\boldsymbol{\mu}}_\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t)]^T\Big\} \\ &= \mathcal{E}\Big\{[\mathbf{f}[\mathbf{u}(t), t] - \dot{\boldsymbol{\mu}}_\mathbf{u}(t)]\,[\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)]^T + \mathbf{G}(t)\mathbf{w}(t)[\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)]^T \\ &\quad + [\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)]\,[\mathbf{f}[\mathbf{u}(t), t] - \dot{\boldsymbol{\mu}}_\mathbf{u}(t)]^T + [\mathbf{u}(t) - \boldsymbol{\mu}_\mathbf{u}(t)]\mathbf{w}^T(t)\mathbf{G}^T(t)\Big\} \end{aligned} \tag{3.67}$$

where again we use the notation $\dot{(.)}$ to indicate differentiation with respect to time. This expression becomes very complicated when taken to high order in the expansion of $\mathbf{f}$ about the mean $\boldsymbol{\mu}_{\mathbf{u}}(t)$. This would lead to an equation for the variance depending on moments of higher order, thus not being a closed equation, just as it happened for the mean equation.

For the sake of simplicity, let us consider the second-order approximation (3.56) for $\mathbf{f}[\mathbf{u}(t),t)]$, as well as the second-order approximation for the evolution of the mean (3.65), so that we can write

$$\begin{aligned}
\mathbf{f}[\mathbf{u}(t),t] - \dot{\boldsymbol{\mu}}_{\mathbf{u}}(t) &= \mathbf{f}[\boldsymbol{\mu}_{\mathbf{u}}(t),t] + \mathcal{F}'[\boldsymbol{\mu}_{\mathbf{u}}(t),t](\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t)) \\
&\quad + \frac{1}{2}\mathcal{F}''[\boldsymbol{\mu}_{\mathbf{u}}(t),t]\,(\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t)) \otimes (\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t)) \\
&\quad - \mathbf{f}[\boldsymbol{\mu}_{\mathbf{u}}(t),t] - \frac{1}{2}\,\mathcal{F}''[\boldsymbol{\mu}_{\mathbf{u}}(t),t]\,vec[\mathbf{P}_{\mathbf{u}}(t)] \\
&= \mathcal{F}'[\boldsymbol{\mu}_{\mathbf{u}}(t),t](\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t)) \\
&\quad + \frac{1}{2}\mathcal{F}''[\boldsymbol{\mu}_{\mathbf{u}}(t),t]\,\{\,(\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t)) \otimes (\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t)) \\
&\qquad\qquad - vec[\mathbf{P}_{\mathbf{u}}(t)]\,\} \qquad (3.68)
\end{aligned}$$

Substituting this result into (3.67) yields

$$\begin{aligned}
\frac{d\mathbf{P}_{\mathbf{u}}(t)}{dt} &= \mathcal{F}'[\boldsymbol{\mu}_{\mathbf{u}}(t),t]\mathbf{P}_{\mathbf{u}}(t) + \mathbf{P}_{\mathbf{u}}(t)\mathcal{F}'^{T}[\boldsymbol{\mu}_{\mathbf{u}}(t),t] \\
&\quad + \frac{1}{2}\mathcal{F}''[\boldsymbol{\mu}_{\mathbf{u}}(t),t]\mathcal{E}\,\{[(\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t)) \otimes (\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t)) \\
&\qquad\qquad - vec[\mathbf{P}_{\mathbf{u}}(t)]]\,(\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t))^{T}\Big\} \\
&\quad + \frac{1}{2}\mathcal{E}\,\{(\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t))\,[(\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t)) \otimes (\mathbf{u}(t)-\boldsymbol{\mu}_{\mathbf{u}}(t)) \\
&\qquad\qquad - vec[\mathbf{P}_{\mathbf{u}}(t)]]^{T}\Big\}\,\mathcal{F}''^{T}[\boldsymbol{\mu}_{\mathbf{u}}(t),t] \\
&\quad + \mathbf{G}(t)\mathcal{E}\{\mathbf{w}(t)\mathbf{u}^{T}(t)\} + \mathcal{E}\{\mathbf{u}(t)\mathbf{w}^{T}(t)\}\mathbf{G}^{T}(t)\,. \qquad (3.69)
\end{aligned}$$

The third and fourth terms of this expression refer to the third-order moments. To determine a closed set of evolution equations for the mean and variance, we ignore moments of order higher than two. Therefore, it follows that

$$\begin{aligned}
\frac{d\mathbf{P}_{\mathbf{u}}(t)}{dt} &= \mathcal{F}'[\boldsymbol{\mu}_{\mathbf{u}}(t),t]\mathbf{P}_{\mathbf{u}}(t) + \mathbf{P}_{\mathbf{u}}(t)\mathcal{F}'^{T}[\boldsymbol{\mu}_{\mathbf{u}}(t),t] \\
&\quad + \mathbf{G}(t)\mathcal{E}\{\mathbf{w}(t)\mathbf{u}^{T}(t)\} + \mathcal{E}\{\mathbf{u}(t)\mathbf{w}^{T}(t)\}\mathbf{G}^{T}(t)\,, \qquad (3.70)
\end{aligned}$$

where the terms containing explicitly the ensemble mean can be evaluated by means of the formal solution (3.53) corresponding to the equation (3.54). That is, by evaluating the term $\mathcal{E}\{\mathbf{w}(t)\mathbf{u}^{T}(t)\}$, we have:

$$\begin{aligned}
\mathcal{E}\{\mathbf{w}(t)\mathbf{u}^{T}(t)\} &= \mathcal{E}\{\mathbf{w}(t)\,\mathbf{u}^{T}(t_0)\} + \int_{t_0}^{t}\mathcal{E}\{\mathbf{w}(t)\mathbf{f}^{T}[\mathbf{u}(s),s]\}\,ds \\
&\quad + \int_{t_0}^{t}\mathcal{E}\{\mathbf{w}(t)\mathbf{w}^{T}(s)\}\mathbf{G}^{T}(s)\,ds \\
&= \int_{t_0}^{t}\mathcal{E}\{\mathbf{w}(t)\mathbf{w}^{T}(s)\}\mathbf{G}^{T}(s)\,ds\,, \qquad (3.71)
\end{aligned}$$

where the second equality is obtained by noticing that the first term at the right of the first equality vanishes since $\mathbf{u}(t_0)$ and $\mathbf{w}(t)$ are uncorrelated; and the second term at the right of the first equality vanishes when we use the expansion of $\mathbf{f}[\mathbf{u}(t), t]$ about the mean, because the process $\mathbf{w}(t)$ has mean zero and we disregard moments of order higher than two, that is,

$$\begin{aligned}\int_{t_0}^{t} \mathcal{E}\{\mathbf{w}(t)\mathbf{f}^T[\mathbf{u}(s), s]\}\, ds &= \int_{t_0}^{t} \mathcal{E}\{\mathbf{w}(t)\}\mathbf{f}^T[\boldsymbol{\mu}(s), s]\, ds \\ &\quad + \int_{t_0}^{t} \mathcal{E}\{\mathbf{w}(t)[\mathbf{u}(s) - \boldsymbol{\mu}_{\mathbf{u}}(s)]\}\mathcal{F}'^T[\boldsymbol{\mu}_{\mathbf{u}}(s), s]\, ds + \cdots \\ &= \int_{t_0}^{t} \mathcal{E}\{\mathbf{w}(t)\mathbf{u}(s)\}\mathcal{F}'^T[\boldsymbol{\mu}_{\mathbf{u}}(s), s]\, ds + \ldots \\ &\approx 0 \end{aligned} \tag{3.72}$$

where the last "equality" invoked second order moment closure.

Therefore, from the definition of the symmetric Dirac delta function in (3.17) we have that

$$\begin{aligned}\mathcal{E}\{\mathbf{w}(t)\mathbf{u}^T(t)\} &= \int_{t_0}^{t} \delta(t-s)\mathbf{Q}(s)\mathbf{G}^T(s)\, ds \\ &= \frac{1}{2}\mathbf{Q}(t)\mathbf{G}^T(t)\,. \end{aligned} \tag{3.73}$$

An analogous expression can be obtained for the transposed term so that the variance equation, to first order, becomes:

$$\frac{d\mathbf{P}_{\mathbf{u}}(t)}{dt} = \mathcal{F}'[\boldsymbol{\mu}_{\mathbf{u}}(t), t]\mathbf{P}_{\mathbf{u}}(t) + \mathbf{P}_{\mathbf{u}}(t)\mathcal{F}'^T[\boldsymbol{\mu}_{\mathbf{u}}(t), t] + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t)\,. \tag{3.74}$$

It is relevant to stress that equation (3.74) for the evolution of the variance $\mathbf{P}_{\mathbf{u}}(t)$ is of second order in the difference $[\mathbf{u}(t) - \boldsymbol{\mu}_{\mathbf{u}}(t)]$. Therefore, it is more consistent to use the second-order expression (3.65), instead of the first-order expression in $[\mathbf{u}(t) - \boldsymbol{\mu}_{\mathbf{u}}(t)]$, given by (3.66), in order to calculate the evolution of both the mean and the variance. This is an important fact that is sometimes ignored in order to reduce the amount of calculation involved involved in solving the equation of the mean, since (3.66) requires less computational effort than (3.65).

3.2.2 Discrete Processes

The nonlinear discrete–time system equivalent to the nonlinear continuous–time system studied in the previous subsection is represented by the equation

$$\mathbf{u}(k+1) = \boldsymbol{\psi}[\mathbf{u}(k), k] + \boldsymbol{\Gamma}(k)\mathbf{w}(k)\,, \tag{3.75}$$

where the m–dimensional process $\{\mathbf{w}(k)\}$ has the same characteristics as that defined for the linear discrete–time case of Section 3.1.2; $\boldsymbol{\psi}[\mathbf{u}(k), k]$ is a nonlinear n–vector function of the n–vector state $\mathbf{u}(t)$. In the more general case, the $n \times m$ matrix $\boldsymbol{\Gamma}(k)$ can be a function of $\mathbf{u}(k)$; however, for reasons analogous to those stated in the continuous–time case, we only consider the simpler situation described by the system above.

By proceeding as in the continuous nonlinear case, it is relatively simple to show that second-order closure produces the following expressions for the evolution equations of the mean and the variance:

$$\boldsymbol{\mu}_{\mathbf{u}}(k+1) = \boldsymbol{\psi}[\boldsymbol{\mu}_{\mathbf{u}}(k),k] + \frac{1}{2}\mathcal{F}''[\boldsymbol{\mu}_{\mathbf{u}}(k),k]vec[\mathbf{P}_{\mathbf{u}}(k)] \tag{3.76a}$$

$$\mathbf{P}_{\mathbf{u}}(k+1) = \mathcal{F}'[\boldsymbol{\mu}_{\mathbf{u}}(k),k]\mathbf{P}_{\mathbf{u}}(k)\mathcal{F}'^{T}[\boldsymbol{\mu}_{\mathbf{u}}(k),k] + \boldsymbol{\Gamma}(k)\mathbf{Q}(k)\boldsymbol{\Gamma}^{T}(k) \tag{3.76b}$$

where

$$\mathcal{F}'[\boldsymbol{\mu}_{\mathbf{u}}(k),k] = \left.\frac{\partial\boldsymbol{\psi}[\mathbf{u}(k),k]}{\partial\mathbf{u}^{T}(k)}\right|_{\mathbf{u}(k)=\boldsymbol{\mu}_{\mathbf{u}}(k)} \tag{3.77a}$$

$$\mathcal{F}''[\boldsymbol{\mu}_{\mathbf{u}}(k),k] = \left.\frac{\partial^{2}\boldsymbol{\psi}[\mathbf{u}(k),k]}{\partial\mathbf{u}^{T}(k)\partial\mathbf{u}^{T}(k)}\right|_{\mathbf{u}(k)=\boldsymbol{\mu}_{\mathbf{u}}(k)} \tag{3.77b}$$

are now the Jacobian and Hessian matrices, respectively. Higher-order equations can also be obtained, however, this goes beyond the scope of an introductory course.

3.3 Stochastic Nonlinear Partial Differential Equations

In this section we are interested in the case where the state variable $\mathbf{u}$ is a function not only of time, but also of space, that is, $\mathbf{u} = \mathbf{u}(\mathbf{r},t)$, with $\mathbf{r} \in R^{n}$. In this case, the equations governing the state evolution are partial differential equations describing the behavior of a stochastic random field. Our goal here is to indicate concisely how to derive evolution equations for the mean and covariance of the random field. A rigorous treatment of stochastic partial differential equations is complicated, especially when boundary conditions are included in order to define the problem completely. In what follows, we only give a formal description of the problem, ignoring the mathematical details. Moreover, we consider only the scalar case, so that there is only one random field to refer to, that is, $u = u(\mathbf{r},t)$. More complete, and mathematically precise descriptions are found in Omatu & Sienfeld [109] and in the collection of articles in Stavroulakis [125]. These treatments are geared toward estimation problems for systems governed by partial differential equations, known as distributed parameter systems. For our simple treatment, the scalar (or univariate) random field $u = u(\mathbf{r},t)$ is continuous in both space and time. Recall that, since u is random we should have in mind that it also depends on a variable ω referring to the realizations of this field. The variable ω is kept implicit in order to maintain the notation as compact as possible, and compatible with our previous notation.

Consider the following system of governing equations:

$$\frac{\partial u(\mathbf{r},t)}{\partial t} = f[u(\mathbf{r},t)] + w(\mathbf{r},t) \tag{3.78}$$

where $f[u(\mathbf{r},t)]$ is a scalar differential operator which involves spatial partial derivatives, possibly nonlinear in the variable u; the scalar function $w(\mathbf{r},t)$ represents a stochastic forcing, which we assume to be white in time with mean $\mu_{w}(\mathbf{r},t)$ and covariance $Q(\mathbf{r},\mathbf{s},t)$:

$$\mathcal{E}\{w(\mathbf{r},t)\} = \mu_{w}(\mathbf{r},t) \tag{3.79a}$$

$$\mathcal{E}\{[w(\mathbf{r},t)-\mu_{w}(\mathbf{r},t)][w(\mathbf{s},\tau)-\mu_{w}(\mathbf{s},\tau)]\} = Q(\mathbf{r},\mathbf{s},t)\delta(t-\tau) \tag{3.79b}$$

with $\mathbf{s} \in R^n$. When referring to Q as covariance, we have in mind its spatial structure; we could refer to this quantity as a variance if we had in mind its temporal structure. Also, because we are dealing with a *scalar* stochastic random field, Q is *scalar* function, and not a matrix.

We assume further that the processes $u(\mathbf{r}, 0)$ and $w(\mathbf{r}, t)$ are uncorrelated, that is,

$$\mathcal{E}\{w(\mathbf{r}, t)u(\mathbf{s}, 0)\} = 0 \tag{3.80}$$

for all $\mathbf{r}, \mathbf{s} \in R^n$, and all times $t \geq 0$.

Proceeding as in the previous section, but now for the univariate case, an equation for the evolution of the mean, written here as $\mu(\mathbf{r}, t) \equiv \mathcal{E}\{u(\mathbf{r}, t)\}$, can be found by applying the ensemble mean operator directly to the governing equation (3.78). Therefore,

$$\frac{\partial \mu(\mathbf{r}, t)}{\partial t} = \mathcal{E}\{f[\mathbf{u}(\mathbf{r}, t)]\} + \mu_w(\mathbf{r}, t) \tag{3.81}$$

Expanding $f[u(\mathbf{r}, t)]$ in a Taylor series about its mean $\mu(\mathbf{r}, t)$ we have

$$\begin{aligned} f[u(\mathbf{r}, t), t] \approx\ & f[\mu(\mathbf{r}, t), t] + \mathcal{F}'[\mu(\mathbf{r}, t)](u(\mathbf{r}, t) - \mu(\mathbf{r}, t)) \\ & + \frac{1}{2}\mathcal{F}''[\mu(\mathbf{r}, t)]\,(u(\mathbf{r}, t) - \mu(\mathbf{r}, t))^2 \end{aligned} \tag{3.82}$$

which is identical to the expansion (3.56), except for the fact that now the Jacobian $\mathcal{F}'$ and the Hessian $\mathcal{F}''$ are functions (differential operators), not matrices. These quantities are defined in an entirely analogous way as to the way we saw in the previous section, that is,

$$\mathcal{F}'[\mu(\mathbf{r}, t)] = \left.\frac{\partial f[u(\mathbf{r}, t)]}{\partial u(\mathbf{r}, t)}\right|_{u(\mathbf{r},t)=\mu(\mathbf{r},t)} \tag{3.83}$$

and

$$\mathcal{F}''[\mu(\mathbf{r}, t)] = \left.\frac{\partial^2 f[u(\mathbf{r}, t)]}{\partial u^2(\mathbf{r}, t)}\right|_{u(\mathbf{r},t)=\mu(\mathbf{r},t)} \tag{3.84}$$

Therefore, the equations for the mean and covariance, with second-order closure, are:

$$\frac{\partial \mu(\mathbf{r}, t)}{\partial t} = f[\mu(\mathbf{r}, t)] + \frac{1}{2}\,\mathcal{F}''[\mu(\mathbf{r}, t), t]\,P(\mathbf{r}, \mathbf{r}, t)\,, \tag{3.85}$$

and

$$\frac{\partial P(\mathbf{r}, \mathbf{s}, t)}{\partial t} = \mathcal{F}'[\mu(\mathbf{r}, t)]P(\mathbf{r}, \mathbf{s}, t) + \mathcal{F}'[\mu(\mathbf{s}, t)]P(\mathbf{r}, \mathbf{s}, t) + Q(\mathbf{r}, \mathbf{s}, t) \tag{3.86}$$

respectively. The equation for the mean involves the variance — covariance $P(\mathbf{r}, \mathbf{s}, t)$, for $\mathbf{s} = \mathbf{r}$. Details in obtaining these equations can be found in Cohn [28]. A simple case, taken from this work and that of Ménard [104] is given in exercises.

Exercises

1. Derive the continuous Lyapunov equation, for the linear system (3.1), in two distinct ways:

(a) Differentiating the solution (3.20) — use Leibnitz integration rule [1].

(b) Differentiating the definition of variance:

$$\mathbf{P}(t) \equiv \mathcal{E}\{[\mathbf{u}(t) - \boldsymbol{\mu}_{\mathbf{u}}(t)][\mathbf{u}(t) - \boldsymbol{\mu}_{\mathbf{u}}(t)]^T\}$$

2. Show that the expressions (3.22) satisfy equation (3.19).

3. Consider general matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of dimensions $n \times m$, $m \times p$, and $p \times q$, respectively. Furthermore, notice that the product of two matrices $\mathbf{A}$ and $\mathbf{B}$ can be written as a column operation according to

$$(\mathbf{AB})_{\bullet j} = \mathbf{A}\mathbf{b}_j$$

where $\mathbf{b}_j$ is the j–th column of $\mathbf{B}$, and the notation $(\bullet j)$ on the left–hand–side stands for the j–th column of the product matrix $(\mathbf{AB})$. Representing the (i,j)-th element of $\mathbf{B}$ as b_{ij}, we can write the product of two matrices in the alternative form

$$(\mathbf{AB})_{\bullet j} = \sum_i (\mathbf{A})_{\bullet i} b_{ij}$$

With that in mind, engage in the following proofs:

(a) Show that:

$$vec(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A}) vec(\mathbf{B})$$

(b) Using the previous result, show that

$$vec(\mathbf{AB}) = (\mathbf{I}_p \otimes \mathbf{A}) vec(\mathbf{B})$$

where $\mathbf{I}_p$ is the $p \times p$ identity matrix.

(c) By noticing that for matrices $\mathbf{A}$ and $\tilde{\mathbf{A}}$, of dimension $n \times m$,

$$vec(\mathbf{A} + \tilde{\mathbf{A}}) = vec(\mathbf{A}) + vec(\tilde{\mathbf{A}}) \,,$$

show that the continuous–time Lyapunov equation (3.21) can be written as

$$vec(\dot{\mathbf{P}}_{\mathbf{u}}) = [\mathbf{F}(t) \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{F}(t)]\, vec(\mathbf{P}_{\mathbf{u}}) + [\mathbf{G}(t) \otimes \mathbf{G}(t)]\, vec(\mathbf{Q})$$

(equivalent to Problem 3.1–1 in Lewis [94].)

(d) Analogously, show that the discrete–time Lyapunov equation (3.40) can be written as

$$\begin{aligned} vec(\mathbf{P}_{\mathbf{u}}(k+1)) &= [\boldsymbol{\Psi}(k+1,k) \otimes \boldsymbol{\Psi}(k+1,k)]\, vec(\mathbf{P}_{\mathbf{u}}(k)) \\ &\quad + [\boldsymbol{\Gamma}(k) \otimes \boldsymbol{\Gamma}(k)]\, vec(\mathbf{Q}_k) \end{aligned}$$

(equivalent to Problem 2.2–1 in Lewis [94].)

[1] Leibnitz integration rule is

$$\frac{d}{dt}\int_{h(t)}^{g(t)} f(t,\tau)\, d\tau = \int_{h(t)}^{g(t)} \frac{\partial f(t,\tau)}{\partial t}\, d\tau + f[t,g(t)]\frac{dg(t)}{dt} - f[t,h(t)]\frac{dh(t)}{dt}$$

4. (Maybeck [101], Problem 2.15)

   (a) Show that, for all t_0, t_1, and t,

   $$\mathbf{\Phi}(t, t_0) \;=\; \mathbf{\Phi}(t, t_1)\mathbf{\Phi}(t_1, t_0)$$

   by showing that both quantities satisfy the same linear differential equation and "initial condition" at time t_1. Thus, the solution of $\dot{\mathbf{u}}(t) = \mathbf{F}(t)\mathbf{u}(t)$ with $\mathbf{u}(t_0) = \mathbf{u}_0$ [i.e., $\mathbf{\Phi}(t, t_0)\mathbf{u}(t_0)$] at any time t_2 can be obtained by forming $\mathbf{u}(t_1) = \mathbf{\Phi}(t_1, t_0)\mathbf{u}(t_0)$ and using it to generate $\mathbf{u}(t_2) = \mathbf{\Phi}(t_2, t_1)\mathbf{u}(t_1)$.

   (b) Since it can be shown that $\mathbf{\Phi}(t, t_0)$ is non–singular, show that the above property implies that

   $$\mathbf{\Phi}^{-1}(t, t_0) \;=\; \mathbf{\Phi}(t_0, t)$$

5. (Mostly from Maybeck [101], Problem 2.18) Given a homogeneous linear differential equation $\dot{\mathbf{u}}(t) = \mathbf{F}(t)\mathbf{u}(t)$, for the n–vector $\mathbf{u}(t)$, the associated "adjoint" differential equation is the differential equation for the n–vector $\mathbf{v}(t)$ such that the inner product of $\mathbf{v}(t)$ with $\mathbf{u}(t)$ is constant for all time:

   $$\mathbf{u}^T(t)\mathbf{v}(t) \;=\; \text{const}$$

   (a) Take the derivative of this expression to show that the adjoint equation associated with $\dot{\mathbf{u}}(t) = \mathbf{F}(t)\mathbf{u}(t)$ is

   $$\dot{\mathbf{v}}(t) = -\mathbf{F}^T(t)\mathbf{v}(t)$$

   (b) If $\mathbf{\Phi}_{\mathbf{u}}(t, t_0)$ is the state transition matrix associated with $\mathbf{F}(t)$ and $\mathbf{\Phi}_{\mathbf{v}}(t, t_0)$ is the state transition matrix associated with $[-\mathbf{F}^T(t)]$, then show that

   $$\mathbf{\Phi}_{\mathbf{v}}(t, t_0) \;=\; \mathbf{\Phi}_{\mathbf{u}}^T(t_0, t) \;=\; [\mathbf{\Phi}_{\mathbf{u}}^T(t, t_0)]^{-1}$$

   To do this, show that $[\mathbf{\Phi}_{\mathbf{v}}^T(t, t_0)\mathbf{\Phi}_{\mathbf{u}}(t, t_0)]$ and $\mathbf{I}$ satisfy the same differential equation and initial condition.

   (c) Show that, as a function of its second argument, $\mathbf{\Phi}_{\mathbf{u}}(t, \tau)$ must satisfy

   $$\frac{\partial \mathbf{\Phi}_{\mathbf{u}}(t, \tau)}{\partial \tau} \;=\; -\mathbf{\Phi}_{\mathbf{u}}(t, \tau)\mathbf{F}(\tau)$$

   or, in other words,

   $$\frac{\partial \mathbf{\Phi}_{\mathbf{u}}^T(t, \tau)}{\partial \tau} \;=\; [-\mathbf{F}^T(\tau)]\mathbf{\Phi}_{\mathbf{u}}^T(t, \tau)$$

   (d) If the inner product to be preserved in time is modified to be

   $$\mathbf{u}^T(t)\mathbf{E}\mathbf{v}(t) \;=\; \text{const}$$

   where the $n \times n$ matrix $\mathbf{E}$ is assumed to be invertible and independent of time, derive the corresponding modification to the adjoint equation in (a).

6. (Mostly from Maybeck [101], Problem 2.17) Let the $n \times n$ matrix $\mathbf{F}$ be constant. Then the evaluation of $\mathbf{\Phi}(t, t_0) = \mathbf{\Phi}(t - t_0)$ can be obtained by

(a) approximating through truncation of series definition of matrix exponential, $e^{\mathbf{F}(t-t_0)}$:

$$e^{\mathbf{F}(t-t_0)} = \mathbf{I} + \mathbf{F}(t-t_0) + \frac{1}{2!}\mathbf{F}^2(t-t_0)^2 + \ldots$$

(b) Laplace methods of solving $\dot{\mathbf{\Phi}}(t-t_0) = \mathbf{F}\mathbf{\Phi}(t-t_0)$, $\mathbf{\Phi}(0) = \mathbf{I}$:

$$\mathbf{\Phi}(t-t_0) = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{F}]^{-1}\}\Big|_{t-t_0}$$

where $\mathcal{L}^{-1}\{.\}|_{(t-t_0)}$ denotes inverse Laplace transform evaluated with time argument equal to $(t-t_0)$.

(c) Cayley–Hamilton theorem (for $\mathbf{F}$ with nonrepeated eigenvalues)

$$\mathbf{\Phi}(t-t_0) = \alpha_0\mathbf{I} + \alpha_1\mathbf{F} + \alpha_2\mathbf{F}^2 + \ldots + \alpha_{n-1}\mathbf{F}^{n-1}$$

To solve for the n functions of $(t-t_0)$, $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$, the n eigenvalues of $\mathbf{F}$ are determined as $\lambda_1, \cdots, \lambda_n$. Then

$$e^{\lambda_i(t-t_0)} = \alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda_i^2 + \ldots + \alpha_{n-1}\lambda_i^{n-1}$$

must be satisfied for each eigenvalue λ_i, for $i = 1, \cdots, n$, yielding n equations for the n unknown α_i's.

(d) Sylvester expansion theorem (for $\mathbf{F}$ with nonrepeated eigenvalues)

$$\mathbf{\Phi}(t-t_0) = \mathbf{F}_1 e^{\lambda_1(t-t_0)} + \mathbf{F}_2 e^{\lambda_2(t-t_0)} + \ldots + \mathbf{F}_n e^{\lambda_n(t-t_0)}$$

where λ_i is the i–th eigenvalue of $\mathbf{F}$ and $\mathbf{F}_i$ is given as the following product of $(n-1)$ factors:

$$\mathbf{F}_i = \left[\frac{\mathbf{F} - \lambda_1\mathbf{I}}{\lambda_i - \lambda_1}\right] \cdots \left[\frac{\mathbf{F} - \lambda_{i-1}\mathbf{I}}{\lambda_i - \lambda_{i-1}}\right] \left[\frac{\mathbf{F} - \lambda_{i+1}\mathbf{I}}{\lambda_i - \lambda_{i+1}}\right] \cdots \left[\frac{\mathbf{F} - \lambda_n\mathbf{I}}{\lambda_i - \lambda_n}\right]$$

The matrix $\mathbf{F}_i$ is a projector onto the direction of the i–th eigenvector of $\mathbf{F}$.

(e) If the eigendecompostion of $\mathbf{F}$ is given by

$$\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$$

where $\mathbf{U}$ is the matrix whose columns are the eigenvectors of $\mathbf{F}$, and $\mathbf{D}$ is a diagonal matrix with the eigenvalues λ_i of $\mathbf{F}$ along the diagonal, that is, $\mathbf{D} = diag(\lambda_1, \ldots, \lambda_n)$, then, the Maclaurin expansion of item (a) above can be used to show that $\mathbf{\Phi}(t, t_0)$ has the same eigenvectors of $\mathbf{F}$ with eigenvalues $e^{\lambda_i(t-t_0)}$, that is,

$$\mathbf{\Phi}(t, t_0) = \mathbf{U} \begin{pmatrix} e^{\lambda_1(t-t_0)} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & \ldots & 0 \\ \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & \ldots & e^{\lambda_n(t-t_0)} \end{pmatrix} \mathbf{U}^{-1}$$

Use the five methods[2] above to evaluate $\mathbf{\Phi}(t,0)$ if $\mathbf{F}$ is given by

$$\mathbf{F} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

7. Consider the Lorenz (1960; [99]) system of equations

$$\begin{aligned} \dot{X} &= aYZ \\ \dot{Y} &= bXZ \\ \dot{Z} &= cXY \end{aligned}$$

where a, b and c are constants to be specified later. Under certain conditions, Lorenz showed that the solution of this system is periodic with predictable period $2K$ for X, and period $4K$ for Y and Z. The expression for K is a function of the initial condition $[X_0\, Y_0\, Z_0]$, and as a consequence, even though the system is deterministic, i.e., non–chaotic, the period of oscillation may change considerably due a small change in the initial condition. The amplitude of the oscillations may change as well.

(a) Assuming there is uncertain knowledge of the initial state, we can think on these equations as a set of stochastic differential equations. Thus, defining the 3–vector $\mathbf{u}(t) = [X\, Y\, Z]^T$ and writing the system of equations as

$$\dot{\mathbf{u}}(t) = \mathbf{f}(\mathbf{u})$$

derive approximate equations for the mean $\boldsymbol{\mu}_{\mathbf{u}}(t)$,

$$\boldsymbol{\mu}_{\mathbf{u}}(t) = \mathcal{E}\{\mathbf{u}(t)\} = [\mu_x\, \mu_y\, \mu_z]^T$$

and for the (co)variance $\mathbf{P}(t)$,

$$\begin{aligned} \mathbf{P}(t) &= \mathcal{E}\{(\mathbf{u}(t) - \boldsymbol{\mu}(t))(\mathbf{u}(t) - \boldsymbol{\mu}(t))^T\} \\ &= \begin{pmatrix} p_x & p_{xy} & p_{xz} \\ p_{xy} & p_y & p_{yz} \\ p_{xz} & p_{yz} & p_z \end{pmatrix} \end{aligned}$$

to second order. Notice that in this exercise we are taking $\mathbf{Q} = \mathbf{0}$. (Hint: We have already derived these equations for a general stochastic system of ordinary differential equations, so this is just an exercise of calculating the appropriate Jacobian and Hessian matrices.)

(b) *Computer Assignment:* [Ehrendorfer (1994a,b; [47, 48]), and Epstein (1969; [50])]

[2]For method (b) the following inverse Laplace transforms are useful:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = \frac{e^{-at} - e^{-bt}}{b-a}$$

for $t > 0$, and $a \neq b$.

i. The solution of a system of ordinary differential equations of the form

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{f}(\mathbf{x}, t)$$

can be approximated by a fourth–order Runge–Kutta method for $\mathbf{x}_{n+1} \approx \mathbf{y}(t_{n+1})$ according to

$$\mathbf{y}_{n+1} \;=\; \mathbf{y}_n \;+\; \frac{\Delta t}{3}\left(\frac{\mathbf{k}_1}{2} + \mathbf{k}_2 + \mathbf{k}_3 + \frac{\mathbf{k}_4}{2}\right)$$

where

$$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}(\mathbf{y}_n, t_n) \\ \mathbf{k}_2 &= \mathbf{f}(\mathbf{y}_n + \tfrac{\Delta t}{2}\mathbf{k}_1, t_n + \tfrac{\Delta t}{2}) \\ \mathbf{k}_3 &= \mathbf{f}(\mathbf{y}_n + \tfrac{\Delta t}{2}\mathbf{k}_2, t_n + \tfrac{\Delta t}{2}) \\ \mathbf{k}_4 &= \mathbf{f}(\mathbf{y}_n + \Delta t\mathbf{k}_3, t_n + \Delta t) \end{aligned}$$

for $t_{n+1} = t_n + \Delta t$, with Δt being the time step (e.g., Press et al. [115], pp. 550–554). Write a Matlab function that, given the initial time t_0, the final time t_f, the time step Δt, and the initial condition $\mathbf{y}_0$, finds the approximate solution of the system of ordinary differential equation for $\mathbf{x}(t)$, according to this differencing method[3].

ii. The solution of the Lorenz system for $\mathbf{u}(t)$ above, with $a = -0.1$, $b = 1.6$, and $c = -0.75$, and initial condition

$$\mathbf{u}_0 \;=\; \begin{pmatrix} X(0) \\ Y(0) \\ Z(0) \end{pmatrix} \;=\; \begin{pmatrix} 0.12 \\ 0.24 \\ 0.10 \end{pmatrix}$$

has approximate period of 23.12 time units in $X(t)$, and approximate period of 46.24 time units in $Y(t)$ and $Z(t)$. Using the Matlab function you created in the previous item, solve the Lorenz system, for the parameters a, b and c, and initial condition given above, from time $t_0 = 0$ to $t_f = 250$, with a time step $\Delta t = 0.5$. Plot the $X(t)$, $Y(t)$ and $Z(t)$, as a function of time.

iii. Now, choose three distinct initial conditions generated as

$$\mathbf{u}(0) \;=\; \mathbf{u}_0 \;+\; \mathbf{w}$$

where the 3–vector $\mathbf{w}$ is normally distributed with mean zero and variance $0.01^2\mathbf{I}$, that is, $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, 0.01^2\mathbf{I})$, with $\mathbf{I}$ being the 3×3 identity matrix. Plot the result of the three corresponding solutions versus time. Comment on the results you obtain here and those obtained in the previous item.

iv. Using the vector notation introduced in this lecture, and exploited in Exercise 3, we can write the mean and (co)variance equations of item (a) simultaneously as

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{\mu}_{\mathbf{u}}(t) \\ vec(\mathbf{P}_{\mathbf{u}}(t)) \end{pmatrix} \;=\; \mathbf{g}\begin{pmatrix} \boldsymbol{\mu}_{\mathbf{u}}(t) \\ vec(\mathbf{P}_{\mathbf{u}}(t)) \end{pmatrix}$$

[3]The Matlab functions ode23 and ode45 are "adaptive mesh" Runge–Kutta solvers for ordinary differential equations. The adaptive nature of these functions provide great accuracy, but do not allow for control of the number of time steps for a given integration interval. Since, in this exercise, we want to have the same number of time steps in all experiments, it is best to write our own Runge–Kutta solver.

where the function $\mathbf{g}$ is defined as

$$\mathbf{g}\left(\begin{array}{c}\boldsymbol{\mu}_{\mathbf{u}}(t)\\ vec(\mathbf{P}_{\mathbf{u}}(t))\end{array}\right) \equiv \left(\begin{array}{l}\mathbf{f}[\boldsymbol{\mu}_{\mathbf{u}}(t)] + \frac{1}{2}\mathcal{F}''[\boldsymbol{\mu}_{\mathbf{u}}(t)]vec(\mathbf{P}_{\mathbf{u}}(t))\\ \{\mathcal{F}'[\boldsymbol{\mu}_{\mathbf{u}}] \otimes \mathbf{I}_3 + \mathbf{I}_3 \otimes \mathcal{F}'[\boldsymbol{\mu}_{\mathbf{u}}]\}\, vec(\mathbf{P}_{\mathbf{u}}(t))\end{array}\right)$$

where $\mathbf{I}_3$ is the 3×3 identity matrix, and we noticed that since $\mathbf{f}$ does not depend explicitly on t, its Jacobian and Hessian matrices also do not depend on t explicitly. Identifying the vector $\mathbf{x}(t)$ as the 12–vector $[\boldsymbol{\mu}_{\mathbf{u}}\, vec(\mathbf{P}_{\mathbf{u}})]^T$, solve the equations for the evolution of the mean and (co)variance above, for the Lorenz system. Plot the results of the evolution of the mean $\boldsymbol{\mu}_{\mathbf{u}} = [\mu_u\, \mu_y\, \mu_z]$ versus time, as well as those for the variances p_x, p_y, p_z, and the cross–covariances p_{xy}, p_{xz} and p_{yz}.

v. Repeat the previous item, when the second–order correction term in the mean equation is neglected. Explain the difference from the results of the previous item.

(c) To really assess to correctness of the means and (co)variances obtained in the previous exercise it is necessary to perform Monte Carlo experiments, with extremely large samples or, alternatively, to solve the Liouville equation — Fokker–Planck equation when $\mathbf{Q} = \mathbf{0}$ — which is an equation for the time evolution of the probability density function related to the stochastic process under consideration. Not surprisingly, this latest approach has been shown by Ehrendorfer (1994a,b; [47, 48]), to provide the most reliable, and efficient, estimate of the moments of the probability distribution. This approach, however, is beyond the scope of our course, and for this reason we will rely on Monte Carlo experiments to assess reliability of the means and (co)variances obtained in the previous exercise.

i. Generating sample initial conditions in the same manner you generated the three distinct initial conditions in (*b.iii*), perform three Monte Carlo experiments which integrate the Lorenz equations for three distinct total number of ensemble members: 50, 100, and 200. Using the Matlab function mean calculate the means $\mu_x^{MC=i}$, $\mu_y^{MC=i}$, and $\mu_z^{MC=i}$, where $i = 50, 100, 200$. Plot the results as a function of time. How do they compare with the evolution of the mean obtained in the previous exercise when the second order correction term was present in the mean equation?

ii. Using the Matlab function cov, calculate the variances $(\sigma_x^{MC=i})^2$, $(\sigma_y^{MC=i})^2$, and $(\sigma_z^{MC=i})^2$, for each sample size $i = 50, 100, 200$. Compare with the results obtained for p_x, p_y, and p_z of the previous exercise.

iii. Still using the same function cov, calculate the cross–covariances $cov(x, y)_{MC=i}$, $cov(x, z)_{MC=i}$, and $cov(y, z)_{MC=i}$ for each sample size $i = 50, 100, 200$. Compare with the results for p_{xy}, p_{xz}, and p_{yz} obtained in the previous exercise.

[Beware: We should really use a sample size of 10^4, or larger, to have a converged Monte Carlo run. However, this would only be feasible if we computed the means, variances, and cross–covariances on–line, that is, while running the time evolution. This would be the way to avoid the memory overload caused when storing the complete time history for each ensemble member, as we are doing in our experiments.]

8. Consider a system governed by the linear differential equation

$$\dot{\mathbf{u}}(t) \;=\; \mathbf{F}(t)\mathbf{u}(t)$$

and assume that stochasticity comes from the fact that we only know the initial condition to a certain degree. That is, the initial condition is

$$\mathbf{u}(0) \;=\; \mathbf{u}_0 \;+\; \boldsymbol{\epsilon}(0)$$

where $\boldsymbol{\epsilon}(0)$ has mean $\boldsymbol{\mu}_0$, and variance $\mathbf{P}_0$. We refer to $\boldsymbol{\epsilon}(0)$ as the initial error.

(a) Show that the error $\boldsymbol{\epsilon}(t)$, at time t, can be determined by

$$\boldsymbol{\epsilon}(t) \;=\; \boldsymbol{\Phi}(t, t_0)\boldsymbol{\epsilon}(0)$$

where $\boldsymbol{\Phi}(t, t_0)$ is the transition matrix related to the governing equation for $\mathbf{u}(t)$. Therefore, for linear dynamics $\mathbf{F}(t)$, the error evolves according to the same "law" as the state vector.

(b) Show that the ensemble average of the error $\boldsymbol{\mu}_{\boldsymbol{\epsilon}}(t) \equiv \mathcal{E}\{\boldsymbol{\epsilon}(t)\}$ evolves according to

$$\boldsymbol{\mu}_{\boldsymbol{\epsilon}}(t) \;=\; \boldsymbol{\Phi}(t, t_0)\boldsymbol{\mu}_{\boldsymbol{\epsilon}}(0)$$

and that the error variance $\mathbf{P}(t) \;\equiv\; \mathcal{E}\{[\boldsymbol{\epsilon}(t) - \boldsymbol{\mu}_{\boldsymbol{\epsilon}}(t)][\boldsymbol{\epsilon}(t) - \boldsymbol{\mu}_{\boldsymbol{\epsilon}}(t)]^T\}$ evolves according to

$$\mathbf{P}(t) \;=\; \boldsymbol{\Phi}(t, 0)\mathbf{P}_0\, \boldsymbol{\Phi}^T(t, 0)$$

This expression is the solution of the Lyapunov equation in the absence of $\mathbf{Q}$, and $\mathbf{P}(t)$ in this case is sometimes referred to as the predictability error (co)variance.

(c) In some applications it is important to determine which perturbations grow fastest within a given period of time. A measure of the growth of initial perturbations can be obtained by defining an amplification factor coefficient $A(t)$ as

$$\begin{aligned} A(t) \;&\equiv\; \frac{||\boldsymbol{\epsilon}(t)||^2}{||\boldsymbol{\epsilon}(0)||^2} \\ &=\; \frac{\boldsymbol{\epsilon}^T(t)\boldsymbol{\epsilon}(t)}{\boldsymbol{\epsilon}^T(0)\boldsymbol{\epsilon}(0)} \end{aligned}$$

(d) (Lacarra & Talagrand [91]) Going back to the time–independent matrix $\mathbf{F}$ of the previous problem, for which you have calculated the corresponding transition matrix $\boldsymbol{\Phi}(t, 0)$, perform the following tasks:

i. Calculate the amplification factor, at time $t = T$, for an initial vector $\boldsymbol{\epsilon}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T$.

ii. What is the amplification factor corresponding to the largest eigenvalue of $\mathbf{F}$?

iii. Show that the eigenvectors of $\mathbf{F}$ are not orthogonal.

9. Consider the linear advection equation in R^1 for a univariate random field $u(x, t)$:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} \;=\; 0$$

with initial condition

$$u(x, t = 0) \;=\; u_0(x)$$

where $U = const.$ represents the advection speed. Thus determine:

(a) The evolution equation for the mean $\mu(x,t) = \mathcal{E}\{u(x,t)\}$.

(b) The evolution equation for the covariance function between two points x e y, that is, $P(x,y,t) = \mathcal{E}\{[u(x,t) - \mu(x,t)][u(y,t) - \mu(y,t)]\}$.

10. (Cohn [28] and Ménard [104]) Consider Burger's equation in one spatial dimension, for $u = u(x,t)$:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2} + w$$

where $\alpha = const.$ and $w = w(x,t)$ is a stochastic forcing term white with zero mean and covariance $Q(x,y,t)$. Obtain the evolution equation for the mean $\mu(x,t)$ and for the covariance $P(x,y,t)$, up to second order. From the covariance equation, obtain the evolution equation for the variance field. (Hint: There is no need to recalculate all the equations as if nothing was known. The intention here is to apply directly the results obtained in the end of this chapter).