Atmospheric Dynamics

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- *Hydrodynamic Field Equations*
- *Hydrostatic Balance*
- *Log-Pressure Height*
- *P.E. for the Earths Atmosphere (β-plane)*
- *Geostrophic Balance and Thermal Wind Balance*
- *Conservation Properties θ and PV*
- *Structure of θ and PV in the Atmosphere*
- *The Utility of PV as a Diagnostic*
- *Rossby-Wave Propagation Mechanism*
- *PV Invertability and Balance*
- *QG PV and its Inversion operator*

Hydrodynamic Field Equations

Conservation of Mass:

$$\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

Conservation Momentum:

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla} P + \boldsymbol{G} + \nu \nabla^2 \boldsymbol{u}$$

Conservation Internal Energy:

$$\frac{T}{\theta}\frac{D\theta}{Dt} = \frac{J}{c_p} + \kappa \nabla^2 T + \Phi$$

Equation of State:

$$P = \rho R T$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$$

$$\boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z^*}\right)$$

6 scalar equations, 6 scalar unknowns:

- ρ — *density* $\left[\mathrm{kg\ m^{-3}}\right]$
- $\boldsymbol{u} = (u, v, w)$ — *velocity* $\left[\mathrm{m\ s^{-1}}\right]$
- T — *temperature* $\left[\mathrm{K^\circ}\right]$
- P — *pressure* $\left[\mathrm{Pa = kg\ m^{-2} s^{-2}}\right]$

$$\theta = T\left(\frac{P}{P_o}\right)^{-R/cp}$$

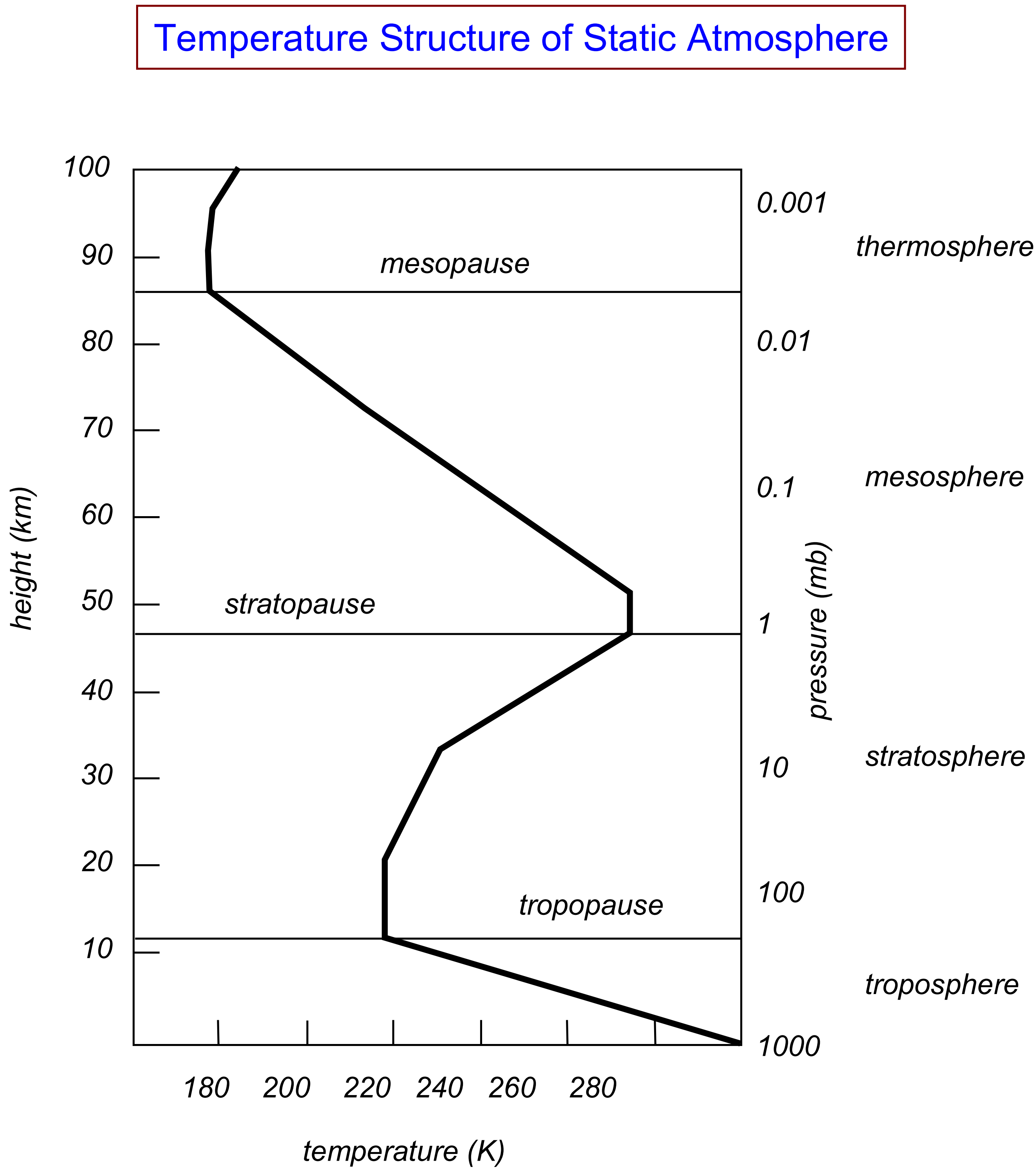
$\boldsymbol{G}$ $\left[\mathrm{m\ s^{-2}}\right]$ — *Body force (e.g., gravity, Coriolis force in noninertial reference frame)*

J $\left[\mathrm{K^\circ Kg^{-1} m^3 s^{-1}}\right]$ — *Local heating (e.g., radiation, latent heat of condensation)*

Φ $\left[\mathrm{K^\circ s^{-1}}\right]$ — *Heating by viscous friction*

ν $\left[\mathrm{m^2 s^{-1}}\right]$ — *Kinematic viscosity*

κ $\left[\mathrm{m^2 s^{-1}}\right]$ — *Coefficient of thermal diffusivity*



Temperature Structure of Static Atmosphere
100
90
80
70
60
50
40
30
20
10
height (km)
0.001
0.01
0.1
1
10
100
1000
pressure (mb)
thermosphere
mesosphere
stratosphere
troposphere
mesopause
stratopause
tropopause
180
200
220
240
260
280
temperature (K)

Hydrostatic Balance

$$\frac{dP}{dz^*} = -\rho g$$

weight of air in column balanced by the vertical pressure gradient force

Choose P *as independent vertical coordinate:* $z^* = z^*(P), \; T = T(P)$

— *introduce geopotential* $\Phi \equiv g\, z^*$

$$z^* = \Phi / g \qquad \textit{(geopotential height)}$$

$$P = \rho R T$$

$$\Rightarrow \quad \frac{dz^*}{dP} = \frac{1}{g}\frac{d\Phi}{dP} = -\frac{RT}{Pg}$$

$$\frac{d\Phi}{d\ln P} = -RT$$

$\rightarrow$ *more intuitive if vertical pressure coordinate is expressed as a height* z

Hydrostatic Balance and Log Pressure Height

$$\frac{dP}{dz^*} = -\rho g$$

$$\int_0^{z^*} dz^* = -\frac{R}{g}\int_{P_s}^{P} T \; d\ln P$$

$$P = \rho R T$$

special case: $T = T_s \;\; (const)$

$$z^* = -\frac{RT_s}{g}\ln\left(\frac{P}{P_s}\right) = -H\ln\left(\frac{P}{P_s}\right) \qquad H \equiv \frac{RT_s}{g}$$

define log-pressure height: $z \equiv -H\ln\left(\frac{P}{P_s}\right)$

$$\Rightarrow \frac{dz}{H} = -\,d\ln P$$

$$H \equiv RT_s/g = 7\text{km}$$

$$T_s = 240\text{K}^\circ$$

$$\frac{d\Phi}{d\ln P} = -RT$$

$$\frac{d\Phi}{dz} = \frac{RT}{H}$$

hydrostatic balance (log-pressure height vertical coordinate)

Reference Pressure $P_o(z)$ and Density $\rho_o(z)$

→ *log-pressure height may be used to derive a reference pressure profile* $P_o(z)$

$$z \equiv -H \ln\left(\frac{P}{P_s}\right) \qquad H \equiv RT_s/g$$

solving for P

$$P_o(z) \equiv P_s\, \mathrm{e}^{-z/H}$$

ideal gas law $\quad P_o(z) = \rho_o(z) RT_s$

$$\rho_o(z) \equiv \rho_s \mathrm{e}^{-z/H}$$

where $\quad P_s = \rho_s RT_s$

Primitive Equations for the Earth's Atmosphere

- *spherical rotating planet (non-inertial reference frame)*

  $\Rightarrow$ *apparent forces (Corilois)*

- *completely general equations can be derived but are more complicated than required*

- *scale analysis simplifies equations*

  e.g., hydrostatic balance dominates vertical momentum equation, can neglect Coriolis force associated with horizontal component of earth's rotation vector

- *use a pressure vertical coordinate*

  e.g., log-pressure height

$$z \equiv -H \ln \left(\frac{P}{P_s} \right)$$

Conservation of Mass / Continuity

$$\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \qquad \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$$

in pressure coordinates (Holton 1979):

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \Big|_P + \frac{\partial \omega}{\partial P} = 0$$

where $\omega \equiv \frac{dP}{dt} = -\rho g w^*$

in log-pressure height coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_o} \frac{\partial \rho_o w}{\partial z} = 0$$

where we have decompose density into reference plus deviation:

$$\rho(x, y, z) = \rho_o(z) + \rho'(x, y, z)$$

noting that: $\rho' / \rho_o \sim 10^{-2}$

Conservation of Momentum

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla} P + \boldsymbol{G} + \nu\nabla^2\boldsymbol{u}$$

horizontal component:

pressure gradient force

$$\frac{1}{\rho}\left(\frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\right)\Bigg|_{z^*} = \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\right)\Bigg|_{z}$$

Coriolis force (noninertial reference frame)

$$f\hat{\boldsymbol{z}} \times \boldsymbol{u} = (-fv, fu, 0)$$

$f = 2\Omega \sin\phi$ — *spherical geometry* (ϕ *latitude)*

$f = f_o + \beta y$ — β*-plane (cartesian geometry)*

$f = f_o$ — f*-plane (cartesian geometry)*

$$f_o \equiv 2\Omega \sin\phi_o \qquad \beta \equiv \frac{2\Omega}{a}\cos\phi_o$$

a *–radius of earth*

PE horizontal momentum equations on β *-plane*

$$\frac{Du}{Dt} - fv + \frac{\partial \Phi}{\partial x} = X$$

$$\frac{Dv}{Dt} + fu + \frac{\partial \Phi}{\partial y} = Y$$

Conservation of Momentum

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \boldsymbol{G} + \nu\nabla^2\boldsymbol{u}$$

vertical component:

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z^*} - g$$

scale analysis: *(midlatitude synoptic scales)*	10^{-7}	10	10	$\mathrm{m\,s^{-2}}$

$\Rightarrow$ *hydrostatic balance dominates*

PE vertical momentum equation:

$$\frac{d\Phi}{dz} = \frac{RT}{H}$$

hydrostatic balance (log-pressure height vertical coordinate)

Primitive Equations on β *-plane*

mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_o}\frac{\partial \rho_o w}{\partial z} = 0$$

momentum:

$$\frac{Du}{Dt} - fv + \frac{\partial \Phi}{\partial x} = X$$

$$\frac{Dv}{Dt} + fu + \frac{\partial \Phi}{\partial y} = Y$$

$$\frac{d\Phi}{dz} = \frac{RT}{H}$$

energy:

$$\frac{D\theta}{Dt} = Q$$

state:

$$P = \rho RT$$

X Y — *dissipative processes, non-conservative processes*

Q — *diabatic heating processes (radiative, latent, chemical heating etc.) thermal conduction*

Geostrophic Balance and Thermal Wind

— *scale analysis of horizontal momentum equations*

midlatitude synoptic scales

$U \sim 10\,\mathrm{m\,s^{-1}}$ $\quad L \sim 10^6\,\mathrm{m}$ *(horizontal)*

$W \sim 1\,\mathrm{m\,s^{-1}}$ $\quad D \sim 10^4\,\mathrm{m}$ *(vertical)*

$\Delta P/\rho \sim 10^3\,\mathrm{m^2\,s^{-2}}$ $\quad L/U \sim 10^5\,\mathrm{s}$

$$\frac{Du}{Dt} - fv + \frac{\partial \Phi}{\partial x} = 0$$

$$\frac{Dv}{Dt} + fu + \frac{\partial \Phi}{\partial y} = 0$$

$$10^{-4} \qquad 10^{-3} \qquad 10^{-3} \qquad \mathrm{m\,s^{-2}}$$

→ *defines geostrophic wind:*

$$(u_g, v_g) \equiv \frac{1}{f_o}\left(-\frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial x}\right)$$

eliminate Φ *using hydrostatic balance:* $\dfrac{\partial \Phi}{\partial z} = \dfrac{RT}{H}$

Thermal Wind Relation:

$$\frac{\partial u_g}{\partial z} = -\left(\frac{R}{Hf_o}\right)\frac{\partial T}{\partial y}$$

$$\frac{\partial v_g}{\partial z} = \left(\frac{R}{Hf_o}\right)\frac{\partial T}{\partial x}$$

Cold Pole ⇔ *Fast Jet*

Material Conservation

Adiabatic $(Q = 0)$ *and frictionless flow* $(X = Y = 0)$

$$\frac{D\theta}{Dt} = 0$$

following the motion, the potential temperature is constant

$\Rightarrow$ *adiabatic flow remains on constant potential temperature (isentropic) surfaces*

- *very useful for tracer transport problems*

θ *is a monotonic function of height and so can be used as a vertical coordinate for fluid parcels*

$\Rightarrow$ *problem of following fluid parcel trajectories reduced from 3D to 2D*

Material Conservation (cont)

Adiabatic $(Q = 0)$ *and frictionless flow* $(X = Y = 0)$

$$\frac{DPV}{Dt} = 0$$

following the motion, the potential vorticity is constant

$$PV = -g\left\{\zeta_\theta + f\right\}\frac{\partial \theta}{\partial P}$$

(Ertel's potential vorticity)

$$1PVU = 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}\mathrm{K\,kg}^{-1}$$

$$\zeta_\theta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Bigg|_\theta$$

(relative vorticity)

$$f = 2\Omega \sin\phi$$

(planetary vorticity)

$$-g\frac{\partial \theta}{\partial P}$$

(measure of static stability)

- *contours of* PV *on isentropic surface are material lines*

  $\Rightarrow$ *fluid parcels are constrained to move along these lines*

Potential vorticity is a powerful dynamical tool. It's conservation provides a connection between relative vorticity, planetary vorticity, and stratification.

In reality, however, we know that in the real atmosphere

$$Q \neq 0 \qquad X \neq 0 \qquad Y \neq 0$$

$\Rightarrow$ *becomes a question of the time scale of interest*

"quasi-conservative" or "long-lived" tracer

PV *and* θ *both well conserved in lower stratosphere for periods of several days*

diabatic and frictional effects typically occur on time scales of weeks in this region of the atmosphere

Mean PV and θ distribution

$-g\frac{\partial\theta}{\partial P}$ *large in Stratosphere / small in Troposphere*

PV *equator to pole increase in f, and stratification (~ 300mb)*

PV = 2 surface correlates well with the extra-tropical tropopause

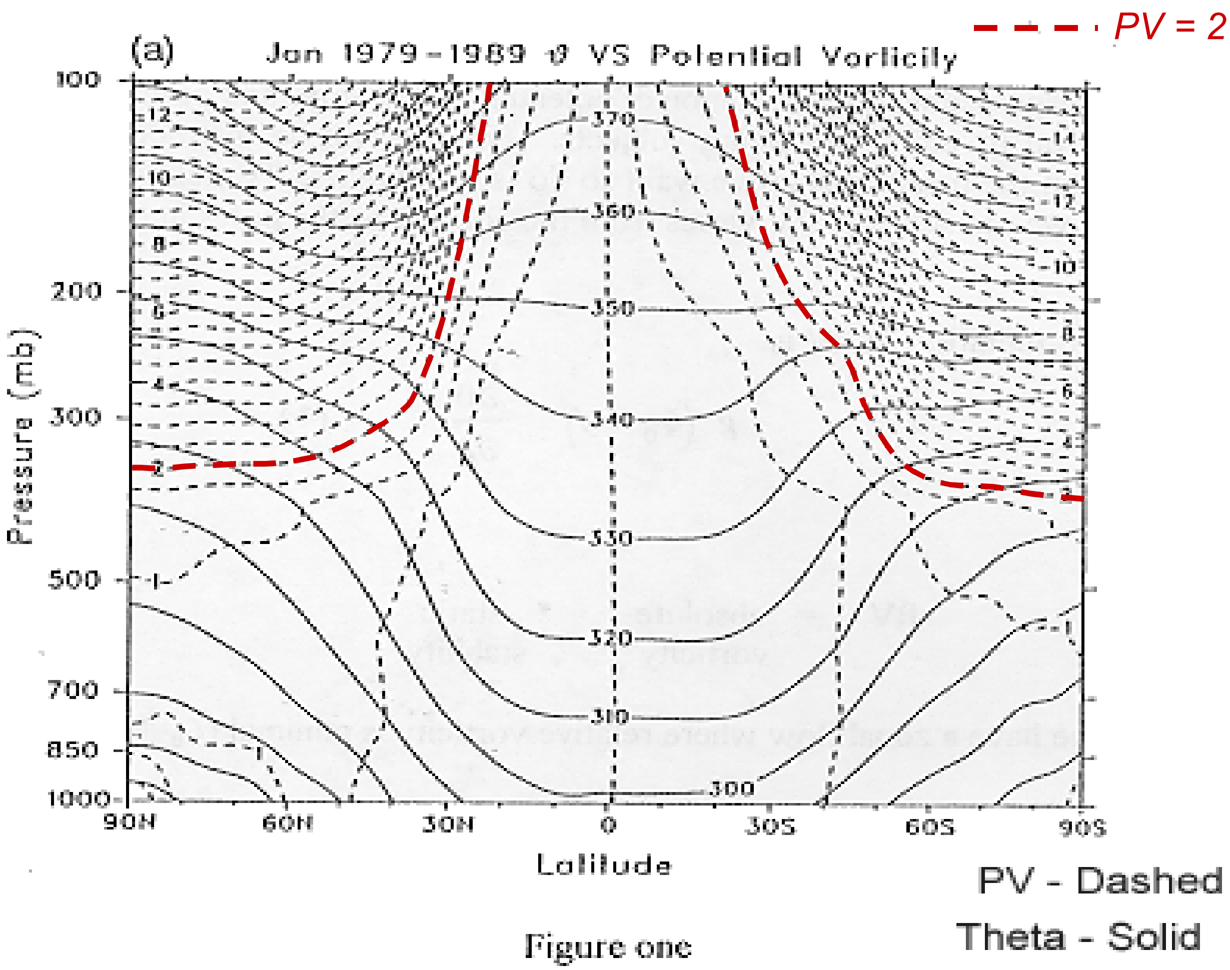
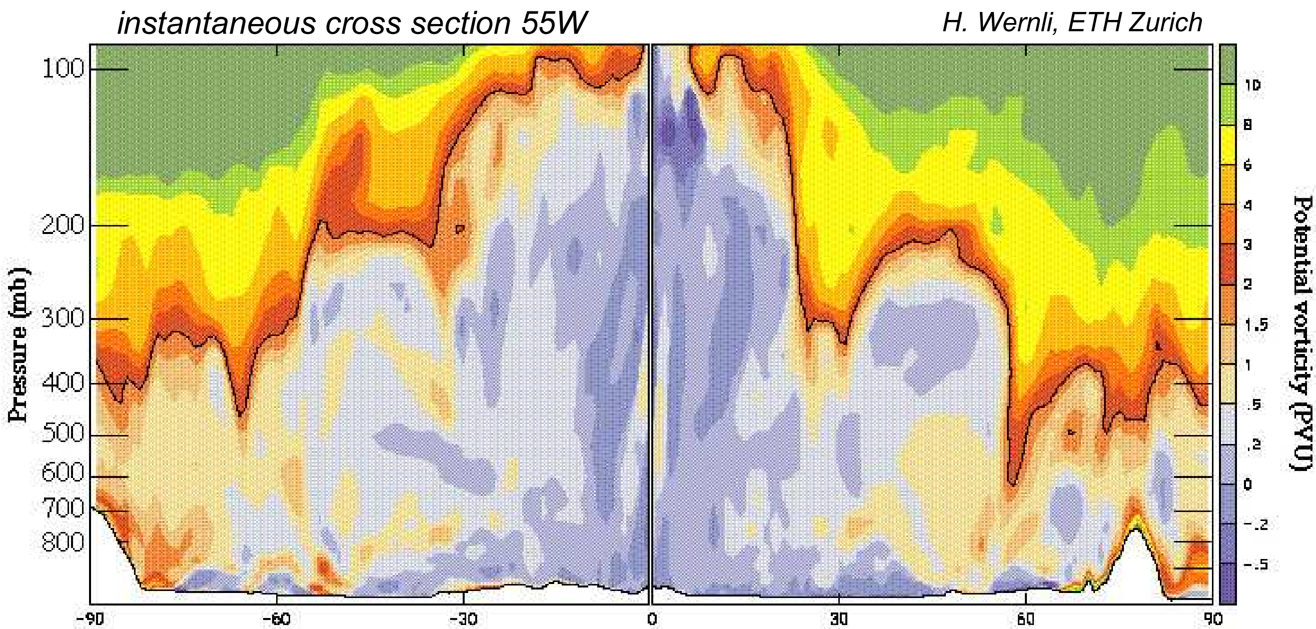
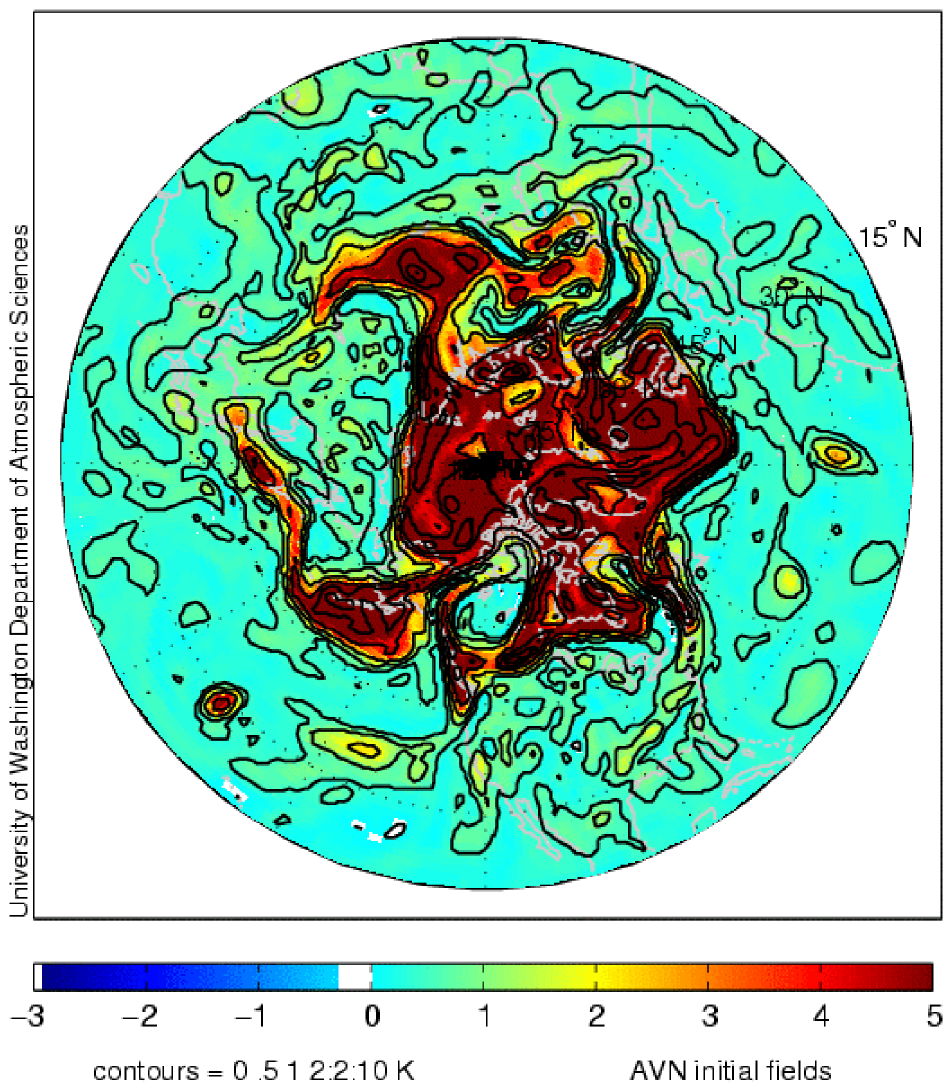


Figure one

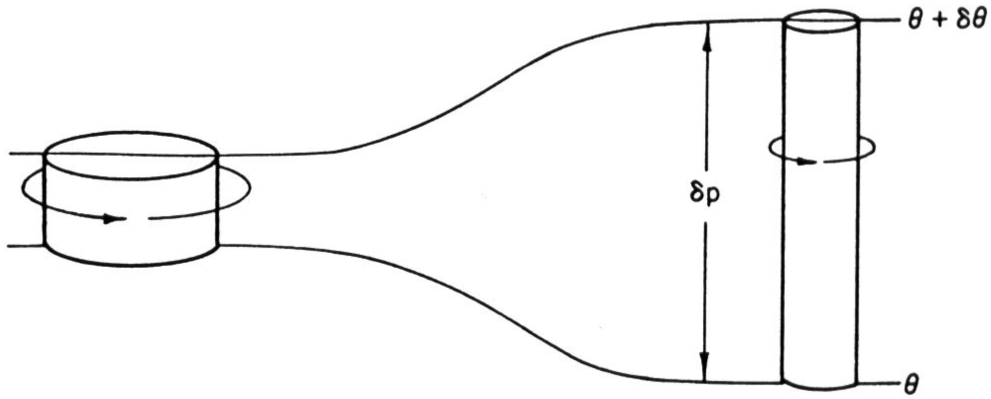


PV on 330K 07/26/2003 0000 UTC



Vortex Stretching

$$PV = -g\left\{\zeta_\theta + f_o\right\}\frac{\partial\theta}{\partial P}$$



Holton (1979)

$\dfrac{\partial\theta}{\partial P}$ ⟶ *decrease*

ζ_θ ⟶ *increase*

PV Conservation and North/South Displacement of Zonal Jet

– *initial flow zonal (westerly or easterly)* $\Rightarrow \quad \zeta_o = 0$

$$\frac{\partial \theta}{\partial P} = const \quad \Rightarrow \quad PV = -g\left\{\zeta + f_o\right\}\frac{\partial \theta}{\partial P} = C \bullet f_o$$

Consider a Northward or Southward displacement of the flow

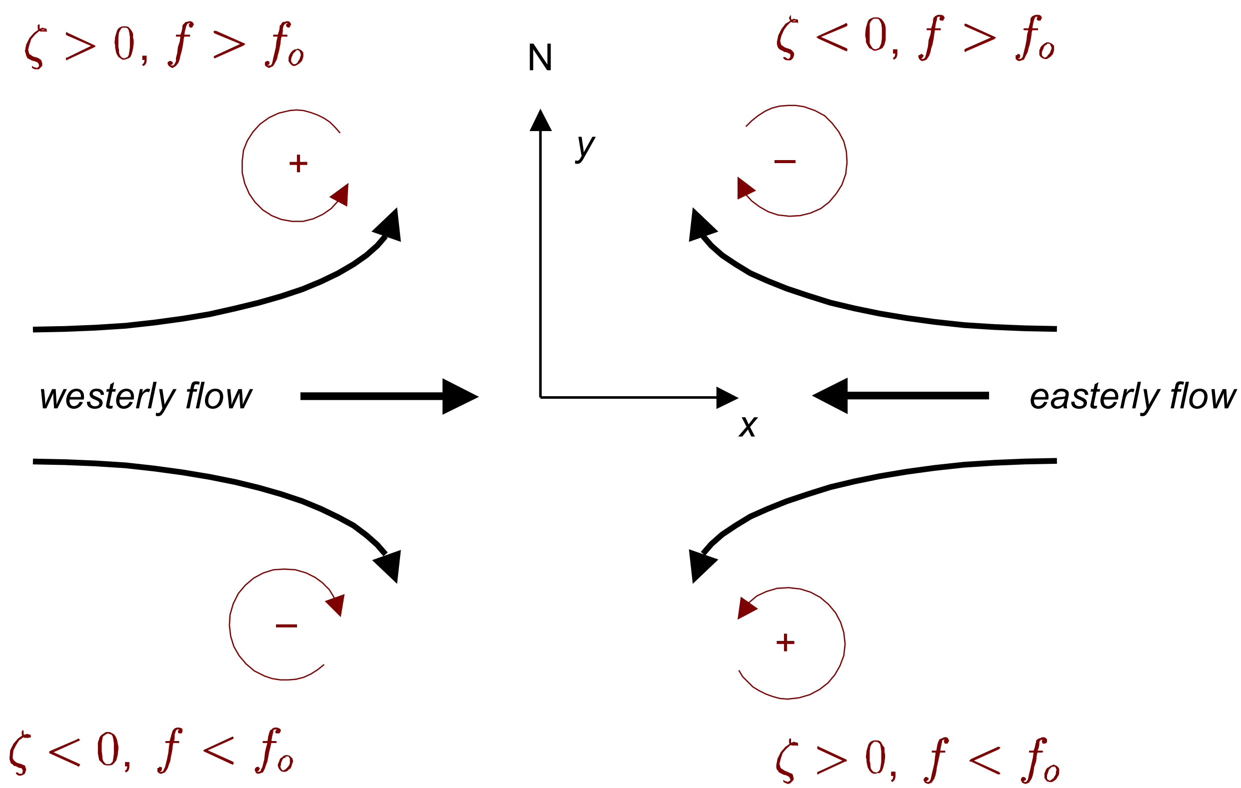
– to conserve PV, ζ and f must change in opposite directions

Westerly Flow

× *conservation not possible*

Easterly Flow

✓ *conservation possible*



Large-Scale Steady Topographic Flows

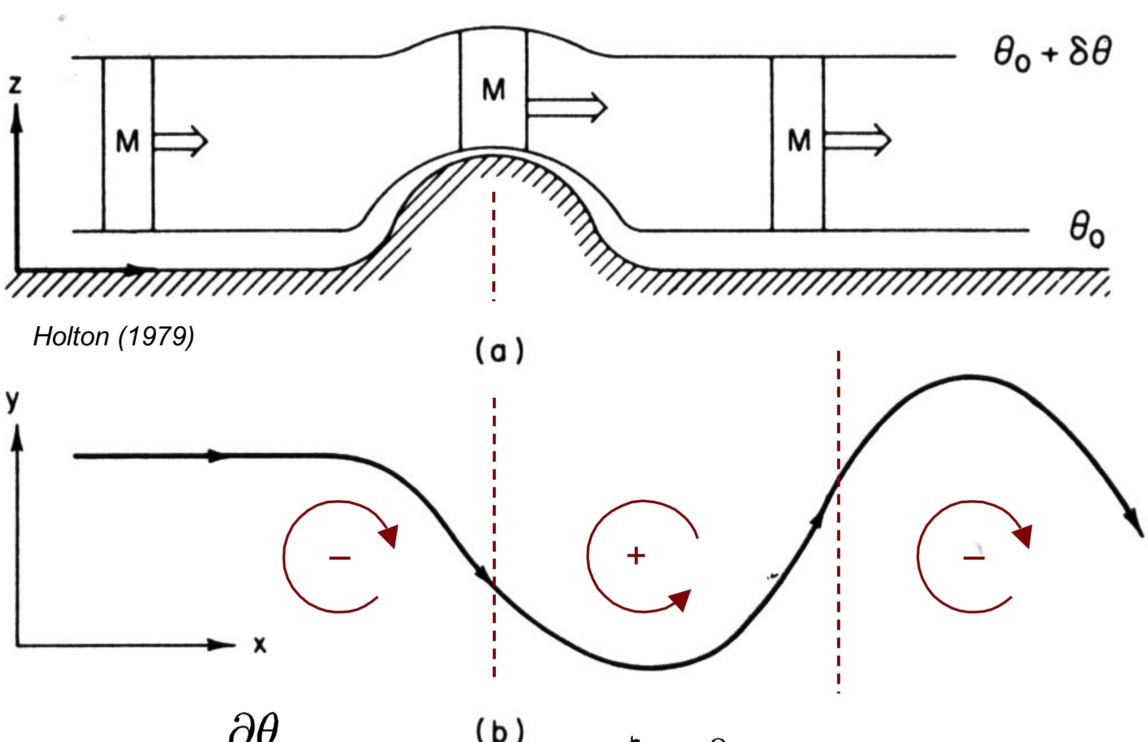
$$PV = -g\left\{\zeta + f_o\right\}\frac{\partial \theta}{\partial P}$$

upstream:

$$\zeta = 0$$

$$f = f_o$$

$$\frac{\partial \theta}{\partial P} = S$$



$$\frac{\partial \theta}{\partial P} > S$$

$$\zeta < 0$$

$$f < f_o$$

$$\zeta > 0$$

$$f_{min}$$

$$\frac{\partial \theta}{\partial P} = S$$

$$\zeta = 0$$

$$f = f_o$$

$$\left.\frac{\partial \theta}{\partial P}\right|_{max}$$

$$\zeta = 0$$

$$f < f_o$$

$$\frac{\partial \theta}{\partial P} = S$$

$$\zeta = 0$$

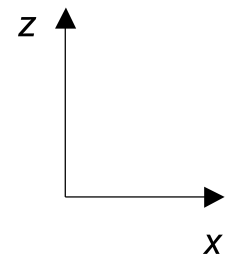
$$f = f_o$$

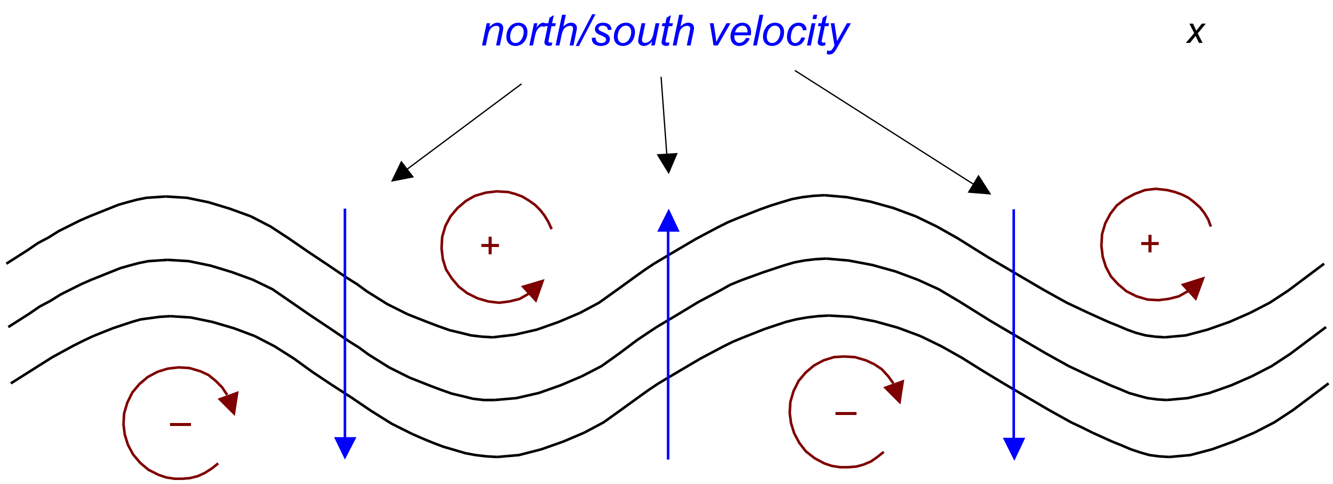
Rossby-Wave Propagation Mechanism

$$PV = -g\left\{\zeta + f_o\right\}\frac{\partial\theta}{\partial P}$$

$$\frac{\partial\theta}{\partial P} = const$$

$$f = f_o$$





$\zeta < 0$	$\zeta > 0$	$\zeta < 0$	$\zeta > 0$
$f > f_o$	$f < f_o$	$f > f_o$	$f < f_o$

Wave pattern propagates to the west

$\Rightarrow$ Rossby waves cannot propagate to the East

(relative to constant background wind)

PV Invertability and Balance

1) PV Conservation or PV evolution equation

$$\frac{DPV}{Dt} = 0 \qquad\qquad \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$$

2) PV Invertability (balance)

– *applies to rotating stably stratified "slow" dynamics*

i.e., Rossby waves, vortex dynamics, blocking, baroclinic and barotropic instability

– *PV contains all dynamical information*

i.e., all dynamical variables $\boldsymbol{u}, T, P$ *are determined by the PV distribution on an isentropic surface at every instant*

– $\boldsymbol{u}, T, P$ *obtained by inversion of PV*

$\Rightarrow$ *Flow has fewer degrees of freedom than a fully general flow*

Invertability $\Leftrightarrow$ *Balance*

Balanced Integration:

Invert PV to obtain $\boldsymbol{u}$

advect / evolve PV

Balance Condition/Relation

Conditions that eliminate "fast" gravity-wave and sound-wave motion and give a complete description of the "slow" fluid motion

A flow is balanced if the velocity field $\boldsymbol{u}$ is functionally related to the spatial distribution of mass throughout the fluid system

or equivalently

The velocity field $\boldsymbol{u}$ can be deduced from the mass field diagnostically (i.e., at any instant t – no integrals or derivatives with respect to t)

through hydrostatic balance then,

knowledge of the mass $\Rightarrow$ implies knowledge of the temperature

balanced system has too few degrees of freedom to describe sound waves or gravity waves

$\Rightarrow$ such waves are said to be "slaved" to the balanced flow or, the dynamical system is confined to a "slow manifold"

Quasi-Geostrophic PV Inversion Operator

balance relation:

geostrophic relation: $$(u_g, v_g) \equiv \frac{1}{f_o}\left(-\frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial x}\right)$$

hydrostatic balance: $$\frac{\partial \Phi}{\partial z} = \frac{RT}{H}$$

quasi-geostrophic PV:

$$PV_g = f_o + \beta y + \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + \rho_o^{-1}\frac{\partial}{\partial z}\left(\rho_o \frac{f_o^2}{N^2(z)}\frac{\partial \psi}{\partial z}\right)$$

where: $\psi \equiv f_o^{-1}(\Phi - \Phi_o)$ *– QG stream function* *geostrophic balance relation is lowest in hierarchy of more accurate balance relations*

However, it highlights many important properties of PV inversion:

$$PV_g = f_o + \beta y + \left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \rho_o^{-1}\frac{\partial}{\partial z}\left(\rho_o \frac{f_o^2}{N^2(z)}\frac{\partial\psi}{\partial z}\right)$$

$$PV_g = \Gamma(\psi)$$

inversion $\Rightarrow \quad \psi = \Gamma^{-1}(PV_g)$

inversion is a non-local process

local knowledge of PV does not imply local knowledge of $\boldsymbol{u}$

inversion operator (elliptic) is a smoothing operator

small-scale features in PV have weak effect on $\boldsymbol{u}$*, while large-scale features in PV have strong effect on* $\boldsymbol{u}$

very full literature of higher-order balance conditions and inversion operators which provide increased accuracy relative to PE solutions

(McIntyre and Norton 2000)

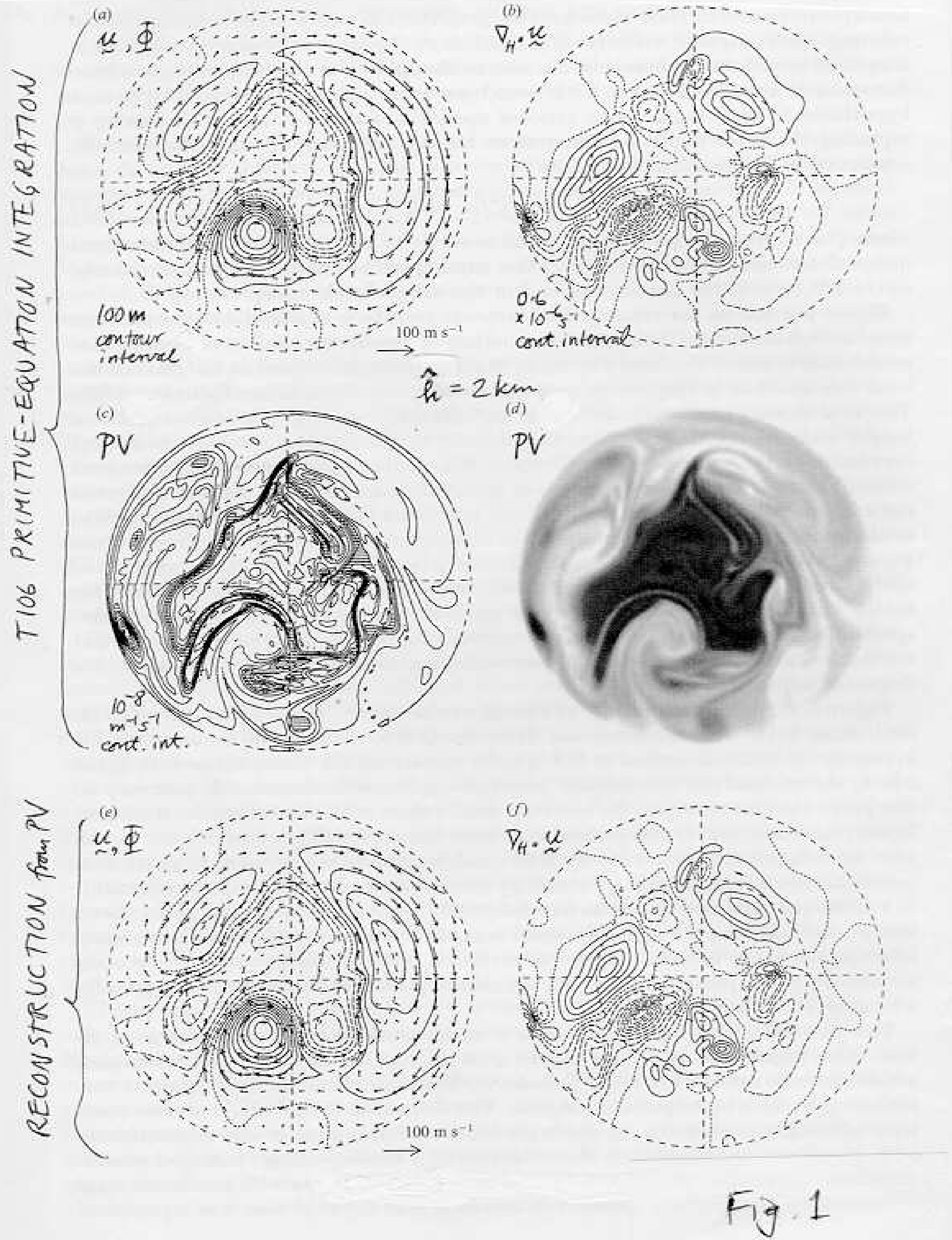
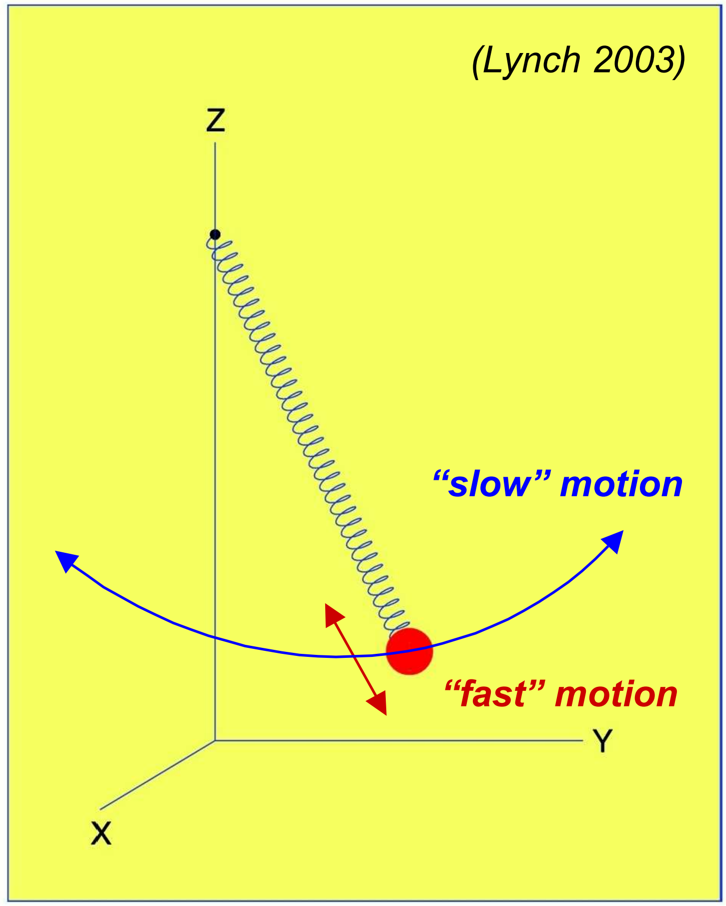


Fig. 1

Does an exact balance condition exist?



Low order dynamical systems (e.g., "stiff" spring pendulum) indicate possibly Yes.

For ocean-atmosphere dynamics the answer is almost certainly No.

Time evolving balanced flows tend to radiate sound and gravity waves ("spontaneous emission") which are, by definition, not described by balance

Balanced Models

– *employ balance condition from the start*

⇒ *imposes slow manifold on the problem*

– *one prognostic equation for "master variable" (e.g., PV)*

Hamiltonian balanced models

– *impose balance condition as a constraint on full dynamics within Hamiltonian framework (Salmon 1988)*

– *Hamiltonian framework allows control over conservation principles (e.g., mass, momentum, and energy)*

trade off:

accuracy of balance model vs conservation

"velocity splitting" (McIntyre and Roulstone 2002)

–*if one requires both accuracy **and** conservation then two velocities are required*

⇒ *one velocity to advect the PV and another to evaluate the PV (Hamiltonian models)*

or

⇒ *one velocity to advect the mass and another to advect the PV (higher-order balance models)*

– *two velocities differ by very little but they must differ to allow for fast dynamics*